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Kyoto University
MIXED HODGE THEORY AND
PREHOMOGENEOUS VECTOR SPACES.

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0.0. In [Sat], M.Sato obtained a formula which describes the Fourier transform of
a complex power of a relatively invariant polynomial of a prehomogeneous vector
space over the real number field, up to an ambiguity of certain exponential factors.
In [Gyo2], I formulated conjectures which would give a finite field analogue of
the theorem of M.Sato, without any ambiguity. Recently, J.Denef and I jointly have
succeeded to prove these conjectures [DG] based on Laumon’s product formula
[Lau]. The purpose of the present paper is to give an alternative approach based
on the mixed Hodge theory. Our main result is Theorem 11, which includes as
a special case Conjecture A of [Gyo2] up to an ambiguity of a constant factor
of absolute value one. Thus our result is less precise than [DG]. The result of
the present paper was obtained around 1986 with help of M.Kashiwara, and thus seems
more or less out of date, but I think it is still of some interest. The content was
announced and outlined in [Gyo2].

0.1. Our argument roughly goes as follows. Fix an isomorphism \((1 - q)^{-1} \mathbb{Z}/\mathbb{Z} \xrightarrow{\sim} \text{Hom}(F_q^\times, \overline{\mathbb{Q}}_l^\times)\), \(\alpha \mapsto \chi_\alpha\), where \(l\) is a prime number \(\neq p\), and \(\overline{\mathbb{Q}}_l\) an algebraic closure of the \(l\)-adic number field.

First, we calculate the weight filtration of \(\mathcal{F}(D f^\alpha)\) \((\alpha \in \mathbb{Q})\) in the sense of the mixed Hodge theory due to M.Saito, where \(D = D_{C^n} = C \left\langle x_1, \cdots, x_n, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right\rangle\) and \(\mathcal{F}\) is the formal Fourier transformation \(x_j \mapsto \sqrt{-1} \frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_j} \mapsto \sqrt{-1} y_j\).

Second, using the result of the first step and by the Riemann-Hilbert correspondence, we calculate the weight filtration of \(\mathcal{F}^+(j_* C f^\alpha[n])\), where \(\mathcal{F}^+\) is the Sato-Fourier transformation and \(j : V \setminus f^{-1}(0) \to V\) is the inclusion mapping.

Third, using the result of the second step and by the reduction modulo \(p\), we calculate the weight filtration of \(\mathcal{F}_\psi(j_* f^* L_{\chi_\alpha}[n])\), where \(\mathcal{F}_\psi\) is the Deligne-Fourier transformation, and \(L_\chi\) is the Kummer torsor associated to \(\chi\).

Finally, we deduce the desired result from the result of the third step, using the trace formula of Grothendieck, the ‘Weil conjecture’ proved by Deligne, and a result of Katz-Laumon on \(\mathcal{F}_\psi\).

0.2. In this paper, we obtain arithmetic result starting from the mixed Hodge theory. The author expects that, following the opposite course, we might be able
to study the mixed Hodge structure starting from the arithmetic result of [DG].
Cf. [Maz], [Kat].

**Notation**

N1. We denote by \( Z, Q, R \) and \( C \) the ring of rational integers, the rational number field, the real number field, and the complex number field, respectively. For a prime number \( p \), \( Q_p \) denotes the \( p \)-adic number field and \( Z_p \) its integer ring.

N2. We always assume that a commutative ring, say \( R \), contains \( 1_R \), a homomorphism \( R \to R' \) sends \( 1_R \) to \( 1_R' \), and \( 1_R \) acts trivially on an \( R \)-module. For a (not necessarily commutative) ring \( A \) with the identity element, \( A^X \) denotes the multiplicative group of the invertible elements. For a commutative ring \( R \), the subscript \( R \) of a symbol corresponding to a scheme or a morphism between schemes (resp. a sheaf) means that it belongs to the category of \( R \)-schemes (resp. sheaves of \( R \)-modules). If \((-)_R \) has been already defined and \( R \) is an \( R' \)-algebra (i.e., a ring homomorphism \( R' \to R \) is given), then \((-)_R := (-)_R \otimes_R R \), unless otherwise stated. If the ring \( R \) can be understood from the context, we omit the subscript. If \( X \) is an affine scheme over \( R \), \( R[X] \) denotes its coordinate ring. If \( X \) is an \( R' \)-scheme and \( R \) is an \( R' \)-algebra, \( X(R) \) denotes the set of \( R \)-rational points.

N3. For a morphism \( F \) between two spaces, the sheaf theoretic pull-back \( F^* \) is sometimes denoted by \( F^{-1} \) to avoid a confusion. For a complex non-singular variety (always assumed to be of pure dimension), let \( D = D_X \) (resp. \( O = O_X \)) denote the sheaf of algebraic differential operators (resp. regular functions). Let \( Mod(D_X) \) denote the category of \( D_X \)-modules, and \( Mod^{qc}(D_X) \) (resp. \( Mod_{rh}(D_X) \)) its full subcategory of \( D_X \)-modules which are quasi-coherent over \( O_X \) (resp. regular holonomic). Let \( D(D_X) \) denote the derived category of \( Mod(D_X) \), and \( D^{b}_{qc}(D_X) \) (resp. \( D^{b}_{rh}(D_X) \)) the full subcategory of \( D(D_X) \) consisting of bounded complexes whose cohomologies are quasi-coherent (resp. regular holonomic). Let \( Mod(C_X) \) denote the category of \( C_X \)-modules, \( D(C_X) \) its derived category, and \( D^{b}_{c}(C_X) \) the full subcategory of \( D(C_X) \) consisting of bounded complexes whose cohomologies are (algebraically) constructible. Let \( O^{an} = O^{an}_X \) be the sheaf of holomorphic functions on the underlying analytic manifold \( X^{an} \) of \( X \), and \( M^{an} := M \otimes O^{an} \) for \( M \in Mod(D_X) \). For \( M \in Mod_{rh}(D_X) \), the de Rham complex is defined by \( DR(M) = DR_X(-) \) (resp. \( M^{an} := DR_X(M) := RHom_{D_{qc}}(O^{an}, M^{an}) \), where \( Hom \) denotes the sheaf of local homomorphisms. Besides, put \( ^pDR_X = DR_X[dim X] \). For a morphism \( F : X \to X' \), we define functors \( D^F, D^{F*}, D^{F!} \) between \( D^{b}_{rh}(D_X) \) and \( D^{b}_{rh}(D_{X'}) \) so that \( ^pDR_X \circ D^F = RF \circ ^pDR_X, ^pDR_X \circ D^{F*} = RF \circ ^pDR_X, ^pDR_X \circ D^{F!} = F' \circ ^pDR_X \). If \( f \in \Gamma(X, O_X) \), and \( i : f^{-1}(0) \to X \) is the inclusion mapping, we define the functors \( D\psi, D\phi \) etc. of \( D^{b}_{rh}(D_X) \) to itself so that \( ^pDR_X \circ D\psi_i = \psi \circ ^pDR_X, ^pDR_X \circ D\phi_i = \phi \circ ^pDR_X \), etc. where \( \psi \) and \( \phi \) are the nearby cycle functor and the vanishing cycle functor, respectively [Del1]. If \( X \) is an affine variety, we put \( D = D_X = \Gamma(X, D_X) \). Let \( Mod(D_X) \) be the category of \( D_X \)-modules, which is equivalent to \( Mod^{qc}(D_X) \). An object of \( Mod_{qc}(D_X) \) corresponding to an object of \( Mod(D_X) \) is denoted by the
Thus $\text{Mod}(D_X) \sim \text{Mod}_{qc}(D_X)$ by $M \mapsto \mathcal{M}$. Using this category equivalence, we define $D_{F_*}, D_F, D_{F^*}, D_{F^!}, D_{\psi_f}, D_{\phi_{f,1}}, \text{supp}$, etc. for $D$-modules (satisfying appropriate conditions) as well. We denote the functors in the category of mixed Hodge modules given in [Sai2] by the same symbol with the superscript $MH$ on the left, e.g., $MH_{F_*}, MH_{\psi},$ etc.

N4. We write, for example $L(\alpha)$ in place of $L_\alpha$ on occasion to avoid multiple indices, or conversely, $L_\alpha$ in place of $L(\alpha)$ on occasion to avoid a confusion with the Tate twist.

1. MONODROMY FILTRATION

Let $\psi$ be an object of some abelian category (e.g., a module), $N$ a nilpotent endomorphism of $\psi$, and $\{W_m\}$ the monodromy filtration associated to $N$, shifted by $w - 1$ ($w \in \mathbb{Z}$). Cf. [Del2]. Namely $\{W_m\}$ is the finite increasing filtration of $\psi$ characterized by the two properties $NW_m \subset W_{m-2}$ and $N^j : \text{gr}_{w-1+j}^W \psi \rightarrow \text{gr}_{w-1-j}^W \psi$ for any $j \geq 0$. More explicitly,

\begin{equation}
W_{w-1+m} = \sum_{i \geq 0, -m} N^i (\text{ker} N^{m+1+2i}).
\end{equation}

The primitive part of $\text{gr}_{w-1+m}^W \psi$ ($m \geq 0$) is by definition

\begin{align*}
P \text{gr}_{w-1+m}^W \psi := \text{ker}(N^{m+1} | \text{gr}_{w-1+m}^W \psi \rightarrow \text{gr}_{w-3-m}^W \psi) \\
= \frac{W_{w-1+m} \cap (N^{m+1})^{-1} W_{w-4-m}}{W_{w-2+m}}.
\end{align*}

Let $m > 0$. By (1.1),

\begin{align*}
(N^{m+1})^{-1} W_{w-4-m} &= (N^{m+1})^{-1} \sum_{i \geq m+3} N^i (\text{ker} N^{m-2+2i}) \\
&= \text{ker} N^{m+1} + \sum_{i \geq 2} N^i (\text{ker} N^{m+2i}) \subset W_{w-1+m}.
\end{align*}

Hence

\begin{align*}
P \text{gr}_{w-1+m}^W \psi &= \frac{\text{ker} N^{m+1} + \sum_{i \geq 2} N^i (\text{ker} N^{m+2i})}{\sum_{i \geq 0} N^i (\text{ker} N^{m+2i})} \\
&= \frac{\text{ker} N^{m+1} + \sum_{i \geq 0} N^i (\text{ker} N^{m+2i})}{\text{ker} N^m + \sum_{i \geq 0} N^i (\text{ker} N^{m+2i})} \rightarrow \frac{\text{ker} N^{m+1} + N\psi}{\text{ker} N^m + N\psi}.
\end{align*}

The last surjection is the natural one, which is easily seen to be also injective. Thus we get

\begin{equation}
P \text{gr}_{w-1+m}^W \psi = \frac{\text{ker} N^{m+1} + N\psi}{\text{ker} N^m + N\psi}
\end{equation}
for $m > 0$. Similar argument shows (1.2) holds for $m = 0$ as well, where $N^0$ should be understood as the identity. Let $m_0$ be the integer such that $N^{m_0} = 0$ and $N^{m_0-1} \neq 0$. Here we understand $m_0 = 0$ (resp. $1$) if $\psi = 0$ (resp. $\psi \neq 0$ and $N = 0$). If $m_0 > 1$,

$$\psi = \ker N^{m_0} + N\psi \supset \ker N^{m_0-1} + N\psi = \ker N^{m_0-1}.$$  

Hence in any case,

(1.3) \[ \max\{m \geq 0 \mid P_{\gr_{w-1+m}}^{W}\psi \neq 0\} = m_0 - 1. \]

Here we understand $\max \phi = -1$.

## 2. $C[s, t]$-Modules

Let $C[s, t, t^{-1}]$ be the algebra defined by the relations $ts = (s+1)t$ and $tt^{-1} = t^{-1}t = 1$. Let $M$ be a $C[s, t]$-module (or more generally, an object of suitable abelian category with a $C[s, t]$-action) such that

\begin{align*}
(2.1) \quad & M \subset M[t^{-1}] := C[s, t, t^{-1}] \otimes_{C[s, t]} M, \quad \text{and} \\
(2.2) \quad & \text{there exists } 0 \neq b(s) \in C[s] \text{ such that } b(s)M \subset tM.
\end{align*}

For two integers $k \leq l$, let $b_{k,l}(s) = \prod_{\gamma \in C} (s-\gamma)^{m(\gamma; k, l)}$ be the monic generator of the ideal $\ker(C[s] \to \End_C(t^kM/t^lM))$. Given $\gamma \in C$. Since $b_{k,l}(s)$ divides $\prod_{k \leq j < l} b(s+j)$, there exists integers $k_0 < l_0$ such that $b_{k,l}(\gamma) \neq 0$ whenever $k \geq k_0$ or $l \leq l_0$. If $k' \leq k \leq k_0$ and $l_0 \leq l \leq l'$, then $b_{k', l'}(s)$ divides $b_{k, l'}(s) b_{k'}(s) b_{l'}(s)$, and $b_{k, l'}(\gamma) b_{l'}(\gamma) \neq 0$. Hence $0 \leq m(\gamma; k', l') \leq m(\gamma; k, l)$. Taking $k_0 \ll 0$ and $l_0 \gg 0$, we may assume that $m(\gamma; k, l)$ is independent of $k$ and $l$ for any $k \leq k_0$ and $l_0 \leq l$. Put

$$m(\gamma) := \lim_{k \to -\infty, l \to +\infty} m(\gamma; k, l).$$

### 2.3. $m(\gamma)$ depends only on $\gamma \mod \mathbb{Z}$.

**Proof.** Since $b_{k,l}(s) = b_{k-1,l-1}(s+1)$, $m(\gamma) = m(\gamma - 1)$.

### 2.4. $s - \gamma$ acts on $t^kM/t^lM$ as an automorphism if $k \geq l_0$ or $l \leq k_0$.

**Proof.** Take $a(s), c(s) \in C[s]$ so that $a(s)b_{k,l}(s) + c(s)(s-\gamma) = 1$. Then $c(s)(s-\gamma) = 1$ on $t^kM/t^lM$.

### 2.5. $\ker((s-\gamma)^m | t^kM/t^lM)$ is independent of $m \geq m(\gamma)$, if $k \leq k_0$ and $l \geq l_0$.

**Proof.** Let $b_{k,l}(s) = (s-\gamma)^{m(\gamma)} d(s)$. For $i > 0$, take $a(s), c(s) \in C[s]$ so that $a(s)(s-\gamma)^i + c(s)d(s) = 1$. If $(s-\gamma)^{m(\gamma)+i} x = 0$ for some $x \in t^kM/t^lM$, then $0 = a(s)(s-\gamma)^i \cdot (s-\gamma)^{m(\gamma)} x + c(s)d(s) \cdot (s-\gamma)^{m(\gamma)} x = (s-\gamma)^{m(\gamma)} x$.
2.6. Let $M_1$ be a module over a polynomial ring $\mathbb{Z}[T]$, and $M_1 \supset M_2 \supset M_3 \supset M_4 \supset \mathbb{Z}[T]$-submodules. Assume that the $T$-actions on $M_1/M_2$ and $M_3/M_4$ are invertible. Then the natural morphism $M_2/M_4 \rightarrow M_2/M_3 \rightarrow M_1/M_3$ induce isomorphisms $\ker(T \mid M_2/M_4) \cong \ker(T \mid M_2/M_3) \cong \ker(T \mid M_1/M_3)$.

Proof. Consider $T$-actions on the exact sequences $0 \rightarrow M_3/M_4 \rightarrow M_2/M_4 \rightarrow M_2/M_3 \rightarrow 0$ and $0 \rightarrow M_2/M_3 \rightarrow M_1/M_3 \rightarrow M_1/M_2 \rightarrow 0$. Then apply the snake lemma. □

By (2.4)–(2.6), we can see that

2.7. $\psi = \psi_\gamma := \ker((s-\gamma)^m \mid t^k M/t^l M)$ does not depend on $m \geq m(\gamma)$, $k \leq k_0$ nor $l \geq l_0$.

Let $N = N_\gamma$ denote the endomorphism of $\psi$ induced by $s-\gamma$.

2.8. If $m(\gamma) > 0$, then $N^m(\gamma) = 0$ and $N^m \neq 0$ for $0 \leq m < m(\gamma)$.

The proof of (2.8) is easy and omitted. Applying (1.3) to the above $(\psi,N)$ and understanding $\max \phi = -1$, we get the monodromy filtration $\{W_m\}$ of $\psi$ shifted by $w-1$, and we get

2.9. $\max \{m \geq 0 \mid P \mathrm{gr}_{W_{-1+m}}^w \psi \neq 0\} = m(\gamma) - 1$.

(Here we include the case $m(\gamma) = 0$.)

2.10. Remark. If $(\deg b_{k,l}(s))/(l-k)$ is independent of $k$ and $l$, then $m(\gamma) = \mathrm{card}\{\alpha \in C \mid b(\alpha) = 0, \alpha \equiv \gamma \mod Z\}$ (including multiplicity).

3. $D$-modules

Let $C[s]$, $C[s,t]$, $C[s,t,t^{-1}]$ be as in §2. Put $D[s] = D \otimes_C C[s]$ etc.

3.1. $D$-Module $D[s](f^s u)$. Let $X$ be a connected non-singular variety over $C$, $0 \neq f \in \Gamma(X,\mathcal{O}_X)$, $X_0 := X \setminus f^{-1}(0)$, $\mathcal{M}$ be a coherent $\mathcal{D}_{X_0}$-module, and $u = (u_1, \cdots, u_p)$ a $p$-tuple of elements of $\Gamma(X_0,\mathcal{M})$. Consider the left $\mathcal{D}_X[s]$-submodule $I$ of $\mathcal{D}_X[s]^p$ consisting of $(P_1(s), \cdots, P_p(s)) \in \mathcal{D}_X[s]^p$ such that $\sum_{i=1}^p(f^{m-s}P_i(s)f^s)u_i = 0$ holds in $C[s] \otimes_C \mathcal{M}$ whenever $m \in Z$ is sufficiently large. Put $N := \mathcal{D}_X[s]^p/I$. Denote by $(f^s u)_i$ the element $((0, \cdots, 0, 1, 0, \cdots, 0) \mod I)$, where $1$ appears at the $i$-th place. Put $f^s u = ((f^s u)_1, \cdots, (f^s u)_p)$.

Then $N = \sum_{i=1}^p \mathcal{D}_X[s](f^s u)_i$. For $\alpha \in C$, put $N(\alpha) := N/(s-\alpha)N$ and $f^\alpha u = ((f^\alpha u)_1, \cdots, (f^\alpha u)_p) := (f^s u)$ mod $(s-\alpha)N$. Then $N(\alpha) = \sum_{i=1}^p \mathcal{D}_X[f^\alpha u]_i$. We often write $N = \mathcal{D}_X[s](f^s u)$, $N(\alpha) = \mathcal{D}_X[f^\alpha u]$, $\sum_{i=1}^p \mathcal{D}_X[s]u_i = \mathcal{D}_X[s]u$ etc.

3.2. $D[s,t]$-Module structure. Define a $D[s,t]$-module structure of $N = \mathcal{D}_X[s](f^s u)$ by $t(\sum_{i=1}^p P_i(s)(f^s u)_i) = \sum_{i=1}^p P_i(s+1)f(f^s u)_i$. Then $N[f^{-1}]$ has a natural $\mathcal{D}_X[s,t,t^{-1}]$-module structure, $N \subset N[f^{-1}]$, and $N[f^{-1}] = N[t^{-1}]$.

3.3. $b$-Function. Assume $\mathcal{D}_{X_0}u$ to be holonomic, and let $B(s, u) = B_f(s, u)$ be the monic minimal polynomial of $s \in \mathrm{End}(N/tN)$. Cf [Kas3].
3.4. **Assumption A1.** We assume that

1. \( \mathcal{D}_{X_0,\mathcal{U}} \) is regular holonomic,
2. the zeros of \( B(s, u) \in \mathbb{C}[s] \) are rational numbers, and
3. there exists \( N \in \mathbb{Z}_{>0} \) and a complex of \( \mathbb{Z}[N^{-1}] \)-sheaves \( K_{0,\mathbb{Z}[N^{-1}]} \) on \( X_0 \) such that \( \text{DR}(\mathcal{D}_{X_0,\mathcal{U}}) \simeq K_{0,\mathbb{C}}. \) (Cf. (N2).)

These assumptions are satisfied if \( \mathcal{M} \) is a regular holonomic \( \mathcal{D}_{X_0} \)-module such that \( \text{DR}(\mathcal{D}_{X_0,\mathcal{U}}) \) is a locally constant sheaf whose monodromy is finite and defined over \( \mathbb{Q} \). (Cf. [Gyo3, (5.14)].) (In the subsequent argument, we assume several conditions including the above one. We have in mind an application to the theory of prehomogeneous vector spaces, where all these conditions are satisfied.)

3.5. **Vanishing cycle sheaf.** Fix \( \gamma \in \mathbb{C} \). Applying the construction of §2 to \( M = \mathcal{D}_X[s,t](f^*\mathcal{U}) = \mathcal{D}_X[s](f^*\mathcal{U}) \), we can define \( m(\gamma) = m(\gamma, \mathcal{U}), \quad D\psi_{f,\mathcal{E}(\gamma)}(\mathcal{D}_{X_0,\mathcal{U}}) := \psi_{\gamma} = \psi, \) and a nilpotent endomorphism \( N = N_\gamma \) of \( \psi \), where \( \mathcal{E}(\gamma) := \exp(2\pi \sqrt{-1}\gamma). \) (Cf. (2.3).) By (3.4, (2)), \( m(\gamma) = 0 \) unless \( \gamma \in \mathbb{Q} \). Put

\[
D\psi_f(\mathcal{D}_{X_0,\mathcal{U}}) := \bigoplus_{\gamma \in \mathbb{Q}/\mathbb{Z}} D\psi_{f,\mathcal{E}(\gamma)}(\mathcal{D}_{X_0,\mathcal{U}}), \quad \text{and}
\]

\[
b_{\mathcal{E}}(t, \mathcal{U}) = b_{\mathcal{E}}^\mathcal{E}(t, \mathcal{U}) = \prod_{\gamma \in \mathbb{Q}/\mathbb{Z}} (t - \mathcal{E}(\gamma))^m(\gamma).
\]

Then

1. \( b_{\mathcal{E}}(t) \) is the minimal polynomial of the endomorphism \( T \) of \( D\psi_f(\mathcal{D}_{X_0,\mathcal{U}}) \) induced by the \( s \)-action on \( M \). (Cf. (2.8).)

Here \( b_{\mathcal{E}}(x, \mathcal{U}) \) is determined only by \( \mathcal{D}_{X_0,\mathcal{U}}, \) and is independent of the special choice of the generator system \( \mathcal{U} \). Thus we sometimes write \( b_{\mathcal{E}}(t, \mathcal{D}_{X_0,\mathcal{U}}) \) or \( b_{\mathcal{E}}(t, \text{DR}_{X_0}(\mathcal{D}_{X_0,\mathcal{U}})) \) for \( b_{\mathcal{E}}^\mathcal{E}(t, \mathcal{U}) \). We follow the similar convention for \( m(\mathcal{E}, \gamma, \mathcal{U}); m(\gamma, \mathcal{U}) = m(\gamma, \mathcal{D}_{X_0,\mathcal{U}}) = m(\gamma, \text{DR}_{X_0}(\mathcal{D}_{X_0,\mathcal{U}})). \) By [Mal], [Kas2], we have

2. \( \text{DR}(D\psi_f(\mathcal{D}_{X_0,\mathcal{U}})) = i_* \psi_f(\text{DR}(\mathcal{D}_{X_0,\mathcal{U}}))[-1], \) and the right hand side is

\[
i_* \psi_f K_{0,\mathbb{Z}[N^{-1}]}[-1] \otimes \mathbb{Z}[N^{-1}] \mathbb{C} \] by (3.4, (3)), where \( i : f^{-1}(0) \rightarrow X \) is the inclusion mapping and \( \psi_f \) is the nearby cycle functor [Del1], and

3. \( T \) corresponds to the Picard-Lefschetz monodromy of \( i_* \psi_f K_0. \)

Hence

4. \( b_{\mathcal{E}}(t, \mathcal{U}) \in \mathbb{Q}[t]. \)

In other words,

5. \( \text{if } \mathcal{E}(\gamma) \text{ and } \mathcal{E}(\gamma') (\gamma, \gamma' \in \mathbb{Q}) \text{ are conjugate over } \mathbb{Q}, \text{ then } m(\gamma) = m(\gamma'). \)

By (3)

6. \( \text{if } \deg B_f^n(s, \mathcal{U})/n \text{ is independent of } n \in \mathbb{Z}_{>0}, \) \( b_{\mathcal{E}}^\mathcal{E}(t, \mathcal{U}) \) has the following simple expression. If \( B_f(s, \mathcal{U}) = \prod(s + \alpha_j), \) then \( b_{\mathcal{E}}^\mathcal{E}(t, \mathcal{U}) = \prod(t - \mathcal{E}(\alpha_j)). \) (Cf. (2.10).)
3.6. Applying (2.9) to $D^j_f,1(D_X(f^a u)) = D^j_f,e(a)(D_X u)$, we get a monodromy filtration $\{W_m\}$ shifted by $w - 1$, and we get

$$\max\{m \geq 0 \mid P \text{gr}_{w-1+m}(D^j_f,1(f^a u)) \neq 0\} = m(-\alpha - 1) - 1.$$ 

By (3.5, (5)), the right hand side is $m(\alpha) - 1$.

4. Mixed Hodge modules

Here we study some (mixed) Hodge modules. As for the Hodge modules (resp. mixed Hodge modules), the basic reference is [Sai1] (resp. [Sai2]). A brief account can be found in [Tan]. We fix a positive integer $c$ throughout §4.

4.1. Locally constant sheaf $H(c)Z$. Put $\zeta = \zeta_c = e(1/c)$, and $\Xi(c) := \{d/c \in c^{-1}Z \mid 0 < d \leq c, (c, d) = 1\}$. For any $\beta \in \Xi(c)$, there is a unique element $\sigma_\beta \in \text{Gal}(\mathbb{Q}(\zeta_c)/\mathbb{Q})$ such that $\sigma_\beta(e(1/c)) = e(\beta)$. Then $\text{Gal}(\mathbb{Q}(\zeta_c)/\mathbb{Q}) = \{\sigma_\beta \mid \beta \in \Xi(c)\}$. Define a locally constant sheaf of $\mathbb{Z}$-modules on $\mathbb{C}^\times = \{t \in \mathbb{C} \mid t \neq 0\}$ by

$$H_{c,Z} = H(c)Z = \{ \sum_{\beta \in \Xi(c)} \sigma_\beta(u)t^\beta \mid u \in \mathbb{Z}[\zeta]\}.$$ 

Here we take the single-valued branches $t^\beta (\beta \in \Xi(c))$ locally on $\mathbb{C}^\times$ as follows. First take any single-valued branch of $t^{1/c}$ locally on $\mathbb{C}^\times$. Then put $t^{d/c} = (t^{1/c})^d$ in the domain where $t^{1/c}$ is defined. Let $T$ be the generator of $\pi_1(\mathbb{C}^\times)$ defined by the oriented circle $\{e(t) \mid t : 0 \rightarrow 1\}$. Consider the natural action of $\pi_1(\mathbb{C}^\times)$ on the set of single-valued branches of $t^\beta$ (monodromy action). Then $T(t^{1/c}) = \zeta ct^{1/c}$ and $T(t^{d/c}) = \zeta^ct^{d/c} = \sigma_{d/c}(\zeta_c)t^{d/c}$. Hence $H_{c,Z}$ is well-defined. For any $\gamma \in \Xi(c)$, define a locally constant sheaf of $\mathbb{Z}[\zeta]$-modules on $\mathbb{C}^\times$ by

$$L_{\gamma,Z[\zeta]} = L(\gamma)Z[\zeta] := Z[\zeta]t^\gamma$$

in the same way as above.

4.2. Variation of Hodge structures $(H(c)Q, F)$ and polarization $S$. For $\beta \in \Xi(c)$, let $\beta' \in \Xi(c)$ be the element such that $\beta + \beta' \in \mathbb{Z}$. (If $c > 1$, $\beta' = 1 - \beta$. If $c = 1$, $\beta = \beta' = 1$.) Then $\sigma_{\beta'}(u) = \overline{\sigma_\beta(u)} (u \in \mathbb{Z}[\zeta])$, and

$$H(c)_R = \{ \sum_{\beta \in \Xi(c)} u_\beta t^\beta \mid u_\beta \in \mathbb{C}, u_{\beta'} = \overline{u_\beta} \}.$$ 

Consider the tensor product of sheaves $H(c)_O := H(c)_Z \otimes \mathbb{Z} O = \bigoplus_{\beta \in \Xi(c)} O t^\beta = \bigoplus_{\beta \in \Xi(c)} D t^\beta$, where $O = O_{\mathbb{C}^\times}$ and $D = D_{\mathbb{C}^\times}$. Define a decreasing filtration $\{F^p\}_{p \in \mathbb{Z}}$ of $H(c)_O$ by $F^p = H(c)_O$ if $p \leq 0$ and $= 0$ if $p > 0$. Then $(H(c)Q, F)$ is a variation of Hodge structures of weight 0 on $\mathbb{C}^\times$. See [Sai1, 5.4] or [Tan, 1.1]. Define a $\mathbb{C}$-bilinear form $S$ on $\sum_{\beta \in \Xi(c)} C t^\beta$ by $S(t^\beta, t^\gamma) := \delta_{\beta', \gamma}$. Then

$$S(\sum_{\beta} \sigma_\beta(u)t^\beta, \sum_{\gamma} \sigma_\gamma(v)t^\gamma) = \sum_{\beta} \sigma_\beta(u\overline{v}) \in \mathbb{Z}$$
for \( u, v \in \mathbb{Z}[\zeta] \), and
\[
S(\sum_{\beta} u_{\beta} t^\beta, \sum_{\beta} u_{\beta} t^\beta) = \sum_{\beta} u_{\beta} u_{\beta'} = \sum_{\beta} |u_{\beta}|^2
\]
for \( \sum_{\beta} u_{\beta} t^\beta \in H(c)_R \). Hence \( S \) gives a polarization of \( (H(c)_Q, F) \). Cf. [Tan, 1.3, (p5)] and [Del4, (2.1.15)]. Let \( MH(X, Q, w)^p \) be the category of polarizable Hodge modules of weight \( w \) [Sai1, 5.1.6]. Cf. [Tan, 1.3]. Put \( \mathcal{H}(c) := (H(c)_O, F, H(c)_Q[1]) \in MH(C^\times, Q, 1)^p \), where \( O = O_{C^\times} \).

4.3. Assumption A2. Besides (A1), assume further that
(1) \( f : X_0 \to C^\times \) is smooth, and
(2) \( D_{X_0} u \) is underlying a pure Hodge module \( MH D_{X_0} u \in MH(X_0, Q, w)^p \).

4.4. Hodge module \( \mathcal{M}(c) \). Let \( \Delta : X_0 \to X_0 \times X_0 \) be the diagonal embedding, and put
\[
\mathcal{M}(c) = \mathcal{M}_{X_0}(c) := MH \Delta^*(MHf^* \mathcal{H}(c) \boxtimes MH D_{X_0} u)[-n],
\]
where \( n = \dim X \). By [Sai2, 2.25], \( \mathcal{M}(c) \in MH(X_0, Q, w) \). By (3.4, (3)), the underlying perverse sheaf of \( \mathcal{M}(c) \) is
\[
(f^*H(c)_Z \otimes Z K_0, Z[N-1][\dim X]) \otimes Z[N-1] Q.
\]
Let \( j : X_0 \to X_0 \times X_0 \) be the inclusion mapping. By the definition of the category \( MHM(X) \) of mixed Hodge modules [Sai2, 4.2.11], \( MH j_* \mathcal{M}(c) \in MHM(X) \) can be defined. Its underlying \( D_X \)-module is \( \bigoplus_{\beta \in \Xi(c)} D_{X_0}(f^\beta u) \otimes \bigoplus_{\beta \in \Xi(c)} D_X(f^\beta u) \) (\( k \gg 0 \)). Cf. [Gyo3, (5.16, (2))].

4.5. Weight filtration of \( MH j_* \mathcal{M}(c) \). We can describe the weight filtration of \( MH j_* \mathcal{M}(c) \) using the description given in the proof of [Sai2, 2.11] as follows. (See also [Sai2, 2.8].)

(1) For any \( m \in \mathbb{Z} \), the weight filtration \( W_m(MH j_* \mathcal{M}(c)) \) is of the form \( \bigoplus_{\beta \in \Xi(c)} W_m(D_{X_0}(f^\beta u)) \) with some \( D_X \)-submodule \( W_m(D_{X_0}(f^\beta u)) \) of \( D_{X_0}(f^\beta u) \).

(2) \( \max\{m \geq 0 | gr^W_{w+m}(D_{X_0}(f^\beta u)) \neq 0 \} = m(\beta) \).

Proof. Put \( \mathcal{M} = \text{image}(MH j_! \mathcal{M}(c) \to MH j_* \mathcal{M}(c)) \). Then \( \mathcal{M} \in MH(X, w)^p \). Apply [Sai2, (2.11.10)] to the present situation, using (3.6). □

Note that \( m(\beta) \) is independent of \( \beta \in \Xi(c) \). (Cf (3.5, (5)).) Put \( m(\Xi(c)) := m(\beta) \) (\( \beta \in \Xi(c) \)). Then

(3) \( \max\{m \geq 0 | gr^W_{w+m}(MH j_*(\mathcal{M}(c)) \neq 0 \} = m(\Xi(c)) \).

4.6. (The description of the weight filtration given in this and the next paragraphs is due to M.Kashiwara, which I learned from him in 1986. Here I include this result...
by permission of M. Kashiwara, to whom I am very grateful. In the present paper, (4.6) and (4.7) will not be used.) Let \( k \ll 0 \) and \( l \gg 0 \). Put \( \mathcal{M}_k := t^k D_X [s, t] (f^S u) \). Then \( \mathcal{M}_k / (s - \beta) \mathcal{M}_k = D_X (f^\beta + k \underline{u}) = D_j_*(D_X 0(f^\beta \underline{u})) \). Put

\[
W'(D_j_*(D_X 0(f^\beta \underline{u}))) := \text{image} \left( \{ u \in \mathcal{M}_k \mid (s - \beta)^m u \in \mathcal{M}_l \} \rightarrow \frac{\mathcal{M}_k}{(s - \beta) \mathcal{M}_k} \right).
\]

Then

\[
\text{gr}^W_{m}(Dj_*(D_X 0(f^\beta \underline{u}))) = \text{Pgr}_{w+m-2}(D^j_*(D_X 0(f^\beta \underline{u}))) \quad \text{by (1.2)}
\]

Hence

\[
W_m(D_j_*(D_X 0(f^\beta \underline{u}))) = W'_m(Dj_*(D_X 0(f^\beta \underline{u}))),
\]

and the underlying \( D_X \)-module of \( W_m(M^H j_*(\mathcal{M}(c))) \) is \( \bigoplus_{\beta \in \Lambda} W'_m(D_j_*(D_X 0(f^\beta \underline{u}))) \).

**4.7. Microlocal structure of the weight filtration.** Let \( \pi : T^* X \rightarrow X \) be the cotangent bundle of \( X \), and \( \mathcal{E} = \mathcal{E}_X \) the sheaf of \( \mathfrak{m}_{\mathfrak{C}} \) \textit{car} \( \text{di} \text{f} \text{e} \text{r} \text{e} \text{n} \text{t} \) \text{a} \text{l} \text{o} \text{p} \text{e} \text{r} \text{a} \text{t} \text{ors} on \( T^* X \). Since \( \mathcal{E} \) is flat over \( \pi^{-1} D \), we can see

\[
\mathcal{E}_X \otimes_{\pi^{-1} D_X} \pi^{-1} W_m(Dj_*(D_X 0(f^\beta \underline{u}))) = \mathcal{E}_X \otimes_{\pi^{-1} D_X} \pi^{-1} W_m(Dj_*(D_X 0(f^\beta \underline{u})))
\]

is equal to the right hand side of (4.6, (1)) with \( M_j \) replaced with \( \mathcal{E}_X \otimes_{\pi^{-1} D_X} \pi^{-1} M_j \).

Now let us consider the case where \( X \) is a complex non-singular algebraic variety on which a connected algebraic group \( G \) acts. Assume that \( f \) is relatively \( G \)-invariant, and that \( \Lambda \) is an irreducible component of the characteristic variety \( \text{ch}(D_X f^\beta) \) on which \( G \) acts prehomogeneously. Let \( b_\Lambda(s) \) be the local \( b \)-function of \( f \) on \( \Lambda \) [SKKO] (cf. [Gyo4, 6.1]). Put \( b_\Lambda(s) = \prod_{j=1}^{d_{A}} (s + \alpha_j) \) and \( w_\Lambda(\alpha) := \# \{ j \mid \beta + \alpha_j \in \mathbb{Z} \} \). Let us apply (2.9) and (4.6, (2)) to the \( \mathcal{E} \)-modules appearing in (1). Then we get

\[
0 = W_0 = \cdots = W_{w_\Lambda(\beta)-1} \subset W_{w_\Lambda(\beta)} = \cdots = \mathcal{E}_X f^\beta-k := \mathcal{E}_X \otimes_{\pi^{-1} D_X} \pi^{-1} (D_X f^\beta-k)
\]

on \( \Lambda \), where \( W_m := \mathcal{E}_X \otimes_{\pi^{-1} D_X} \pi^{-1} W_m(Dj_*(D_X 0(f^\beta))) \) for \( m \in \mathbb{Z} \). (Note that

\[
\mathcal{E}_X \otimes_{\pi^{-1} D_X} \pi^{-1} Dj_*(D_X 0(f^\beta)) = \mathcal{E}_X \otimes_{\pi^{-1} D_X} \pi^{-1} (D_X f^\beta-k) \quad (k \ll 0)
\]

is simple holonomic on \( \Lambda \) [SKKO], and hence (2) is equivalent to

\[
\text{gr}^W_{w_\Lambda(\beta)} \neq 0
\]

on \( \Lambda \). In terms of characteristic variety, (2) can be also expressed as

\[
\text{ch}(W_m(Dj_*(D_X 0(f^\beta)))) \supset \Lambda \Leftrightarrow w_\Lambda(\alpha) \leq m,
\]

or

\[
\text{ch}(\text{gr}^W_m(Dj_*(D_X 0(f^\beta)))) \supset \Lambda \Leftrightarrow w_\Lambda(\alpha) = m.
\]
5. Fourier Transformation of D-modules

5.1. Let \( V = \mathbb{C}^n \) \((n > 0)\), \( V^\vee \) be its dual space, \( x = (x_1, \ldots, x_n) \) a linear coordinate of \( V \) and \( y = (y_1, \ldots, y_n) \) its dual coordinate of \( V^\vee \). Then \( D_V = \mathbb{C}\langle x_1, \ldots, x_n, \partial/\partial x_1, \ldots, \partial/\partial x_n \rangle \) and \( D_{V^\vee} = \mathbb{C}\langle y_1, \ldots, y_n, \partial/\partial y_1, \ldots, \partial/\partial y_n \rangle \). (Cf. (N3).) Define an algebra isomorphism \( F: D_V \to D_{V^\vee} \) by

\[
F(x_j) = \sqrt{-1} \frac{\partial}{\partial y_j} \quad \text{and} \quad F\left( \frac{\partial}{\partial x_j} \right) = \sqrt{-1} y_j.
\]

For a \( D_V \)-module \( M \), define a \( D_{V^\vee} \)-module structure on \( M \) using this isomorphism \( F \). We shall denote this \( D_{V^\vee} \)-module by \( F(M) \), and call it the Fourier transform of \( M \).

5.2. Assumption A3. Let \( f \in \mathbb{C}[V] \), \( f^\vee \in \mathbb{C}[V^\vee] \), \( \Omega = V \setminus f^{-1}(0) \), \( \Omega^\vee = V^\vee \setminus f^\vee^{-1}(0) \), \( j: \Omega \to V \) and \( j^\vee: \Omega^\vee \to V^\vee \) be the inclusion mappings. Besides (A1) and (A2), assume that there exists a simple \( D_\Omega \)-module \( 0 \neq M_\Omega = M^\vee(\alpha) \in \text{mod}_r(\Omega^\vee) \) such that \( F(j_* D(f^\alpha \underline{u})) = j_!^\vee M \).

5.3. Put \( \mathcal{F}(W_{w+m}(Dj_* D(f^\alpha \underline{u}))) =: M_{m}^\vee \). Then \( M_{m}^\vee \subset Dj_{*}^\vee M_{m}^\vee \subset M^\vee \), and

\[
\begin{align*}
&j^\vee M_{m}^\vee \neq 0 \Leftrightarrow j^\vee M_{m}^\vee = M^\vee \Leftrightarrow j^{-1}(Dj_{*}^\vee M_{m}^\vee / M_{m}^\vee) = 0 \\
&\Leftrightarrow Dj_{*}^\vee M_{m}^\vee / M_{m}^\vee = 0 \Leftrightarrow M_{m}^\vee = Dj_{*}^\vee M_{m}^\vee.
\end{align*}
\]

(The first and the second lines are equivalent since \( Dj_{*}^\vee M_{m}^\vee \) does not have a non-zero quotient supported by \( f^\vee^{-1}(0) \). Note that \( Dj_{*}^\vee M_{m}^\vee \in \text{mod}(D_{V^\vee}) \), since \( Rj_{*}^\vee PDR(M^\vee) \) is a perverse sheaf [BBD, 4.1.3].) Hence

\[
\begin{align*}
&\mathcal{F}(\text{gr}^W_{w+m}(Dj_* D(f^\alpha \underline{u})))|\Omega^\vee \neq 0 \\
&\Leftrightarrow j^\vee M_{m}^\vee / M_{m-1}^\vee \neq 0 \\
&\Leftrightarrow j^\vee M_{m}^\vee \neq 0 \quad \text{and} \quad j^\vee M_{m-1}^\vee = 0 \\
&\Leftrightarrow M_{m-1}^\vee \subset M_{m}^\vee = Dj_{*}^\vee M_{m}^\vee \\
&\Leftrightarrow W_{w+m-1}(Dj_* D(f^\alpha \underline{u})) \subset W_{w+m}(Dj_* D(f^\alpha \underline{u})) = Dj_* D(f^\alpha \underline{u}) \\
&\Leftrightarrow m = m(\alpha).
\end{align*}
\]

(The second and the third lines are equivalent because of the simplicity of \( M^\vee \). The equivalence of the third and the fourth lines follows from (1). The last equivalence follows from (4.5, (2)).)

6. Fourier Transformation of Sheaves on \( \mathbb{C}^n \)

6.1. Sato-Fourier transformation. ([BMV], [Bry]). Let notation be as in §4. Let \( Z = \{ z \in \mathbb{C} \mid \text{Re}(z) \leq 0 \} \), \( i: Z \to \mathbb{C} \) be the inclusion mapping, \( L^+ = i_* \mathbb{C}_Z, \)
$pr : V^\vee \times V \to V$ and $pr^\vee : V^\vee \times V \to V^\vee$ be the projections, and $( ) : V^\vee \times V \to \mathbb{C}$ the natural pairing. For $K \in D^b(C_V)$, put $\mathcal{F}^+(K) := Rpr_!^{\vee}(pr^*(K) \otimes (\langle \rangle^* L^+))[n]$, which is called the Sato-Fourier transform of $K$ [SKK], [KasSch], [BMV], [Bry], [HK].

6.2. Define $h : C^\times \times V \to V$ by $h(t, v) = tv$. Let $c$ be a positive integer, $\alpha \in \Xi(c)$, and put $D_{mon, \alpha}^b(V) := \{K \in D^b(C_V) | h^*K = L_{\alpha, C} \}$.

(See (4.1) for $L_{\alpha} = L_{\alpha, C}$.) If a subvariety $X \subset V$ is $C^\times$-stable, we can similarly define $D_{mon, \alpha}^b(X)$.

6.3. Put $\tau^+(\alpha) = R\Gamma_c(C^\times, L_{\alpha} \otimes L^+)[1]$. Since $C^\times \cap Z$ is homeomorphic to $(0, \infty) \times \left[\frac{3}{2}, \frac{3}{2}\right]$, $\tau^+(\alpha) = H^1_c(C^\times, L_{\alpha} \otimes L^+) \simeq \mathbb{C}$.

6.4. Let $Q := \{(v^\vee, v) \in V^\vee \times V \mid \langle v^\vee, v \rangle = 1\}$, $e : Q \to V^\vee \times V$ be the inclusion mapping, consider the natural morphism $\gamma : e!e^!C_{V^\vee \times V} \to C_{V^\vee \times V}$, and let $\omega$ be its mapping cone; $\omega := \text{cone} (\gamma)$.

6.5. **Radon transformation.** For any $K \in D_{mon, \alpha}^b(V)$,

$$\mathcal{F}^+(K) \simeq \mathcal{F}^+(K) \otimes \tau^+ \simeq Rpr_!^{\vee}(pr^*(K) \otimes \omega)[n] =: \mathcal{R}(K).$$

(This isomorphism is not canonical.) The right member is called the Radon transform of $K$. We omit the proof, since the proof of (7.7) below is essentially the same, and the latter is included in [DG].

6.6. **Assumption** A4. (1) $K_{0, c} \in D_{mon, \alpha_0}^b(\Omega)$ for some $\alpha_0 \in C$.

(2) $f$ and $f^\vee$ are homogeneous polynomials.

6.7. Define a filtration $\{W_m\}$ of the perverse sheaf $Rj_*(f^*L_{-\alpha} \otimes K_{0, c})[n]$ (cf. [BBD]) by $pDR(W_m(Dj_*D\Omega(f^\alpha \underline{u}))):= W_m(Rj_*(f^*L_{-\alpha} \otimes K_{0, c})[n])$. Then

$$m = m(\alpha)
\Rightarrow \mathcal{F}(\text{gr}_w^{W}(Dj_*D\Omega(f^\alpha \underline{u})))|\Omega^\vee \neq 0 \quad \text{by (5.3, (2))}
\Rightarrow \mathcal{F}^+(\text{gr}_w^{W}(Rj_*(f^*L_{-\alpha} \otimes K_{0, c})[n]))|\Omega^\vee \neq 0 \quad \text{by (A4) in (6.6), and [HK]}
\Rightarrow \mathcal{R}(\text{gr}_w^{W}(Rj_*(f^*L_{-\alpha} \otimes K_{0, c})[n]))|\Omega^\vee \neq 0 \quad \text{by (6.5)}.$$
7.2. **Artin-Schreier torsor.** Fix a non-trivial additive character \( \psi \in \text{Hom}(\mathbb{F}_q, \overline{\mathbb{Q}}_l^\times) \). Let \( L_\psi \) be the Artin-Schreier torsor on \( \mathbb{A}^1 \). The Frobenius endomorphism \( \text{Frob}_q \) acts on \( L_{\psi, x} (x \in \mathbb{F}_q) \) as the multiplication by \( \psi(x) \). See [Del3, 1.4].

7.3. **Lang torsor.** For \( \chi \in \text{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}}_l^\times) \), let \( L_\chi \) be the Kummer torsor of order \( q-1 \) on \( G_m = \mathbb{A}^1 \setminus \{0\} \). The Frobenius endomorphism \( \text{Frob}_q \) acts on \( L_{\chi, x} (x \in \mathbb{F}_q^\times) \) as the multiplication by \( \chi(x) \). Fixing an isomorphism \( \mathbb{F}_q^\times / \mathbb{Z} / \mathbb{Z} \simeq \text{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}}_l^\times) \), \( \alpha \mapsto \chi_\alpha \), we sometimes denote \( L_\alpha = L(\alpha) \) for \( L_{\chi_\alpha} \). See [Del3, 1.4].

7.4. **Deligne-Fourier transformation.** ([KL].) Define a functor \( \mathcal{F}_\psi : D^b_c(\mathbb{F}_q, \overline{\mathbb{Q}}_l) \to D^b_c(\mathbb{F}_q^\vee, \overline{\mathbb{Q}}_l) \) by \( \mathcal{F}_\psi(-) = \text{Rpr}_! (\text{pr}^*(-) \otimes \langle \rangle^* L_\psi)[n] \), which is called the Deligne-Fourier transformation.

7.5. Define \( h : G_{m, \mathbb{F}_q} \times V_{\mathbb{F}_q} \to V_{\mathbb{F}_q} \) by \( h(t, v) = tv \). For \( \chi \in \text{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}}_l^\times) \), put

\[
D^b_{\text{mon}, \chi}(V_{\mathbb{F}_q}) = \{ K \in D^b_c(\mathbb{F}_q^\vee, \overline{\mathbb{Q}}_l) \mid h^* K = L_\chi \boxtimes K \}.
\]

7.6. Put \( \tau(\chi, \psi) = \text{R}\Gamma_c(G_{m, \mathbb{F}_q}, L_\chi \boxtimes L_\psi)[1] \). Then we get a (non-canonical) isomorphism \( \tau(\chi, \psi) = H^1_c(G_{m, \mathbb{F}_q}, L_\chi \boxtimes L_\psi) \simeq \overline{\mathbb{Q}}_l \) (cf. [Del3, 4.2]).

7.7. **Radon transformation.** (Cf. (6.5).) Consider the natural morphism \( \gamma : e_1 e_1^! \overline{\mathbb{Q}}_l \to \overline{\mathbb{Q}}_l \). Put \( \omega := \text{cone}(\gamma) \). For \( K \in D^b_{\text{mon}, \chi}(V_{\mathbb{F}_q}) \), define its Radon transform by \( \text{Rpr}_! (\text{pr}^* K \otimes \omega)[n] =: \mathcal{R}(K) \). In [DG], we get

\[
\mathcal{F}_\psi(K) \simeq \mathcal{F}_\psi(K) \otimes \tau(\overline{\chi}, \overline{\psi}) \simeq \mathcal{R}(K) \text{ (non-canonically)}. 
\]

8. **Specialization from** \( D^b_c(V(\mathbb{C}), \mathbb{C}) \) **to** \( D^b_c(V_{\mathbb{F}_q} \overline{\mathbb{Q}}_l) \)

8.1. Since \( H_{c, \mathbb{Q}_l} = H_{c, \mathbb{Z} \otimes \mathbb{Q}_l} \) and \( K_{0, \mathbb{Q}_l} = K_{0, \mathbb{Z} \otimes \mathbb{Q}_l} \), we can consider their \('\text{reduction modulo } p'\) (\( p \gg 0 \)). See [BBD, 6.1]. (See (A1, (3)) in (3.4) for \( K_0 \).) We denote the resulting \( \mathbb{Q}_l \)-sheaves on \( V_{\mathbb{F}_q} \) by the same letter and thus each of such symbols has two meanings.

8.2. **Assumption A5.** Assume that

1. \( f \in \mathbb{Q}[V], f^\vee \in \mathbb{Q}[V^\vee] \), and
2. \( K_{0, \mathbb{Q}_l}[n] \in D^b_c(V_{\mathbb{F}_q}, \mathbb{Q}_l) \) is pure of weight \( w \).

8.3. \( W_m(Rj_* (f^* H_{c, \mathbb{Q}_l} \otimes K_{0, \mathbb{Q}_l})[n]) \in D^b_c(V(\mathbb{C}), \mathbb{Q}_l) \) (= the perverse sheaf underlying the weight filtration of the mixed Hodge module, whose coefficient ring is extended to \( \mathbb{Q}_l \)) specializes (by the comparison theorem and by \('\otimes \mathbb{F}_q'\)) to the perverse sheaf of the same name in \( D^b_c(V_{\mathbb{F}_q}, \mathbb{Q}_l) \) (= the weight filtration of the mixed perverse sheaf) if \( \text{char}(\mathbb{F}_q) \gg 0 \) and if \( l \) is a prime which does not divide \( qN \). Here \( N \) is the integer appeared in (3.4, (3)).
Proof. Let $\hat{V}$ be a smooth scheme over $\mathbb{Z}[N_{1}^{-1}]$ ($N_{1} \in \mathbb{Z}_{>0}$), $\pi : \hat{V} \to V$ a projective morphism such that $\hat{f} := f \circ \pi$ is normal crossing relative to $\mathbb{Z}[N_{1}^{-1}]$ and $\pi$ induces an isomorphism $\hat{\Omega} := V \setminus \hat{f}^{-1}(0) \to \Omega$ [Hir]. Let $\tilde{j} : \hat{\Omega} \to \hat{V}$ be the inclusion morphism. Then it is easy to see that (8.3) holds if $f$ and $j$ are replaced with $\hat{f}$ and $\hat{j}$. Put $\hat{K} := Rj_{*}(f^{*}H_{c},Q \otimes \pi^{*}K_{0,Q_{l}})$ and $K := Rj_{*}(f^{*}H_{c}, Q \otimes K_{0,Q_{l}})$. Then $R\pi_{*}\hat{K} = K$. From the exact sequence of perverse sheaves

$$0 \to W_{\leq m}(\hat{K}[n]) \to \hat{K}[n] \to W_{>m}(\hat{K}[n]) \to 0,$$

we get an exact sequence

$$pH^{0}(R\pi_{*}(W_{\leq m}(\hat{K}[n]))) \alpha \to \hat{K}[n] \to pH^{0}(R\pi_{*}(W_{>m}(\hat{K}[n])))$$

and consequently, $\text{image}(\alpha) = W_{\leq m}(K[n])$ in the both sense of (8.1). Thus we get (8.3). □

8.4. Continue to assume (A1)–(A5). By (6.7) and by the scalar restriction, we get

$$m = m(\Xi(c)) \Leftrightarrow \mathcal{R}(\text{gr}^{W}_{w+m}(Rj_{*}(f^{*}H_{c,Q_{l}} \otimes K_{0,Q_{l}}))[n]))|\Omega^{\vee} \neq 0 \text{ in } D_{c}^{b}(V^{\vee}(C,Q_{l})).$$

Since the Radon transformation $\mathcal{R}(-)$ is compatible with the `reduction modulo $p^{r}$' ($p \gg 0$), we may understand the right member of the above equivalence in $D_{c}^{b}(V_{F_{q}}^{\vee}, Q_{l})$. Since $H_{c,Q_{l}} = \bigoplus_{\text{ord } x = c} L_{x}$,

$$\mathcal{R}(\text{gr}^{W}_{w+m}(Rj_{*}(f^{*}L_{x} \otimes K_{0,Q_{l}}))[n]))|\Omega^{\vee} \neq 0 \text{ for some } \chi \text{ of order } c$$

(2)

$$\Rightarrow \mathcal{R}(\text{gr}^{W}_{w+m}(Rj_{*}(f^{*}H_{c,Q_{l}} \otimes K_{0,Q_{l}}))[n]))|\Omega^{\vee} \neq 0$$

Take $l$ so that $\mathbb{Z}[\zeta_{c}] \otimes Q_{l}$ is an integral domain. Then $Q_{l}(\zeta_{c}) = \mathbb{Z}[\zeta_{c}] \otimes Q_{l} \text{ and it acts on } H_{c,Q_{l}} \text{ in several ways, giving } Q_{l}(\zeta_{c}) \text{-sheaves } L_{x} \text{ for all } \chi \text{ of order } c.$ Hence

(3) $$\Rightarrow \mathcal{R}(\text{gr}^{W}_{w+m}(Rj_{*}(f^{*}L_{x} \otimes K_{0,Q_{l}}))[n]))|\Omega^{\vee} \neq 0 \text{ for any } \chi \text{ of order } c,$$

for the specific $l$ as above, and hence the same holds for any $l$.

Theorem 9. Assume (A1)–(A5) and that $\text{char}(F_{q}) \gg 0$. Then $\mathcal{F}_{\psi}(Rj_{*}(f^{*}L_{x} \otimes K_{0,Q_{l}}))[n]))|\Omega^{\vee}$ is pure of weight $w + n + m$ with $m = m(\Xi(\text{ord } \chi))$.

Proof. By [KL, (2.2.1)], $\mathcal{F}_{\psi}(\text{gr}^{W}_{w+m}(-)) = \text{gr}^{W}_{w+n+m}(\mathcal{F}_{\psi}(-))$. Hence (8.4, (1)–(3)), (8.3), and (7.7, (1)) yields the result. □

10. PREHOMOGENEOUS VECTOR SPACES

10.1. Let us review [Gyo1] and [Gyo3]. Let $G$ be a connected reductive group, $\rho : G \to GL(V)$ a finite dimensional rational representation, and assume that everything is defined over an algebraic number field $K (\subset \mathbb{C})$. Assume that $(G, \rho, V) \otimes \mathbb{C}$ is a prehomogeneous vector space, i.e., $G(\mathbb{C})$ has a dense orbit in $V(\mathbb{C})$. Let
\( \rho^\vee : G \to GL(V^\vee) \) be the contragradient representation of \( \rho, \phi \in \text{Hom}(G, G_m) \), \( 0 \neq f \in Q[V], 0 \neq f^\vee \in Q[V^\vee] \), and assume that \( f(gv) = \phi(g)f(v) \) and \( f^\vee(gv) = \phi(g)^{-1}f^\vee(v) \) for any \( g \in G(C), v \in V(C) \) and \( v^\vee \in V^\vee(C) \). Put \( \Omega = V \setminus f^{-1}(0) \) and \( \Omega^\vee = V^\vee \setminus f^\vee^{-1}(0) \). Then there exists a unique closed \( G(C) \)-orbit \( O_1(C) \) in \( \Omega(C) \) (resp. \( \Omega^\vee(C) \)). Let \( i : O_1 \to \Omega \) and \( i^\vee : \Omega^\vee \to \Omega^\vee \) be the inclusion morphisms. Put \( n = \dim V = \dim V^\vee \) and \( m = \dim O_1 = \dim \Omega^\vee \). Define \( F : \Omega \to V^\vee \) and \( F^\vee : \Omega^\vee \to V \) by \( F = \text{grad log} f \) and \( F^\vee = \text{grad log} f^\vee \). Then \( F \) and \( F^\vee \) induce smooth morphisms \( F : \Omega \to O_1^\vee \) and \( F^\vee : \Omega^\vee \to O_1 \). Let \( \pi^\vee : \hat{O}_1^\vee \to O_1^\vee \) be the two fold covering defined in [Gyo1, 3.14]. Let \( L(\omega^\vee) \) be the cokernel of \( \mathcal{Z}O_1^\vee \to \pi^\vee \mathcal{Z}\hat{O}_1^\vee \). Then \( L(\omega^\vee) \) is a locally constant sheaf on \( O_1^\vee(C) \).

Consider a \( D \)-module \( \mathcal{D}_{O_1^\vee}u^\vee \) satisfying the following condition.

10.2. **Assumption A6.** \( \mathcal{D}_{O_1^\vee}u^\vee \) is a regular holonomic \( D \)-module such that \(-Au^\vee = \chi(A)u^\vee \) \((A \in \text{Lie}(G)) \) and \( \text{DR}(\mathcal{D}_{O_1^\vee}u^\vee) \) is a locally constant sheaf of rank one. \( \)\( \) (Note that \( A \in \text{Lie}(G) \) induces a vector field on \( O_1^\vee \), which can be regarded as a differential operator of first order.) Put \( K_0^\vee := \text{DR}(\mathcal{D}_{O_1^\vee}u^\vee), K^\vee := f^\vee \ast L_\alpha \otimes K_0^\vee, \) and \( K^\alpha := F \ast K^\vee. \)

**Lemma 10.3.** [Gyo3, 6.21]. Assume (A6). Then (1) \( \mathcal{F}^+(Rj_*F^\vee K_0^\vee[n]) = j^\vee i^\vee (K_0^\vee \otimes L(\omega^\vee))[m] \) and (2) \( \mathcal{F}^+(Rj^\vee i^\vee (K_0^\vee \otimes L(\omega^\vee))[m]. \)

10.4. Assume that \( \text{char}(F_q) \neq 2 \) and let \( \chi_{1/2} \) be the unique character of \( F_q^\times \) of order 2. For \( v^\vee \in V^\vee(F_q) \), let \( h^\vee(v^\vee) \) be the discriminant of the quadratic form determined by \( \left( \frac{\partial^2 \log L^\vee (v^\vee)}{\partial y_i \partial y_j} \right) \). (Cf. [Gyo2, §7].) By [DG, 3.5.4], we have

\[
\text{trace}(\text{Frob}_q \mid L(\omega^\vee)_{Q_i}) = \chi_{1/2}(h^\vee(v^\vee)) \times C_1, \text{ for } v^\vee \in O_1^\vee(F_q)
\]

for any Frobenius action on \( L(\omega^\vee)_{Q_i} \), where \( C_1 \) is some constant independent of \( v^\vee \). In the situation of the following theorem, all the assumptions (A1)–(A6) are satisfied.

**Theorem 11** Assume that \( f \in Q[V], f^\vee \in Q^\vee[V^\vee] \) and \( \text{char}(F_q) \gg 0. \) Fix an isomorphism \( \left( \frac{1}{1-q} \right) Z \cong F_q^\times, \alpha \mapsto \chi_\alpha, \) and put \( L_\alpha := L_{\chi_\alpha}. \) Then the following holds in \( D_c^b(F_q^\times, Q_l) \).

(a) If we forget the Frobenius action, then

(1) \( \mathcal{F}^+(Rj_*f^\vee L_{-\alpha}[n]) \cong j^\vee i^\vee (f^\vee \ast L_\alpha \otimes L(\omega^\vee))[m], \)

(2) \( \mathcal{F}^+(Rj_*L_{-\alpha} \otimes F^\vee \ast L(\omega^\vee))[n]) \cong j^\vee i^\vee f^\vee \ast L_\alpha[m], \)

(3) \( \mathcal{F}^+(Rj^\vee i^\vee (f^\vee \ast L_{-\alpha} \otimes L(\omega^\vee))[m]) \cong j^\vee f^\vee \ast L_\alpha[n], \)

(4) \( \mathcal{F}^+(Rj^\vee i^\vee f^\vee \ast L_{-\alpha}[m]) \cong j^\vee (f^\ast L_\alpha \otimes F^\ast \ast L(\omega^\vee))[n], \)

(1') \( \mathcal{F}^+(Rj^\vee i^\vee (f^\ast \ast L_{-\alpha} \otimes L(\omega^\vee)))[m]) \cong Rj^\vee f^\ast \ast L_{-\alpha}[n], \)

(2') \( \mathcal{F}^+(Rj^\vee i^\vee f^\ast \ast L_{-\alpha}[m]) \cong Rj^\vee f^\ast \ast L_{-\alpha}[m], \)

(3') \( \mathcal{F}^+(Rj^\vee i^\vee f^\ast \ast L_{-\alpha}[m]) \cong Rj^\vee f^\ast \ast L_{-\alpha}[n], \)

(4') \( \mathcal{F}^+(Rj^\vee i^\vee f^\ast \ast L_{-\alpha}[m]) \cong Rj^\vee f^\ast \ast L_{-\alpha}[m]. \)
(b) Assume that the Frobenius action on $L_{\alpha}$ and $L(\omega)$ are the natural ones. Then the left hand sides of (1)–(4) and (1')–(4') restrict to pure sheaves on $\Omega$ or $\Omega^\vee$. The weights are

(1) $n + m(\alpha, C_{1}[n])$,
(2) $n + m(\alpha, F^*L(\omega)[n])$,
(3) $n + m(\alpha, i_{\vee}^\vee L(\omega)[m])$,
(4) $n + m(\alpha, i_{\vee}^\vee C_{O_Y}[m])$,

for $\alpha, \beta, \gamma, \omega, \nu, \nu'$.

The $\text{wei}, \text{ghts}$ are

(1) $n + m(\alpha, C_{O_Y}[m])$,
(2) $n + m(\alpha, F^*L(\omega)[n])$,
(3) $n + m(\alpha, i_{\vee}^\vee L(\omega)[m])$,
(4) $n + m(\alpha, i_{\vee}^\vee C_{O_Y}[m])$,

respectively. (For $m(\alpha, -)$, see the lines following (3.5, (1)).)

Proof. (a) (1) and (2) follows from (10.3) as in [Gyo2]. By the Verdier duality, (i) $\Leftrightarrow (i')$ for $i = 1, \cdots, 4$. By $'F^\psi = F^\psi^{-1}$, (1') $\Leftrightarrow (3)$ and (2') $\Leftrightarrow (4)$.

(b) (1)–(4) follows from Theorem 9, from which (1')–(4') follows by the Verdier duality. \(\square\)

**Corollary 12** Keep the notations and the assumptions of the above theorem.

(a) There exist constants $C_1, \cdots, C_4$ such that

(1) $q^{-n} \sum_{v \in O(F_q)} \chi_\alpha(f(v))\psi((v^\vee, v))$

$$= \begin{cases} C_1 \cdot \chi_\alpha(f^*(v^\vee)^{-1}) \cdot \chi_{1/2}(h^\vee(v^\vee)) & \text{for } v^\vee \in \Omega^\vee(F_q) \\ 0 & \text{for } v^\vee \in (\Omega^\vee \setminus \Omega^\vee)(F_q). \end{cases}$$

(2) $q^{-n} \sum_{v \in O(F_q)} \chi_\alpha(f(v))\chi_{1/2}(h^\vee(F(v)))\psi((v^\vee, v))$

$$= \begin{cases} C_2 \cdot \chi_\alpha(f^*(v^\vee)^{-1}) & \text{for } v^\vee \in \Omega^\vee(F_q) \\ 0 & \text{for } v^\vee \in (\Omega^\vee \setminus \Omega^\vee)(F_q). \end{cases}$$

(3) $q^{-m} \sum_{v^\vee \in \Omega^\vee(F_q)} \chi_\alpha(f^\vee(v^\vee))\chi_{1/2}(h^\vee(v^\vee))\psi((v^\vee, v))$

$$= C_3 \cdot \chi_\alpha(f(v)^{-1}) \quad \text{for } v \in \Omega(F_q).$$

(4) $q^{-m} \sum_{v^\vee \in \Omega^\vee(F_q)} \chi_\alpha(f^\vee(v^\vee))\psi((v^\vee, v))$

$$= C_4 \cdot \chi_\alpha(f(v)^{-1})\chi_{1/2}(h^\vee(F(v))) \quad \text{for } v \in \Omega(F_q).$$

The constants $C_1$ and $C_2$ (resp. $C_3$ and $C_4$) are independent of $v^\vee \in \Omega^\vee(F_q)$ (resp. $v \in \Omega(F_q)$), but depend on the other parameters. (see [DG] for the precise formula.)

(b) The values of $w_i := \log |C_i|/\log \sqrt{q}$ are given by the following formula.

(1) $w_1 = -m - m(\alpha, C_{O_Y}[n])$.
(2) $w_2 = -m - m(\alpha, F^*L(\omega)[n])$.
(3) $w_3 = -m - m(\alpha, i_{\vee}^\vee L(\omega)[m])$.
(4) $w_4 = -m - m(\alpha, i_{\vee}^\vee C_{O_Y}[m])$. 

\(\square\)
Corollary 13. \( b^\text{exp}(t, C_n[n]) = b^\text{exp}(t, F^*L(\omega^\vee)[n]) = b^\text{exp}(t, i^\vee L(\omega^\vee)[m]) = b^\text{exp}(t, i^\vee C_OY^T[m]) \).

Proof. By [DG], we can write down explicitly the constants \( C_i \) \((1 \leq i \leq 4)\), and we get \( C_1 = C_4 \) and \( C_2 = C_3 \). Then by (1) and (4) of (12, (b)), and by (3.5, (7)), we get

\[
b^\text{exp}(t, C_n[n]) = b^\text{exp}(t, i^\vee C_OY^T[m]), \quad b^\text{exp}(t, F^*L(\omega^\vee)[n]) = b^\text{exp}(t, i^\vee L(\omega^\vee)[m]).
\]

By [Gyo3, 6.19], \((-1)^s B_f(-s - 1, 1) = B_f^\vee(s, F(f^0))\). Hence \( b^\text{exp}_f(s, C_n[n]) = b^\text{exp}_f(s, 1) = b^\text{exp}_f(s, F(f^0)) \). Hence \( b^\text{exp}_f(s, i^\vee L(\omega^\vee)[m]) \).

References


