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On semi-reduced quadratic forms, continued fractions and class number

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§1. Introduction

Let m be a square-free integer > 1 and let Δ be the discriminant of the real quadratic field $h(\mathbb{Q}(\sqrt{m}))$. Define

$$\Delta = \begin{cases} m, \\ 4m, \end{cases} \quad \tilde{\omega} = \begin{cases} \frac{1}{2}+\frac{1}{2}\sqrt{m}, \\ \sqrt{m}, \end{cases} \text{if} \begin{cases} m \equiv 1 \pmod 4, \\ m \equiv 2,\,3 \pmod 4, \end{cases}$$

$$\theta = \begin{cases} 0 \text{ if } m \equiv 1 \pmod 4, \ \ell \text{ even and } 2 \not| k_{\ell/2}, \\ 1 \text{ if } \begin{cases} m \equiv 1 \pmod 4, & \ell \text{ even and } 2 | k_{\ell/2}, \\ m \equiv 1 \pmod 4, & \ell \text{ odd}, \\ m \equiv 2,\,3 \pmod 4, & \ell \text{ even and } 2 \not| k_{\ell/2}, \end{cases} \\ 2 \text{ if } \begin{cases} m \equiv 2,\,3 \pmod 4, & \ell \text{ even and } 2 | k_{\ell/2}, \\ m \equiv 2,\,3 \pmod 4, & \ell \text{ odd}. \end{cases} \end{cases}$$

Write $\tilde{\omega}$ as a continued fraction:

$$\tilde{\omega} = \left[k_0, \overline{k_1, \ldots, k_\ell}\right].$$

Let $\lambda_1(m)$ (resp. $\lambda_2(m)$) be the number of solutions of

$$x^2 + 4yz = \Delta \quad (\text{resp. } x^2 + 4y^2 = \Delta)$$

with x, y, $z \in I\!N = \{0, 1, 2, \ldots\}$. Then H. Lu [Lu] proved the following result.

Theorem (Lu). *The class number h_Δ of $\mathbb{Q}(\sqrt{m})$ is equal to 1 if and only if*

$$\theta + \sum_{i=1}^{\ell} k_i = \lambda_1(m) + \lambda_2(m).$$

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Problem. Generalize Lu's result. More precisely, use continued fractions to list the elements of the set

$$\left\{aX^2+bXY+cY^2 : \ a,b,c \in \mathbb{Z},\ b^2-4ac=\Delta \ \text{ with } \ 0\le b<\sqrt{\Delta}\right\}$$

of *semi-reduced quadratic forms* of discriminant Δ and relate h_Δ^+ (resp. h_Δ) to the cardinality of this last set, i.e., to the cardinality of the set

$$\left\{(a,b,c)\ \in\ \mathbb{Z}^3 :\ \Delta=b^2-4ac \ \text{ with } \ 0\le b<\sqrt{\Delta}\right\}.$$

§2. Preliminaries

A quadratic form f is a homogeneous polynomial of the form

$$f=f(X,Y)\ =<a,b,c>=\ aX^2+bXY+cY^2, \ \text{ with } \ a,\ b,\ c\ \in \mathbb{Z},$$

which we may write in matrix form as

$$f(X,Y)=\begin{pmatrix} X & Y\end{pmatrix}\begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c\end{pmatrix}\begin{pmatrix} X \\ Y\end{pmatrix}.$$

We say that the matrix of f is

$$M_f=\begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c\end{pmatrix}.$$

By definition, the *discriminant* of f is

$$\Delta=\Delta_f=b^2-4ac;$$

moreover f is *primitive* if $pgcd\,(a,b,c)=1$; f is *definite positive* if $\Delta<0,\ a>0,\ c>0$; and f is *indefinite* if $\Delta>0$.

Consider

$$\mathcal{F}_\Delta=\{\text{primitive quadratic forms of discriminant } \Delta\}.$$

We need

$$GL_2(\mathbb{Z})=\left\{\begin{pmatrix} r & s \\ t & u\end{pmatrix} :\ r,\ s,\ t,\ u\ \in \mathbb{Z},\ \ ru-st=\pm 1\right\},$$

$$SL_2(\mathbb{Z})=\left\{\begin{pmatrix} r & s \\ t & u\end{pmatrix} :\ r,\ s,\ t,\ u\ \in \mathbb{Z},\ \ ru-st=+1\right\}.$$

We can define an action of $A \in GL_2(\mathbb{Z})$ on f by stating that

$$g = Af \text{ where } M_g = AM_fA^t.$$

Moreover we say:

$$f \sim g \ \ (f \text{ is equivalent to } g) \iff g = Af \ \text{ for some } A \in GL_2(\mathbb{Z}),$$

$$f \approx g \ \ (f \text{ is strictly equivalent to } g) \iff g = Af \ \text{ for some } A \in SL_2(\mathbb{Z}).$$

It turns out that the *class group* (resp. *strict class group*) of $\mathbb{Q}(\sqrt{\Delta}$ is $\mathcal{F}_\Delta/\sim$ (resp. $\mathcal{F}_\Delta/\approx$), its cardinality being denoted the *class number* h_Δ (resp. the *strict class number* h_Δ^+) of $\mathbb{Q}(\sqrt{\Delta})$.

We are extensively working with the matrices

$$E(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad A(u) = \begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix}.$$

We associate to f the quadratic numbers

$$\omega = \omega(f) = \frac{b+\sqrt{\Delta}}{2|c|} \text{ and } \Omega = \Omega(f) = \frac{b+\sqrt{\Delta}}{2|a|},$$

and define

$$\sigma(f) = \text{sign}(a).$$

Gauss defined the *right neighbour* $R(f)$ and the *left neighbour* $L(f)$ of f:

$$Rf = A(e)f = <c, -b-2ce, a+be+ce^2> \qquad \text{with } \ e = -\text{sign}(c)\,[\omega(f)],$$

$$Lf = A(E)^{-1}f = <c+bE+aE^2, -b-2aE, a> \ \text{ with } \ E = -\text{sign}(a)\,[\Omega(f)].$$

We say that $f = <a,b,c>$ with $\Delta > 0$ is *reduced* if

$$\begin{cases} (i) & 0 < b < \sqrt{\Delta}, \\ (ii) & \sqrt{\Delta} - b < 2|c| < \sqrt{\Delta} + b, \end{cases}$$

which is equivalent to saying

$$\begin{cases} (i) & 0 < b < \sqrt{\Delta}, \\ (ii) & \sqrt{\Delta} - b < 2|a| < \sqrt{\Delta} + b. \end{cases}$$

Gauss showed how to calculate the class number h_Δ^+. With the continued fraction algorithm, write the finite set of reduced quadratic forms of discriminant $\Delta > 0$ as a union of h_Δ disjoint cycles, by picking up a reduced quadratic form $f = \langle a, b, c \rangle$ and taking the continued fraction of

$$\omega_0 = \omega(f) = \frac{b+\sqrt{\Delta}}{2|c|} = \frac{P_0+\sqrt{\Delta}}{2Q_0},$$

i.e., for $i \geq 0$,

$$\begin{cases} \omega_i &= \dfrac{P_i+\sqrt{\Delta}}{2Q_i}, \\ P_{i+1} &= 2k_iQ_i - P_i \quad \text{with } k_i = [\omega_i], \\ Q_{i+1} &= \dfrac{\Delta - P_{i+1}^2}{4Q_i}, \end{cases}$$

where [ ] is the greatest integer function. Hence

P_0	P_1	$\dots$	$P_{\ell-1}$	P_ℓ	$P_{\ell+1}$	$\dots$	P_{2l-1}	$\dots$
Q_0	Q_1	$\dots$	$Q_{\ell-1}$	Q_ℓ	$Q_{\ell+1}$	$\dots$	$Q_{2\ell-1}$	$\dots$
k_0	k_1	$\dots$	$k_{\ell-1}$	k_ℓ	$k_{\ell+1}$	$\dots$	$k_{2\ell-1}$	$\dots$

It is well known that f is reduced if and only if $\omega(f)$ is a reduced quadratic number (i.e., the continued fraction expansion of $\omega(f)$ is purely periodic).

Let

$$\tilde{\ell} = \begin{cases} \ell & \text{if } \ell \text{ is even,} \\ 2\ell & \text{if } \ell \text{ is odd.} \end{cases}$$

The *cycle* of f is

$$[f_1, f_2, \dots, f_{\tilde{\ell}}],$$

where

$$f = f_1 = \langle a, b, c \rangle = \langle \sigma(f)Q_0, P_1, -\sigma(f)Q_1 \rangle$$

and where for $i = 1, \dots, \tilde{\ell} - 1$,

$$\begin{aligned} f_{i+1} &= R(f_i) \\ &= \left\langle (-1)^i\sigma(f)Q_i, P_{i+1}, (-1)^{i+1}\sigma(f)Q_{i+1} \right\rangle \\ &\doteq A_i f_i \end{aligned}$$

with

$$A_i = A\left(\sigma(f_i)k_i\right) = A\left((-1)^{i-1}\sigma(f)k_i\right).$$

In symbols,

$$\begin{cases} f_{i+1} &= Rf_i \quad \text{for } 1 \leq i \leq \tilde{\ell} \text{ with } f_1 = Rf_{\tilde{\ell}}, \\ f_j &= Lf_{j+1} \quad \text{for } 0 \leq j < \tilde{\ell} \text{ with } f_{\tilde{\ell}} = Lf_1, \end{cases}$$

and $k_i = [\omega(f_i)]$.

Fact: h^+_Δ is the number of cycles. Moreover $h^+_\Delta = h_\Delta$ (resp. $h^+_\Delta = 2h_\Delta$) if the length of the continued fraction of $\omega(f)$ for any reduced quadratic form f is odd (resp. even).

See section 7 for examples of calculations of cycles.

§3. Semi-reduced forms

A quadratic form $f = <a, b, c>$ of discriminant Δ is *semi-reduced* if

$$0 \leq b < \sqrt{\Delta},$$

and f is said to be *intermediate* if f est semi-reduced without being reduced.

In each of the following examples, we give the list of semi-reduced quadratic forms, the reduced forms being written in boldface, the intermediate forms appearing in roman style.

Example 3.1. Let $m = 14$. There are 24 semi-reduced quadratic forms of discriminant $\Delta = 56$:

$<1,0,-14>$	$<-1,0,14>$	$<2,0,-7>$	$<-2,0,7>$	$<7,0,-2>$	$<-7,0,2>$
$<1,2,-13>$	$<-1,2,13>$	$<14,0,-1>$	$<-14,0,1>$	$<13,2,-1>$	$<-13,2,1>$
$<1,4,-10>$	$<-1,4,10>$	$\mathbf{<2,4,-5>}$	$\mathbf{<-2,4,5>}$	$\mathbf{<5,4,-2>}$	$\mathbf{<-5,4,2>}$
$<10,4,-1>$	$<-10,4,1>$	$\mathbf{<1,6,-5>}$	$\mathbf{<-1,6,5>}$	$\mathbf{<5,6,-1>}$	$\mathbf{<-5,6,1>}$

Example 3.2. Let $m = 19$. There are 36 semi-reduced quadratic forms of discriminant $\Delta = 76$:

$<1,0,-19>$	$<-1,0,19>$	$<19,0,-1>$	$<-19,0,1>$	$<1,2,-18>$	$<-1,2,18>$
$<2,2,-9>$	$<-2,2,9>$	$<3,2,-6>$	$<-3,2,6>$	$<6,2,-3>$	$<-6,2,3>$
$<9,2,-2>$	$<-9,2,2>$	$<18,2,-1>$	$<-18,2,1>$	$<1,4,-15>$	$<-1,4,15>$
$\mathbf{<3,4,-5>}$	$\mathbf{<-3,4,5>}$	$\mathbf{<5,4,-3>}$	$\mathbf{<-5,4,3>}$	$<15,4,-1>$	$<-15,4,1>$
$<1,6,-10>$	$<-1,6,10>$	$\mathbf{<2,6,-5>}$	$\mathbf{<-2,6,5>}$	$\mathbf{<5,6,-2>}$	$\mathbf{<-5,6,2>}$
$<10,6,-1>$	$<-10,6,1>$	$\mathbf{<1,8,-3>}$	$\mathbf{<-1,8,3>}$	$\mathbf{<3,8,-1>}$	$\mathbf{<-3,8,1>}$

Example 3.3. Let $m = 26$. There are 36 semi-reduced quadratic forms of discriminant $\Delta = 104$:

$<1,0,-26>$	$<-1,0,26>$	$<2,0,-13>$	$<-2,0,13>$	$<13,0,-2>$	$<-13,0,2>$
$<26,0,-1>$	$<-26,0,1>$	$<1,2,-25>$	$<-1,2,25>$	$\mathbf{<5,2,-5>}$	$\mathbf{<-5,2,5>}$
$<25,2,-1>$	$<-25,2,1>$	$<1,4,-22>$	$<-1,4,22>$	$<2,4,-11>$	$<-2,4,11>$
$<11,4,-2>$	$<-11,4,2>$	$<22,4,-1>$	$<-22,4,1>$	$<1,6,-17>$	$<-1,6,17>$
$<17,6,-1>$	$<-17,6,1>$	$<1,8,-10>$	$<-1,8,10>$	$\mathbf{<2,8,-5>}$	$\mathbf{<-2,8,5>}$
$\mathbf{<5,8,-2>}$	$\mathbf{<-5,8,2>}$	$<10,8,-1>$	$<-10,8,1>$	$\mathbf{<1,10,-1>}$	$\mathbf{<-1,10,1>}$

Example 3.4. Let $m = 33$. There are 20 semi-reduced quadratic forms of discriminant $\Delta = 33$:

$<1, 1, -8>$	$<-1, 1, 8>$	$<2, 1, -4>$	$<-2, 1, 4>$	$<4, 1, -2>$
$<-4, 1, 2>$	$<8, 1, -1>$	$<-8, 1, 1>$	$<1, 3, -6>$	$<-1, 3, 6>$
$\mathbf{<2, 3, -3>}$	$\mathbf{<-2, 3, 3>}$	$\mathbf{<3, 3, -2>}$	$\mathbf{<-3, 3, 2>}$	$<6, 3, -1>$
$<-6, 3, 1>$	$\mathbf{<1, 5, -2>}$	$\mathbf{<-1, 5, 2>}$	$\mathbf{<2, 5, -1>}$	$\mathbf{<-2, 5, 1>}$

Example 3.5. Let $m = 35$. There are 40 semi-reduced quadratic forms of discriminant $\Delta = 140$:

$<19, 8, -1>$	$<-19, 8, 1>$	$\mathbf{<1, 10, -10>}$	$\mathbf{<-1, 10, 10>}$	$<1, 8, -19>$
$<-1, 8, 19>$	$<1, 0, -35>$	$<-1, 0, 35>$	$<5, 0, -7>$	$<-5, 0, 7>$
$<7, 0, -5>$	$<-7, 0, 5>$	$<35, 0, -1>$	$<-35, 0, 1>$	$<1, 2, -34>$
$<-1, 2, 34>$	$<2, 2, -17>$	$<-2, 2, 17>$	$<1, 4, -31>$	$<-1, 4, 31>$
$<17, 2, -2>$	$<-17, 2, 2>$	$<34, 2, -1>$	$<-34, 2, 1>$	$<31, 4, -1>$
$<-31, 4, 1>$	$<1, 6, -26>$	$<-1, 6, 26>$	$<2, 6, -13>$	$<-2, 6, 13>$
$<13, 6, -2>$	$<-13, 6, 2>$	$<26, 6, -1>$	$<-26, 6, 1>$	$\mathbf{<2, 10, -5>}$
$\mathbf{<-2, 10, 5>}$	$\mathbf{<5, 10, -2>}$	$\mathbf{<-5, 10, 2>}$	$\mathbf{<10, 10, -1>}$	$\mathbf{<-10, 10, 1>}$

Let $f = <a, b, c>$ be a reduced form of discriminant Δ such that $e = \sigma(f)[\omega(f)]$. Consider the quadratic forms associated to f and given by the following two cases (the first may be empty):

Case (i)	**Case (ii)**
$E(u)f = <a + bu + cu^2, b + 2cu, c>$ with $1 \leq \sigma(f)u \leq \dfrac{b}{2\lvert c\rvert}$	$A(u)f = <c, -b - 2cu, a + bu + cu^2>$ with $\dfrac{b}{2\lvert c\rvert} \leq \sigma(f)u \leq \sigma(f)e = [\omega(f)]$

We can show that the quadratic forms of **Cases (i)** and **(ii)** are all semi-reduced (i.e., are either reduced or intermediate) and that the intermediate forms $<a', b', c'>$ of **Cases (i)** and **(ii)** respectively verify the following properties:

$$\textbf{Case (i)} \begin{cases} a'c' < 0, \\ |a'| > |c'|, \\ R<a', b', c'> = Rf, \end{cases} \qquad \textbf{Case (ii)} \begin{cases} a'c' < 0, \\ |a'| < |c'|, \\ L<a', b', c'> = f. \end{cases}$$

These last properties characterize intermediate quadratic forms. We can show that intermediate forms come from reduced forms. More precisely, let $g = <a', b', c'>$ be an intermediate quadratic form. Then the following properties hold true:

(*i*) If $|a'| > |c'|$, then the forms $f_1 = Rg$ and $f = Lf_1 = <a, b, c>$ are both reduced and

$$g = E(u)f = <a + bu + cu^2, b + 2cu, c>$$

with

$$1 \leq \sigma(f)u \leq \frac{b}{2|c|}.$$

(*ii*) If $|a'| < |c'|$, then the form $f = Lg = <a, b, c>$ is reduced and

$$g = A(u)f = <c, -b - 2cu, a + bu + cu^2>$$

with

$$\frac{b}{2|c|} \leq \sigma(f)u \leq \sigma(f)e = [\omega(f)].$$

We can now state our first result.

Theorem. (1) *For each reduced form $f = <a, b, c>$ of discriminant Δ, the quadratic forms associated to f and defined by*

$$\begin{cases} E(u)f & \text{with } \ 1 \leq \sigma(f)u \leq \frac{b}{2|c|}, \\ A(u)f & \text{with } \ \frac{b}{2|c|} \leq \sigma(f)u \leq [\omega(f)], \end{cases}$$

are all different from one another and form a set $I(f)$ of cardinality

$$\#I(f) = \begin{cases} [\omega(f)] & \text{if } \ (2c) \nmid b, \\ 1 + [\omega(f)] & \text{if } \ (2c) | b. \end{cases}$$

(2) *If f and g are two different reduced forms, then $I(f) \cap I(g) = \phi$.*

(3) *Moreover the (disjoint) union of the $I(f)$'s when f runs through the set Red(Δ) of reduced forms is a set equal to the set of semi-reduced forms and is of cardinality*

$$2\left(\sum_{\substack{0 \leq t < \sqrt{\Delta} \\ t \equiv \Delta \ (mod\ 2)}} \tau\left(\frac{\Delta - t^2}{4}\right)\right),$$

where τ is the number of divisors *function.*

§5. The palindrome level

Suppose that the continued fraction of a quadratic number α is purely periodic (i.e. α is reduced):

$$\alpha = \left[\overline{k_1, k_2, \ldots, k_\ell}\right]$$

with ℓ minimal. We associate to α the word

$$w(\alpha) = k_1 k_2 \ldots k_\ell$$

and say that it is defined up to a cyclic permutation (of order ℓ).

Recall that a *palindrome* (*KA I BU N*) is a word which once read from left to right or from right to left is the same.

Examples. In french, english, german and japanese:

(*i*) ELU PAR CETTE CRAPULE

(*ii*) NAME NO ONE MAN

(*iii*) EIN NEGER MIT GAZELLE ZAGT IM REGEN NIE

(*iv*) さくはなの　のもきよきかに　かおるよる
おかにかきよき　もののなはくさ

(*v*) てきをのし　いさましいし　まさいしのおきて

(*vi*) たけやぶ　やけた

When ℓ is odd, we say that α has 1 (resp. 0) *central element* if $w(\alpha)$ is (resp. is not) a palindrome, the central element being the center of $w(\alpha)$. When ℓ is even, we say α has 2 *central elements* (resp. 0 *central element*) if after an eventual cyclic permutation, $w(\alpha)$ is (resp. is not) the concatenation of a palindrome of length $\ell - 1$ and of a palindrome of length 1, the two central elements being the two centers of the concatenated palindromes.

Definition. The *palindrome level* $s = s(\alpha) \in \{0, 1, 2\}$ of a reduced quadratic number is equal to the number of even central elements.

In examples 7.1 to 7.5, s is respectively equal to 2, 1, 1, 0, 0.

§6. The main results

We can now relate the class number h_Δ of $\mathbb{Q}(\sqrt{\Delta})$ to the cardinality of the set of semi-reduced forms by writing this cardinality in terms of the partial quotients of non equivalent reduced quadratic numbers.

Theorem. *Let Δ be the discriminant of a real quadratic field $\mathbb{Q}(\sqrt{\Delta})$. Put*

$$g = \begin{cases} 0 & \text{if } \Delta \equiv 1 \quad (mod\ 4), \\ 2^{r+1} & \text{if } \Delta \equiv 4 \text{ ou } 12 \quad (mod\ 16), \\ 2^{r+2} & \text{if } \Delta \equiv 8 \quad (mod\ 16), \end{cases}$$

where r is the number of odd primes dividing Δ. Then the class number h_Δ of $\mathbb{Q}(\sqrt{\Delta})$ is equal to h_0 if and only if there exist h_0 reduced quadratic numbers $\alpha^{(1)}$, $\alpha^{(2)}$, ..., $\alpha^{(h_0)}$ associated to h_0 quadratic forms non equivalent to one another such that the number of all semi-reduced forms of discriminant Δ is

$$2\left(s_\Delta + \sum_{j=1}^{h_0} \sum_{t=1}^{\ell_j} k_t^{(j)}\right),$$

where for $j = 1, \ldots, h_0$,

$$\alpha^{(j)} = \left[\overline{k_1^{(j)}, k_2^{(j)}, \ldots, k_{\ell_j}^{(j)}}\right]$$

with ℓ_j minimal, and where the palindrome level s_Δ of Δ, defined as the sum of all the palindrome levels of the $\alpha^{(j)}$'s, is equal to

$$s_\Delta = \sum_{j=1}^{h_0} s(\alpha^{(j)}) = \frac{g}{2}.$$

Theorem. *Let Δ be the discriminant of a real quadratic field $\mathbb{Q}(\sqrt{\Delta})$. Put*

$$g_1 = \begin{cases} 0 & \text{if } \Delta \equiv 1 \quad (mod\ 4), \\ 2^{r} & \text{if } \Delta \equiv 4 \text{ ou } 12 \quad (mod\ 16), \\ 2^{r+1} & \text{if } \Delta \equiv 8 \quad (mod\ 16), \end{cases}$$

where r is the number of odd primes dividing Δ. Then the class number h_Δ of $\mathbb{Q}(\sqrt{\Delta})$ is equal to h_1 if and only if there exist h_1 reduced quadratic numbers $\beta^{(1)}$, $\beta^{(2)}$, ..., $\beta^{(h_1)}$ associated to h_1 quadratic forms non equivalent to one another such that

$$\#\{(A, B, C) \in I\!N^3 : \Delta = B^2 + 4AC\} = s_\Delta + \sum_{j=1}^{h_1} \sum_{t=1}^{\ell_j} k_t^{(j)},$$

where for $j = 1, \ldots, h_1$,

$$\beta^{(j)} = \left[\overline{k_1^{(j)}, k_2^{(j)}, \ldots, k_{\ell_j}^{(j)}}\right]$$

with ℓ_j minimal, and where the palindrome level s_Δ of Δ, defined as the sum of all the palindrome levels of the $\beta^{(j)}$'s, is

$$s_\Delta = \sum_{j=1}^{h_1} s(\beta^{(j)}) = g_1.$$

§7. Five examples

One has a chance of verifying the last theorems with the following examples.

Example 7.1. Let $m = 14$, so $\Delta = 56$. Here $h_\Delta^+ = 2$ and $h_\Delta = 1$. The continued fraction expansion of $\omega = \frac{4+\sqrt{56}}{2\cdot 2}$ is $[\overline{2,1,6,1}]$:

$$\omega = \frac{2+2\sqrt{10}}{2\cdot 3} \longleftrightarrow$$

4	4	6	6	4	...
2	5	1	5	2	...
2	1	6	1	2	...

Since $w(\omega) = \underbrace{1,\boxed{6},1}, \underbrace{\boxed{2}}$, we have $s(\omega) = 2$. There are two cycles of reduced forms (in boldface) to which are attached the intermediate forms:

$$\begin{array}{llllll}
\langle -7,0,2\rangle & \leftarrow\!\!\top & \boldsymbol{\langle -5,4,2\rangle} & \Rightarrow & \boldsymbol{\langle 2,4,-5\rangle} & \\
\langle 2,0,-7\rangle & \leftarrow\!\!\lrcorner & \Uparrow & & \Downarrow & \\
 & & \boldsymbol{\langle 1,6,-5\rangle} & \Leftarrow & \boldsymbol{\langle -5,6,1\rangle} & \to \langle -10,4,1\rangle \\
 & & & & & \to \langle -13,2,1\rangle \\
 & & & & & \to \langle -14,0,1\rangle \\
 & & & & & \to \langle 1,0,-14\rangle \\
 & & & & & \to \langle 1,2,-13\rangle \\
 & & & & & \to \langle 1,4,-10\rangle
\end{array}$$

$$\begin{array}{llllll}
\langle 7,0,-2\rangle & \leftarrow\!\!\top & \boldsymbol{\langle 5,4,-2\rangle} & \Rightarrow & \boldsymbol{\langle -2,4,5\rangle} & \\
\langle -2,0,7\rangle & \leftarrow\!\!\lrcorner & \Uparrow & & \Downarrow & \\
 & & \boldsymbol{\langle -1,6,5\rangle} & \Leftarrow & \boldsymbol{\langle 5,6,-1\rangle} & \to \langle 10,4,-1\rangle \\
 & & & & & \to \langle 13,2,-1\rangle \\
 & & & & & \to \langle 14,0,-1\rangle \\
 & & & & & \to \langle -1,0,14\rangle \\
 & & & & & \to \langle -1,2,13\rangle \\
 & & & & & \to \langle -1,4,10\rangle
\end{array}$$

Example 7.2. Let $m = 19$, so $\Delta = 76$. Here $h^+_\Delta = 2$ and $h_\Delta = 1$. The continued fraction expansion of $\omega = \frac{4+\sqrt{76}}{2\cdot 5}$ is $[\overline{1,3,1,2,8,2}]$:

$$\omega = \frac{4+\sqrt{76}}{2\cdot 5} \longleftrightarrow \begin{array}{|cccccc|cc|} \hline 4 & 6 & 6 & 4 & 8 & 8 & 4 & \dots \\ 5 & 2 & 5 & 3 & 1 & 3 & 5 & \dots \\ \hline 1 & 3 & 1 & 2 & 8 & 2 & 1 & \dots \\ \hline \end{array}$$

Since $w(\omega) = \underbrace{1,2,\boxed{8},2,1}, \underbrace{\boxed{3}},$ we have $s(\omega) = 1$. There are two cycles of reduced forms (in boldface) to which are attached the intermediate forms:

$$\begin{array}{lclcllcl}
\langle 9,2,-2\rangle & \leftarrow & \langle \mathbf{5,6,-2}\rangle & \Rightarrow & \langle \mathbf{-2,6,5}\rangle & & \\
\langle -2,2,9\rangle & \leftarrow & \Uparrow & & \Downarrow & & \\
 & & \langle \mathbf{-3,4,5}\rangle & & \langle \mathbf{5,4,-3}\rangle & \longrightarrow & \langle -3,2,6\rangle \\
 & & \Uparrow & & \Downarrow & & \\
\langle 6,2,-3\rangle & \longleftarrow & \langle \mathbf{1,8,-3}\rangle & \Longleftarrow & \langle \mathbf{-3,8,1}\rangle & \rightarrow & \langle -10,6,1\rangle \\
 & & & & & \rightarrow & \langle -15,4,1\rangle \\
 & & & & & \rightarrow & \langle -18,2,1\rangle \\
 & & & & & \rightarrow & \langle -19,0,1\rangle \\
 & & & & & \rightarrow & \langle 1,0,-19\rangle \\
 & & & & & \rightarrow & \langle 1,2,-18\rangle \\
 & & & & & \rightarrow & \langle 1,4,-15\rangle \\
 & & & & & \rightarrow & \langle 1,6,-10\rangle
\end{array}$$

$$\begin{array}{lclcllcl}
\langle -9,2,2\rangle & \leftarrow & \langle \mathbf{-5,6,2}\rangle & \Rightarrow & \langle \mathbf{2,6,-5}\rangle & & \\
\langle 2,2,-9\rangle & \leftarrow & \Uparrow & & \Downarrow & & \\
 & & \langle \mathbf{3,4,-5}\rangle & & \langle \mathbf{-5,4,3}\rangle & \longrightarrow & \langle 3,2,-6\rangle \\
 & & \Uparrow & & \Downarrow & & \\
\langle -6,2,3\rangle & \longleftarrow & \langle \mathbf{-1,8,3}\rangle & \Longleftarrow & \langle \mathbf{3,8,-1}\rangle & \rightarrow & \langle 10,6,-1\rangle \\
 & & & & & \rightarrow & \langle 15,4,-1\rangle \\
 & & & & & \rightarrow & \langle 18,2,-1\rangle \\
 & & & & & \rightarrow & \langle 19,0,-1\rangle \\
 & & & & & \rightarrow & \langle -1,0,19\rangle \\
 & & & & & \rightarrow & \langle -1,2,18\rangle \\
 & & & & & \rightarrow & \langle -1,4,15\rangle \\
 & & & & & \rightarrow & \langle -1,6,10\rangle
\end{array}$$

Example 7.3. Let $m = 26$, so $\Delta = 104$. Here $h_\Delta^+ = h_\Delta = 2$. The continued fraction expansion of $\omega_1 = \frac{8+\sqrt{124}}{2\cdot 2}$ is $\left[\overline{4,1,1}\right]$:

$$\omega_1 = \frac{8+\sqrt{104}}{2\cdot 2} \longleftrightarrow \begin{array}{|ccc|cc|} \hline 8 & 8 & 2 & 8 & \cdots \\ 2 & 5 & 5 & 2 & \cdots \\ \hline \hline 4 & 1 & 1 & 4 & \cdots \\ \hline \end{array}$$

Since $w(\omega_1) = \underbrace{1,\boxed{4},1,}$ we have $s(\omega_1) = 1$.

The continued fraction expansion of $\omega_2 = \frac{10+\sqrt{124}}{2\cdot 1}$ is $\left[\overline{10}\right]$:

$$\omega_2 = \frac{10+\sqrt{104}}{2\cdot 1} \longleftrightarrow \begin{array}{|c|c|c|} \hline 10 & 10 & \cdots \\ 1 & 1 & \cdots \\ \hline \hline 10 & 10 & \cdots \\ \hline \end{array}$$

Since $w(\omega_2) = \underbrace{\boxed{10},}$ we have $s(\omega_2) = 1$. There are two cycles of reduced forms (in boldface) to which are attached the intermediate forms:

$$\begin{array}{llcccll}
\langle -11,4,2\rangle & \leftarrow & \langle \mathbf{-5,8,2}\rangle & \Rightarrow & \langle \mathbf{2,8,-5}\rangle & & \\
\langle -13,0,2\rangle & \leftarrow & \Uparrow & & \Downarrow & & \\
\langle 2,0,-13\rangle & \leftarrow & \langle \mathbf{5,2,-5}\rangle & & \langle \mathbf{-5,2,5}\rangle & & \\
\langle 2,4,-11\rangle & \leftarrow & \Uparrow & & \Downarrow & & \\
 & & \langle \mathbf{-2,8,5}\rangle & \Leftarrow & \langle \mathbf{5,8,-2}\rangle & \rightarrow & \langle 11,4,-2\rangle \\
 & & & & & \rightarrow & \langle 13,0,-2\rangle \\
 & & & & & \rightarrow & \langle -2,0,13\rangle \\
 & & & & & \rightarrow & \langle -2,4,11\rangle
\end{array}$$

$$
\begin{array}{lcccccl}
 & & \mathbf{<1,10,-1>} & \Rightarrow & \mathbf{<-1,10,1>} & & \\
 & & \| & & \| & & \\
<10,8,-1> & \leftarrow & \mathbf{<1,10,-1>} & \Leftarrow & \mathbf{<-1,10,1>} & \rightarrow & <-10,8,1> \\
<17,6,-1> & \leftarrow & & & & \rightarrow & <-17,6,1> \\
<22,4,-1> & \leftarrow & & & & \rightarrow & <-22,4,1> \\
<25,2,-1> & \leftarrow & & & & \rightarrow & <-25,2,1> \\
<26,0,-1> & \leftarrow & & & & \rightarrow & <-26,0,1> \\
<-1,0,26> & \leftarrow & & & & \rightarrow & <1,0,-26> \\
<-1,2,25> & \leftarrow & & & & \rightarrow & <1,2,-25> \\
<-1,4,22> & \leftarrow & & & & \rightarrow & <1,4,-22> \\
<-1,6,17> & \leftarrow & & & & \rightarrow & <1,6,-17> \\
<-1,8,10> & \leftarrow & & & & \rightarrow & <1,8,-10>
\end{array}
$$

Example 7.4. Let $m = 33$, so $\Delta = 33$. Here $h_\Delta^+ = 2$ and $h_\Delta = 1$. The continued fraction expansion of $\omega = \frac{5+\sqrt{33}}{2\cdot 1}$ is $\left[\overline{5,2,1,2}\right]$:

$$
\omega = \frac{5+\sqrt{33}}{2\cdot 1} \longleftrightarrow
\begin{array}{|cccc|cc|}
\hline
5 & 5 & 3 & 3 & 5 & \dots \\
1 & 2 & 3 & 2 & 1 & \dots \\
\hline
5 & 2 & 1 & 2 & 5 & \dots \\
\hline
\end{array}
$$

Since $w(\omega) = \underbrace{2,\boxed{1},2}, \underbrace{\boxed{5}}$, we have $s(\omega) = 0$. There are two cycles of reduced forms (in boldface) to which are attached the intermediate forms:

$$
\begin{array}{lcccccl}
 & & \mathbf{<-2,3,3>} & \Rightarrow & \mathbf{<3,3,-2>} & \longrightarrow & <-2,1,4> \\
 & & \Uparrow & & \Downarrow & & \\
<4,1,-2> & \longleftarrow & \mathbf{<1,5,-2>} & \Leftarrow & \mathbf{<-2,5,11>} & \rightarrow & <-6,3,1> \\
 & & & & & \rightarrow & <-8,1,1> \\
 & & & & & \rightarrow & <1,1,-8> \\
 & & & & & \rightarrow & <1,3,-6>
\end{array}
$$

$$
\begin{array}{ccccccc}
 & & \mathbf{<2,3,-3>} & \Rightarrow & \mathbf{<-3,3,2>} & \longrightarrow & <2,1,-4> \\
 & & \Uparrow & & \Downarrow & & \\
<-4,1,2> & \longleftarrow & \mathbf{<-1,5,2>} & \Leftarrow & \mathbf{<2,5,-1>} & \longmapsto & <6,3,-1> \\
 & & & & & \mapsto & <8,1,-1> \\
 & & & & & \mapsto & <-1,1,8> \\
 & & & & & \mapsto & <-1,3,6>
\end{array}
$$

Example 7.5. Let $m = 41$, so $\Delta = 41$. We will see $h_\Delta^+ = h_\Delta = 1$. The continued fraction expansion of $\omega = \frac{5+\sqrt{41}}{2\cdot 1}$ is $[\overline{5,1,2,2,1}]$:

$$
\omega = \frac{5+\sqrt{41}}{2\cdot 1} \longleftrightarrow
\begin{array}{|ccccc|cc|}
\hline
5 & 5 & 3 & 5 & 3 & 5 & \ldots \\
1 & 4 & 2 & 2 & 4 & 1 & \ldots \\
\hline
5 & 1 & 2 & 2 & 1 & 5 & \ldots \\
\hline
\end{array}
$$

Since $w(\omega) = \underbrace{2, 1, \boxed{5}, 1, 2,}$ we have $s(\omega) = 0$. There is only one cycle of reduced forms (in boldface) to which are attached the intermediate forms.

$$
\begin{array}{ccccccc}
<-8,3,1> & \longleftarrow & \mathbf{<-4,5,1>} & \Rightarrow & \mathbf{<1,5,-4>} & & \\
<-10,1,1> & \leftarrow & \Uparrow & & \Downarrow & & \\
<1,1,-10> & \leftarrow & \mathbf{<2,3,-4>} & & \mathbf{<-4,3,2>} & \longrightarrow & <2,1,-5> \\
<1,3,-8> & \leftarrow & \Uparrow & & \Downarrow & & \\
<-5,1,2> & \longleftarrow & \mathbf{<-2,5,2>} & & \mathbf{<2,5,-2>} & \longrightarrow & <5,1,-2> \\
 & & \Uparrow & & \Downarrow & & \\
<-2,1,5> & \longleftarrow & \mathbf{<4,3,-2>} & & \mathbf{<-2,3,4>} & & \\
 & & \Uparrow & & \Downarrow & & \\
 & & \mathbf{<-1,5,4>} & \Leftarrow & \mathbf{<4,5,-1>} & \longmapsto & <8,3,-1> \\
 & & & & & \mapsto & <10,1,-1> \\
 & & & & & \mapsto & <-1,1,10> \\
 & & & & & \mapsto & <-1,3,8>
\end{array}
$$

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