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Kyoto University
On semi-reduced quadratic forms, continued fractions and class number

Claude LEVESQUE

§1. Introduction

Let $m$ be a square-free integer $> 1$ and let $\Delta$ be the discriminant of the real quadratic field $h(\mathbb{Q}(\sqrt{m}))$. Define

$$\Delta = \begin{cases} m, & \text{if } m \equiv 1 \pmod{4}, \\ 4m, & \text{if } m \equiv 2, 3 \pmod{4}, \end{cases}$$

$$\tilde{\omega} = \begin{cases} \frac{1}{2} + \frac{1}{2}\sqrt{m}, & \text{if } m \equiv 1 \pmod{4}, \\ \sqrt{m}, & \text{if } m \equiv 2, 3 \pmod{4}, \end{cases}$$

$$\theta = \begin{cases} 0 & \text{if } m \equiv 1 \pmod{4}, \ell \text{ even and } 2 \nmid k_{\ell/2}, \\ 1 & \text{if } \begin{cases} m \equiv 1 \pmod{4}, \ell \text{ even and } 2 \nmid k_{\ell/2}, \\ m \equiv 1 \pmod{4}, \ell \text{ odd}, \\ m \equiv 2, 3 \pmod{4}, \ell \text{ even and } 2 \nmid k_{\ell/2}, \end{cases} \\ 2 & \text{if } \begin{cases} m \equiv 2, 3 \pmod{4}, \ell \text{ even and } 2 \nmid k_{\ell/2}, \\ m \equiv 2, 3 \pmod{4}, \ell \text{ odd}. \end{cases} \end{cases}$$

Write $\tilde{\omega}$ as a continued fraction:

$$\tilde{\omega} = [k_0, \overline{k_1, \ldots, k_{\ell}}].$$

Let $\lambda_1(m)$ (resp. $\lambda_2(m)$) be the number of solutions of

$$x^2 + 4yz = \Delta \quad \text{(resp. } x^2 + 4y^2 = \Delta)$$

with $x, y, z \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Then H. Lu [Lu] proved the following result.

Theorem (Lu). The class number $h_{\Delta}$ of $\mathbb{Q}(\sqrt{m})$ is equal to 1 if and only if

$$\theta + \sum_{i=1}^{\ell} k_i = \lambda_1(m) + \lambda_2(m).$$

---

1Written version of a lecture (based on a joint work with E. DUBOIS) given in Kyoto at RIMS on November 27, 1996, during the symposium Algebraic Number Theory and Related Topics. Let me take this opportunity to express my deepest gratitude to professor Dr. Masanobu KANEKO for his kind invitation and his support: "Kanekosensei, arigato gozaimasu". Thanks are also due to professor Toru NAKAHARA and to professor Hiroki SUMIDA.
Problem. Generalize Lu's result. More precisely, use continued fractions to list the elements of the set
\[ \{ aX^2 + bXY + cY^2 : a, b, c \in \mathbb{Z}, b^2 - 4ac = \Delta \text{ with } 0 \leq b < \sqrt{\Delta} \} \]
of semi-reduced quadratic forms of discriminant \( \Delta \) and relate \( h^+_\Delta \) (resp. \( h_\Delta \)) to the cardinality of this last set, i.e., to the cardinality of the set
\[ \{(a, b, c) \in \mathbb{Z}^3 : \Delta = b^2 - 4ac \text{ with } 0 \leq b < \sqrt{\Delta} \}. \]

§2. Preliminaries

A quadratic form \( f \) is a homogeneous polynomial of the form
\[ f = f(X, Y) = \langle a, b, c \rangle = aX^2 + bXY + cY^2, \text{ with } a, b, c \in \mathbb{Z}, \]
which we may write in matrix form as
\[ f(X, Y) = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \]

We say that the matrix of \( f \) is
\[ M_f = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}. \]

By definition, the discriminant of \( f \) is
\[ \Delta = \Delta_f = b^2 - 4ac, \]
moreover \( f \) is primitive if \( \gcd(a, b, c) = 1 \); \( f \) is definite positive if \( \Delta < 0, a > 0, c > 0 \); and \( f \) is indefinite if \( \Delta > 0 \).

Consider
\[ \mathcal{F}_\Delta = \{ \text{primitive quadratic forms of discriminant } \Delta \}. \]

We need
\[ GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} r & s \\ t & u \end{pmatrix} : r, s, t, u \in \mathbb{Z}, ru - st = \pm 1 \right\}, \]
\[ SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} r & s \\ t & u \end{pmatrix} : r, s, t, u \in \mathbb{Z}, ru - st = +1 \right\}. \]
We can define an action of $A \in GL_2(\mathbb{Z})$ on $f$ by stating that

$$g = Af \text{ where } M_g = AM_f A^t.$$ \hfill (1)

Moreover we say:

$$f \sim g \text{ (f is equivalent to g) } \iff g = Af \text{ for some } A \in GL_2(\mathbb{Z}),$$

$$f \approx g \text{ (f is strictly equivalent to g) } \iff g = Af \text{ for some } A \in SL_2(\mathbb{Z}).$$ \hfill (2)

It turns out that the class group (resp. strict class group) of $\mathbb{Q}(\sqrt{\Delta})$ is $\mathcal{F}_\Delta/\sim$ (resp. $\mathcal{F}_\Delta/\approx$), its cardinality being denoted the class number $h_\Delta$ (resp. the strict class number $h_\Delta^+$) of $\mathbb{Q}(\sqrt{\Delta})$.

We are extensively working with the matrices

$$E(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad A(u) = \begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix}. \hfill (3)$$

We associate to $f$ the quadratic numbers

$$\omega = \omega(f) = \frac{b + \sqrt{\Delta}}{2|c|}, \quad \Omega = \Omega(f) = \frac{b + \sqrt{\Delta}}{2|a|}, \hfill (4)$$

and define

$$\sigma(f) = \text{sign}(a).$$

Gauss defined the right neighbour $R(f)$ and the left neighbour $L(f)$ of $f$:

$$Rf = A(e)f = \langle c, -b - 2ce, a + be + ce^2 \rangle \quad \text{ with } \quad e = -\text{sign}(c)[\omega(f)],$$

$$Lf = A(E)^{-1}f = \langle c + bE + aE^2, -b - 2aE, a \rangle \quad \text{ with } \quad E = -\text{sign}(a)[\Omega(f)].$$ \hfill (5)

We say that $f = \langle a, b, c \rangle$ with $\Delta > 0$ is reduced if

$$\begin{cases} (i) & 0 < b < \sqrt{\Delta}, \\ (ii) & \sqrt{\Delta} - b < 2|c| < \sqrt{\Delta} + b, \end{cases}$$

which is equivalent to saying

$$\begin{cases} (i) & 0 < b < \sqrt{\Delta}, \\ (ii) & \sqrt{\Delta} - b < 2|a| < \sqrt{\Delta} + b. \end{cases}$$ \hfill (6)

Gauss showed how to calculate the class number $h_\Delta^+$. With the continued fraction algorithm, write the finite set of reduced quadratic forms of discriminant $\Delta > 0$ as a
union of \( h_\Delta \) disjoint cycles, by picking up a reduced quadratic form \( f = \langle a, b, c \rangle \) and taking the continued fraction of

\[
\omega_0 = \omega(f) = \frac{b + \sqrt{\Delta}}{2|c|} = \frac{P_0 + \sqrt{\Delta}}{2Q_0},
\]
i.e., for \( i \geq 0 \),

\[
\begin{align*}
\omega_i &= \frac{P_i + \sqrt{\Delta}}{2Q_i}, \\
P_{i+1} &= 2k_iQ_i - P_i \quad \text{with } k_i = [\omega_i], \\
Q_{i+1} &= \frac{\Delta - P_{i+1}^2}{4Q_i},
\end{align*}
\]
where \([ ]\) is the greatest integer function. Hence

\[
\begin{array}{cccccccc}
P_0 & P_1 & \ldots & P_{\ell-1} & P_{\ell} & P_{\ell+1} & \ldots & P_{2\ell-1} & \ldots \\
Q_0 & Q_1 & \ldots & Q_{\ell-1} & Q_{\ell} & Q_{\ell+1} & \ldots & Q_{2\ell-1} & \ldots \\
k_0 & k_1 & \ldots & k_{\ell-1} & k_{\ell} & k_{\ell+1} & \ldots & k_{2\ell-1} & \ldots \\
\end{array}
\]

It is well known that \( f \) is reduced if and only if \( \omega(f) \) is a reduced quadratic number (i.e., the continued fraction expansion of \( \omega(f) \) is purely periodic).

Let

\[
\tilde{\ell} = \begin{cases} 
\ell & \text{if } \ell \text{ is even,} \\
2\ell & \text{if } \ell \text{ is odd.}
\end{cases}
\]

The cycle of \( f \) is

\[
[f_1, f_2, \ldots, f_{\tilde{\ell}}],
\]

where

\[
f = f_1 = \langle a, b, c \rangle = \langle \sigma(f)Q_0, P_1, -\sigma(f)Q_1 \rangle
\]

and where for \( i = 1, \ldots, \tilde{\ell} - 1 \),

\[
f_{i+1} = R(f_i) = \langle (-1)^i\sigma(f)Q_i, P_{i+1}, (-1)^{i+1}\sigma(f)Q_{i+1} \rangle = A_if_i
\]

with

\[
A_i = A(\sigma(f_i)k_i) = A((-1)^{i-1}\sigma(f)k_i).
\]

In symbols,

\[
\begin{align*}
f_{i+1} &= Rf_i \quad \text{for } 1 \leq i \leq \tilde{\ell} \quad \text{with } f_1 = Rf_{\tilde{\ell}}, \\
f_j &= Lf_{j+1} \quad \text{for } 0 \leq j < \tilde{\ell} \quad \text{with } f_{\tilde{\ell}} = Lf_1,
\end{align*}
\]
and \( k_i = [\omega(f_i)] \).

Fact: \( h_{\Delta}^+ \) is the number of cycles. Moreover \( h_{\Delta}^+ = h_{\Delta} \) (resp. \( h_{\Delta}^+ = 2h_{\Delta} \)) if the length of the continued fraction of \( \omega(f) \) for any reduced quadratic form \( f \) is odd (resp. even).

See section 7 for examples of calculations of cycles.

\section*{§3. Semi-reduced forms}

A quadratic form \( f = <a, b, c> \) of discriminant \( \Delta \) is semi-reduced if

\[ 0 \leq b < \sqrt{\Delta}, \]

and \( f \) is said to be intermediate if \( f \) est semi-reduced without being reduced.

In each of the following examples, we give the list of semi-reduced quadratic forms, the reduced forms being written in boldface, the intermediate forms appearing in roman style.

Example 3.1. Let \( m = 14 \). There are 24 semi-reduced quadratic forms of discriminant \( \Delta = 56 \):

\[
\begin{align*}
&<1, 0, -14> &<1, 0, 14> &<2, 0, -7> &<2, 0, 7> &<7, 0, -2> &<7, 0, 2> \\
&<1, 2, -13> &<1, 2, 13> &<14, 0, -1> &<14, 0, 1> &<13, 2, -1> &<13, 2, 1> \\
&<1, 4, -10> &<1, 4, 10> &<2, 4, -5> &<2, 4, 5> &<5, 4, -2> &<5, 4, 2> \\
&<10, 4, -1> &<10, 4, 1> &<1, 6, -5> &<1, 6, 5> &<5, 6, -1> &<5, 6, 1>
\end{align*}
\]

Example 3.2. Let \( m = 19 \). There are 36 semi-reduced quadratic forms of discriminant \( \Delta = 76 \):

\[
\begin{align*}
&<1, 0, -19> &<1, 0, 19> &<19, 0, -1> &<19, 0, 1> &<1, 2, -18> &<1, 2, 18> \\
&<2, 2, -9> &<2, 2, 9> &<3, 2, -6> &<3, 2, 6> &<6, 2, -3> &<6, 2, 3> \\
&<9, 2, -2> &<9, 2, 2> &<18, 2, -1> &<18, 2, 1> &<1, 4, -15> &<1, 4, 15> \\
&<3, 4, -5> &<3, 4, 5> &<5, 4, -3> &<5, 4, 3> &<15, 4, -1> &<15, 4, 1> \\
&<1, 6, -10> &<1, 6, 10> &<2, 6, -5> &<2, 6, 5> &<5, 6, -2> &<5, 6, 2> \\
&<10, 6, -1> &<10, 6, 1> &<1, 8, -3> &<1, 8, 3> &<3, 8, -1> &<3, 8, 1>
\end{align*}
\]

Example 3.3. Let \( m = 26 \). There are 36 semi-reduced quadratic forms of discriminant \( \Delta = 104 \):

\[
\begin{align*}
&<1, 0, -26> &<1, 0, 26> &<2, 0, -13> &<2, 0, 13> &<13, 0, -2> &<13, 0, 2> \\
&<26, 0, -1> &<26, 0, 1> &<1, 2, -25> &<1, 2, 25> &<5, 2, -5> &<5, 2, 5> \\
&<25, 2, -1> &<25, 2, 1> &<1, 4, -22> &<1, 4, 22> &<2, 4, -11> &<2, 4, 11> \\
&<11, 4, -2> &<11, 4, 2> &<22, 4, -1> &<22, 4, 1> &<1, 6, -17> &<1, 6, 17> \\
&<17, 6, -1> &<17, 6, 1> &<1, 8, -10> &<1, 8, 10> &<2, 8, -5> &<2, 8, 5> \\
&<5, 8, -2> &<5, 8, 2> &<10, 8, -1> &<10, 8, 1> &<1, 10, -1> &<1, 10, 1>
\end{align*}
\]
Example 3.4. Let $m=33$. There are 20 semi-reduced quadratic forms of discriminant $\Delta=33$:
\[
\begin{align*}
<1,1,-8> & \quad <-1,1,8> & <2,1,-4> & \quad <-2,1,4> & <4,1,-2> \\
<-4,1,2> & \quad <8,1,-1> & <-8,1,1> & <1,3,-6> & <-1,3,6> \\
<2,3,-3> & \quad <-2,3,3> & <3,3,-2> & <-3,3,2> & <6,3,-1> \\
<-6,3,1> & \quad <1,5,-2> & <-1,5,2> & <2,5,-1> & <-2,5,1>
\end{align*}
\]

Example 3.5. Let $m=35$. There are 40 semi-reduced quadratic forms of discriminant $\Delta=140$:
\[
\begin{align*}
<19,8,-1> & \quad <-19,8,1> & <1,10,-10> & <-1,10,10> & <1,8,-19> \\
<-1,8,19> & <1,0,-35> & <-1,0,35> & <5,0,-7> & <-5,0,7> \\
<7,0,-5> & <-7,0,5> & <35,0,-1> & <-35,0,1> & <1,2,-34> \\
<-1,2,34> & <2,2,-17> & <-2,2,17> & <1,4,-31> & <-1,4,31> \\
<17,2,-2> & <-17,2,2> & <34,2,-1> & <-34,2,1> & <31,4,-1> \\
<-31,4,1> & <1,6,-26> & <-1,6,26> & <2,6,-13> & <-2,6,13> \\
<13,6,-2> & <-13,6,2> & <26,6,-1> & <-26,6,1> & <2,10,-5> \\
<-2,10,5> & <5,10,-2> & <-5,10,2> & <10,10,-1> & <-10,10,1>
\end{align*}
\]

Let $f=<a,b,c>$ be a reduced form of discriminant $\Delta$ such that $e=\sigma(f)[\omega(f)]$. Consider the quadratic forms associated to $f$ and given by the following two cases (the first may be empty):

<table>
<thead>
<tr>
<th>Case (i)</th>
<th>Case (ii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(u)f = &lt;a + bu + cu^2, b + 2cu, c&gt;$ with $1 \leq \sigma(f)u \leq \frac{b}{2</td>
<td>c</td>
</tr>
</tbody>
</table>

We can show that the quadratic forms of Cases (i) and (ii) are all semi-reduced (i.e., are either reduced or intermediate) and that the intermediate forms $<a', b', c'>$ of Cases (i) and (ii) respectively verify the following properties:

<table>
<thead>
<tr>
<th>Case (i)</th>
<th>Case (ii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a'c' &lt; 0$, $</td>
<td>a'</td>
</tr>
</tbody>
</table>

These last properties characterize intermediate quadratic forms. We can show that
intermediate forms come from reduced forms. More precisely, let \( g = \langle a', b', c' \rangle \) be an intermediate quadratic form. Then the following properties hold true:

(i) If \( |a'| > |c'| \), then the forms \( f_1 = Rg \) and \( f = Lf_1 = \langle a, b, c \rangle \) are both reduced and

\[
 g = E(u)f = \langle a + bu + cu^2, b + 2cu, c \rangle
\]

with

\[
1 \leq \sigma(f)u \leq \frac{b}{2|c|}.
\]

(ii) If \( |a'| < |c'| \), then the form \( f = Lg = \langle a, b, c \rangle \) is reduced and

\[
 g = A(u)f = \langle c, -b - 2cu, a + bu + cu^2 \rangle
\]

with

\[
\frac{b}{2|c|} \leq \sigma(f)u \leq \sigma(f)e = [\omega(f)].
\]

We can now state our first result.

**Theorem.** (1) For each reduced form \( f = \langle a, b, c \rangle \) of discriminant \( \Delta \), the quadratic forms associated to \( f \) and defined by

\[
\begin{cases}
    E(u)f & \text{with } 1 \leq \sigma(f)u \leq \frac{b}{2|c|}, \\
    A(u)f & \text{with } \frac{b}{2|c|} \leq \sigma(f)u \leq [\omega(f)],
\end{cases}
\]

are all different from one another and form a set \( I(f) \) of cardinality

\[
\#I(f) = \begin{cases}
    [\omega(f)] & \text{if } (2c) \equiv b, \\
    1 + [\omega(f)] & \text{if } (2c) \not\equiv b.
\end{cases}
\]

(2) If \( f \) and \( g \) are two different reduced forms, then \( I(f) \cap I(g) = \emptyset \).

(3) Moreover the (disjoint) union of the \( I(f) \)'s when \( f \) runs through the set \( \text{Red}(\Delta) \) of reduced forms is a set equal to the set of semi-reduced forms and is of cardinality

\[
2 \left( \sum_{0 \leq t < \sqrt{\Delta} \atop \tau \equiv \Delta \pmod{2}} \tau \left( \frac{\Delta - t^2}{4} \right) \right),
\]

where \( \tau \) is the number of divisors function.
§5. The palindrome level

Suppose that the continued fraction of a quadratic number $\alpha$ is purely periodic (i.e. $\alpha$ is reduced):

$$\alpha = [k_1, k_2, \ldots, k_\ell]$$

with $\ell$ minimal. We associate to $\alpha$ the word

$$w(\alpha) = k_1 k_2 \ldots k_\ell$$

and say that it is defined up to a cyclic permutation (of order $\ell$).

Recall that a palindrome $(KA I BU N)$ is a word which once read from left to right or from right to left is the same.

Examples. In french, english, german and japanese:

(i) ELU PAR CETTE CRAPULE
(ii) NAME NO ONE MAN
(iii) EIN NEGER MIT GAZELLE ZAGT IM REGEN NIE
(iv) さくはなの もきよきかに かおるるる おかにかきよき もののなはくさ
(v) てきをのし いさましいし まさいしのおきて
(vi) たけやぶ やけた

When $\ell$ is odd, we say that $\alpha$ has 1 (resp. 0) central element if $w(\alpha)$ is (resp. is not) a palindrome, the central element being the center of $w(\alpha)$. When $\ell$ is even, we say $\alpha$ has 2 central elements (resp. 0 central element) if after an eventual cyclic permutation, $w(\alpha)$ is (resp. is not) the concatenation of a palindrome of length $\ell - 1$ and of a palindrome of length 1, the two central elements being the two centers of the concatenated palindromes.

Definition. The palindrome level $s = s(\alpha) \in \{0, 1, 2\}$ of a reduced quadratic number is equal to the number of even central elements.

In examples 7.1 to 7.5, $s$ is respectively equal to 2, 1, 1, 0, 0.

§6. The main results

We can now relate the class number $h_\Delta$ of $\mathbb{Q}(\sqrt{\Delta})$ to the cardinality of the set of semi-reduced forms by writing this cardinality in terms of the partial quotients of non equivalent reduced quadratic numbers.
**Theorem.** Let $\Delta$ be the discriminant of a real quadratic field $\mathbb{Q}(\sqrt{\Delta})$. Put

$$g = \begin{cases} 
0 & \text{if } \Delta \equiv 1 \pmod{4}, \\
2^r+1 & \text{if } \Delta \equiv 4 \text{ ou } 12 \pmod{16}, \\
2^{r+2} & \text{if } \Delta \equiv 8 \pmod{16},
\end{cases}$$

where $r$ is the number of odd primes dividing $\Delta$. Then the class number $h_\Delta$ of $\mathbb{Q}(\sqrt{\Delta})$ is equal to $h_0$ if and only if there exist $h_0$ reduced quadratic numbers $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(h_0)}$ associated to $h_0$ quadratic forms non equivalent to one another such that the number of all semi-reduced forms of discriminant $\Delta$ is

$$2\left(s_\Delta + \sum_{j=1}^{h_0} \sum_{t=1}^{\ell_j} k_t^{(j)}\right),$$

where for $j = 1, \ldots, h_0$,

$$\alpha^{(j)} = [k_1^{(j)}, k_2^{(j)}, \ldots, k_{\ell_j}^{(j)}]$$

with $\ell_j$ minimal, and where the palindrome level $s_\Delta$ of $\Delta$, defined as the sum of all the palindrome levels of the $\alpha^{(j)}$'s, is equal to

$$s_\Delta = \sum_{j=1}^{h_0} s(\alpha^{(j)}) = \frac{g}{2}.$$ 

**Theorem.** Let $\Delta$ be the discriminant of a real quadratic field $\mathbb{Q}(\sqrt{\Delta})$. Put

$$g_1 = \begin{cases} 
0 & \text{if } \Delta \equiv 1 \pmod{4}, \\
2^r & \text{if } \Delta \equiv 4 \text{ ou } 12 \pmod{16}, \\
2^{r+1} & \text{if } \Delta \equiv 8 \pmod{16},
\end{cases}$$

where $r$ is the number of odd primes dividing $\Delta$. Then the class number $h_\Delta$ of $\mathbb{Q}(\sqrt{\Delta})$ is equal to $h_1$ if and only if there exist $h_1$ reduced quadratic numbers $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(h_1)}$ associated to $h_1$ quadratic forms non equivalent to one another such that

$$\# \{(A, B, C) \in \mathbb{N}^3 : \Delta = B^2 + 4AC\} = s_\Delta + \sum_{j=1}^{h_1} \sum_{t=1}^{\ell_j} k_t^{(j)},$$

where for $j = 1, \ldots, h_1$,

$$\beta^{(j)} = [k_1^{(j)}, k_2^{(j)}, \ldots, k_{\ell_j}^{(j)}]$$

with $\ell_j$ minimal, and where the palindrome level $s_\Delta$ of $\Delta$, defined as the sum of all the palindrome levels of the $\beta^{(j)}$'s, is

$$s_\Delta = \sum_{j=1}^{h_1} s(\beta^{(j)}) = g_1.$$
§7. Five examples

One has a chance of verifying the last theorems with the following examples.

Example 7.1. Let \( m = 14 \), so \( \Delta = 56 \). Here \( h_\Delta^+ = 2 \) and \( h_\Delta = 1 \). The continued fraction expansion of \( \omega = \frac{4 + \sqrt{56}}{2} \) is \([2, 1, 6, 1]\):

\[
\omega = \frac{2 + 2\sqrt{10}}{2 \cdot 3} \quad \leftrightarrow \quad \[ \begin{array}{cccc} 4 & 4 & 6 & 6 \\ 2 & 5 & 1 & 5 \\ 2 & 1 & 6 & 1 \\ \end{array} \]
\]

Since \( w(\omega) = 1, \underline{6}, 1, \underline{2}, \) we have \( s(\omega) = 2 \). There are two cycles of reduced forms (in boldface) to which are attached the intermediate forms:

\[
\begin{array}{c}
\langle -7, 0, 2 \rangle \quad \langle -5, 4, 2 \rangle \quad \Rightarrow \quad \langle 2, 4, -5 \rangle \\
\langle 2, 0, -7 \rangle \quad \uparrow \quad \downarrow \\
\langle 1, 6, -5 \rangle \quad \Leftarrow \quad \langle -5, 6, 1 \rangle \\
\end{array}
\]
\[
\begin{array}{c}
\langle -7, 0, -2 \rangle \quad \langle 5, 4, -2 \rangle \quad \Rightarrow \quad \langle -2, 4, 5 \rangle \\
\langle -2, 0, 7 \rangle \quad \uparrow \quad \downarrow \\
\langle -1, 6, 5 \rangle \quad \Leftarrow \quad \langle 5, 6, -1 \rangle \\
\end{array}
\]
Example 7.2. Let \( m = 19 \), so \( \Delta = 76 \). Here \( h_\Delta^+ = 2 \) and \( h_\Delta = 1 \). The continued fraction expansion of \( \omega = \frac{4 + \sqrt{76}}{2 \cdot 5} \) is \([1, 3, 1, 2, 8, 8]\):

\[
\omega = \frac{4 + \sqrt{76}}{2 \cdot 5} \leftrightarrow \begin{array}{cccccccc}
4 & 6 & 6 & 4 & 8 & 8 & 4 & \ldots \\
5 & 2 & 5 & 3 & 1 & 3 & 5 & \ldots \\
1 & 3 & 1 & 2 & 8 & 2 & 1 & \ldots 
\end{array}
\]

Since \( w(\omega) = 1, 2, [8], 2, 1, [3] \), we have \( s(\omega) = 1 \). There are two cycles of reduced forms (in boldface) to which are attached the intermediate forms:

\[
\begin{align*}
<9, 2, -2> & \quad <5, 6, -2> \quad \Rightarrow \quad <-2, 6, 5> \\
<-2, 2, 9> & \quad \uparrow \quad \downarrow \\
& \quad <-3, 4, 5> \quad <5, 4, -3> \quad \rightarrow \quad <-3, 2, 6> \\
& \quad \uparrow \quad \downarrow \\
<6, 2, -3> & \quad <1, 8, -3> \quad \leftarrow \quad <-3, 8, 1> \quad \rightarrow \quad <-10, 6, 1> \\
& \quad \rightarrow \quad <-15, 4, 1> \\
& \quad \rightarrow \quad <-18, 2, 1> \\
& \quad \rightarrow \quad <-19, 0, 1> \\
& \quad \rightarrow \quad <-1, 0, -19> \\
& \quad \rightarrow \quad <-1, 2, -18> \\
& \quad \rightarrow \quad <-1, 4, -15> \\
& \quad \rightarrow \quad <-1, 6, -10>
\end{align*}
\]
Example 7.3. Let \( m = 26 \), so \( \Delta = 104 \). Here \( h^+_\Delta = h_\Delta = 2 \). The continued fraction expansion of \( \omega_1 = \frac{8 + \sqrt{124}}{2 \cdot 2} \) is \( [4,1,1] \):

\[
\omega_1 = \frac{8 + \sqrt{104}}{2 \cdot 2} \rightarrow \begin{array}{c}
8 & 8 & 2 & 8 & \ldots \\
2 & 5 & 5 & 2 & \ldots \\
4 & 1 & 1 & 4 & \ldots 
\end{array}
\]

Since \( w(\omega_1) = 1,4,1 \), we have \( s(\omega_1) = 1 \).

The continued fraction expansion of \( \omega_2 = \frac{10 + \sqrt{124}}{2 \cdot 1} \) is \( [10] \):

\[
\omega_2 = \frac{10 + \sqrt{104}}{2 \cdot 1} \rightarrow \begin{array}{c}
10 & 10 & \ldots \\
1 & 1 & \ldots \\
10 & 10 & \ldots 
\end{array}
\]

Since \( w(\omega_2) = 10 \), we have \( s(\omega_2) = 1 \). There are two cycles of reduced forms (in boldface) to which are attached the intermediate forms:

\[
\begin{array}{c}
< -11, 4, 2 > \\
<-13, 0, 2> \\
< 2, 0, -13 > \\
< 2, 4, -11 > \\
< -2, 8, 5 > \\
< -5, 8, 2 > \\
< 5, 2, -5 > \\
< 5, 8, -2 > \\
< -11, 4, -2 > \\
< -2, 0, 13 > \\
< -2, 4, 11 > \\
< 13, 0, -2 > \\
\end{array}
\]
Example 7.4. Let $m = 33$, so $\Delta = 33$. Here $h_\Delta^+ = 2$ and $h_\Delta = 1$. The continued fraction expansion of $\omega = \frac{5 + \sqrt{33}}{2 \cdot 1}$ is $[5, 2, 1, 2]$:

$$\omega = \frac{5 + \sqrt{33}}{2 \cdot 1} \rightarrow$$

Since $w(\omega) = 2, 1, 2, 5$, we have $s(\omega) = 0$. There are two cycles of reduced forms (in boldface) to which are attached the intermediate forms:
Example 7.5. Let $m = 41$, so $\Delta = 41$. We will see $h^+=h_\Delta = 1$. The continued fraction expansion of $\omega = \frac{5 + \sqrt{41}}{2 \cdot 1}$ is $\overline{5, 1, 2, 2, 1}$:

$$
\omega = \frac{5 + \sqrt{41}}{2 \cdot 1} \rightarrow \begin{array}{c|c|c|c|c|c}
5 & 5 & 3 & 5 & 3 & 5 \\
\hline
1 & 4 & 2 & 2 & 4 & 1 \\
5 & 1 & 2 & 2 & 1 & 5 \\
\end{array}$$

Since $w(\omega) = 2, 1, [5], 1, 2$, we have $s(\omega) = 0$. There is only one cycle of reduced forms (in boldface) to which are attached the intermediate forms.
References


[Lu] Lu, H., *On the class number of real quadratic fields*, Scientia Sinica II (special number, 1979), 118-130.


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