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<td>SAITO, TAKESHI</td>
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DETERMINANT OF $\ell$-ADIC COHOMOLOGY

TAKESHI SAITO (斎藤義)

Department of Mathematical Sciences
University of Tokyo (東大 数理)

We consider the following problem.

Let $\mathcal{F}$ be a smooth $\ell$-adic sheaf on a smooth scheme $U$ over a field $k$ of characteristic $\neq \ell$. Determine the 1-dimensional $\ell$-adic representation

$$
\det R\Gamma_c(U_{\overline{k}}, \mathcal{F}) = \bigotimes_q (\wedge^{\dim H^q_c(U_{\overline{k}}, \mathcal{F})) \otimes (-1)^q}
$$

of the absolute Galois group $\text{Gal}(\overline{k}/k)$.

Under certain mild assumptions, the answers are roughly given as follows.

(1) When the sheaf $\mathcal{F}$ is constant, it is determined by the discriminant of the de Rham cohomology.

(2) In general, it is the tensor product of the following 3 contributions.

(i) That for the constant sheaf raised to its rank-th power.

(ii) The determinant $\det \mathcal{F} = \wedge^{\text{rank}} \mathcal{F}$ "evaluated" at the canonical cycle.

(iii) The Jacobi sum Hecke character determined by the ramification at the boundary.

In this report, only basic ideas will be sketched because the papers [S1], [S2] are already published. There is a Hodge-de Rham version as announced in [S-T].

1. Basic examples.

A. Fermat curve and Jacobi sum.

We consider the $\ell$-adic sheaf $\mathcal{F}$ on $U = \mathbb{P}^1 - \{0, 1, \infty\} = \{(u : v : w) \in \mathbb{P}^2 | u + v + w = 0, uvw \neq 0\}$ over a field $k$ containing a primitive $m$-th root of unity defined by the covering by a Fermat curve

$$X = \{(x : y : z) \in \mathbb{P}^2 | x^m + y^m + z^m = 0\} \rightarrow \mathbb{P}^1 = \{(u : v : w) \in \mathbb{P}^2 | u + v + w = 0\}

(x : y : z) \mapsto (x^m : y^m : z^m)

unramified on $U$. Let $a = (a, b, c) \in \text{Ker}((\mathbb{Z}/m)^3 \xrightarrow{\text{sum}} \mathbb{Z}/m)$, $a, b, c \neq 0$ be a character of $\text{Gal}(X/\mathbb{P}^1) = \mu_m^3/\text{diag}$, and let $\mathcal{F}_a$ be the corresponding smooth $\ell$-adic sheaf of rank 1 on $U$. Since $H^q_c(U_{\overline{k}}, \mathcal{F}_a) = 0$ except for $q = 1$ and is of dimension 1 for $q = 1$, the determinant $\det R\Gamma_c(U_{\overline{k}}, \mathcal{F}_a)$ is the dual of $H^1_c(U_{\overline{k}}, \mathcal{F}_a)$. It is a well-known fact that the 1-dimensional $\ell$-adic representation $H^1_c(U_{\overline{k}}, \mathcal{F}_a)$ is that defined
by the Jacobi sum Hecke character $J_a$. When $k = \mathbb{Q}(\zeta_m)$, for a finite place $p \mid m$, the algebraic Hecke character $J_a$ is defined by

$$J_a(p) = -\sum_{(u:v:w) \in V(\kappa(p))} \left(\frac{u}{p}\right)^a_m \left(\frac{v}{p}\right)^b_m \left(\frac{w}{p}\right)^c_m$$

where $(\frac{p}{m})$ denotes the $m$-th power residue symbol at $v$ and $V = \{(u : v : w) \in \mathbb{F}^2 | u + v + w = 0, uvw \neq 0\}$. The fact above is a consequence of the Grothendieck trace formula. Here we note that $a, b, c$ appearing in the definition of $J_a$ determine the restriction of the character $a$ to the inertial groups $\mu_m \subset \mu_m^3 / \text{diag}$, at the points $u = 0, v = 0, w = 0$ respectively. As a conclusion, we see the contribution (iii) of the ramification at boundary in this case.

B. Unramified case (cf. [SS]).

In case A above, we only get the contribution of the ramification. However, in a general case, we have contributions (i) and (ii) of global invariants. This is found by Shuji Saito in the case where the base field is finite.

**Theorem.** (Shuji Saito) Let $\mathcal{F}$ be a smooth $\ell$-adic sheaf on a projective smooth variety $X$ over a finite field $k$ of characteristic $\neq \ell$. Then the action of the geometric Frobenius $Fr_k \in \text{Gal}(k_{\text{sep}}/k)$ on $\det R\Gamma_c(U_{\overline{k}}, \mathcal{F})$ is given by

$$\det(Fr_k : R\Gamma_c(X_{\overline{k}}, \mathcal{F})) = \det(Fr_k : R\Gamma_c(X_{\overline{k}}, \mathbb{Q}_\ell))^{\text{rank} \mathcal{F}} \times \det \mathcal{F}(c_X).$$

Here $\det \mathcal{F}(c_X)$ denotes the value of the $\ell$-adic character of the arithmetic fundamental group $\pi_1(X)^{ab}$ corresponding to $\det \mathcal{F} = \wedge^{\text{rank} \mathcal{F}}$ evaluated at the image of the canonical class $c_X = (-1)^n c_n(\Omega_X^1) \in CH^n(X), n = \dim X$, by the reciprocity map $CH^n(X) \rightarrow \pi_1(X)^{ab}$ of the class field theory.

2. Constant coefficient.

If we assume that our $U$ admits a smooth compactification $X$ such that the complement $D = X - U$ is a divisor with simple normal crossings, then the determinant $\det R\Gamma_c(U_{\overline{k}}, \mathbb{Q}_\ell)$ is the alternating product of $\det R\Gamma(D_{x, k}, \mathbb{Q}_\ell)$ where the intersections $D_J = \bigcap_{i \in J} D_i$ of the irreducible components of $D = \bigcup_{i \in I} D_i$ are proper and smooth. In the sequel, we consider the case where $U = X$ is projective and smooth. By Poincaré duality, we see

$$\det R\Gamma(X_{\overline{k}}, \mathbb{Q}_\ell)^{\otimes 2} \simeq \mathbb{Q}_\ell(-n\chi)$$

where $n = \dim X$ and $\chi$ is the Euler characteristic of $X_{\overline{k}}$. Hence there is a character $\epsilon$ of $\text{Gal}(k_{\text{sep}}/k)$ of order 2 such that

$$\det R\Gamma(X_{\overline{k}}, \mathbb{Q}_\ell) \simeq \epsilon(-\frac{n\chi}{2}).$$

When the dimension $n$ is odd, since the cup-product on $H^n$ is a non-degenerate alternating form, the dimension of $H^n$ and hence the Euler number $\chi$ are even and $\epsilon$ is trivial. Therefore the only non-trivial problem is to determine $\epsilon$ when $n$ is even. The answer is the following.
Theorem 1. Assume char $k \neq 2$ and let $X$ be a projective smooth variety over $k$ of even dimension $n = 2m$. Then the character $\epsilon$ corresponds to the square roots of

$(-1)^{n-1} \cdot \text{disc } H^1_{dR}$

where $\chi$ is the Euler number, $b^- = \sum_{g \leq n} H^g_{dR}(X/k)$ and $\text{disc } H^1_{dR}$ is the discriminant of the cup-product of the de Rham cohomology of the middle degree.

Proof is done by taking a Lefschetz pencil and by computing the vanishing cycles by the Picard-Lefschetz formula.

3. With coefficient.

Our result gives an answer under the following rather mild assumption.

(1) The ramification of $\mathcal{F}$ along the boundary is tame. More precisely, we take a smooth compactification $X$ of $U$ such that the complement $D = X - U$ is a divisor with simple normal crossings and, at each irreducible component of $D$, the pro-$p$ Sylow subgroup of the inertia group acts trivially on the stalk of $\mathcal{F}$.

(2) There is a subring $A \subset k$ finitely generated over $\mathbb{Z}$ such that $\mathcal{F}$ is defined on a model of $U$ on $A$.

The condition (1) is satisfied if char $k = 0$ and (2) is satisfied if $\mathcal{F}$ is defined geometrically.

Under the hypothesis (2), by the Cebotarev density, the problem is reduced to the residue fields of the maximal ideals of $A$ and hence, for simplicity, we will assume $k$ is finite in the sequel.

First we describe the formula for curve. Let $U$ be a smooth curve over a finite field $k$ of order $q$ and $X$ be the smooth compactification. Let $\mathcal{F}$ be a smooth $\ell$-adic sheaf ($\ell \nmid q$) on $U$ at most tamely ramified at the boundary $D = X - U$. For simplicity, we assume that the points $x_i \in D = \{x_i\}$ are rational over $k$ and that, for each $x_i$, the representation of the inertia group $I_i$ on the stalk of $\mathcal{F}$ is the direct sum $\mathcal{F}|_{I_i} \simeq \bigoplus_j \chi_{i,j}$ of characters of the quotient $I_i \to k^\times : \sigma \mapsto \sigma(\pi_i^{-1})/\pi_i^{q-1}$ where $\pi_i$ is a uniformizer at $x_i$. In this case, the product formula [L] of Laumon gives us

**Theorem.** (Laumon) Let $U$ be a smooth curve over a finite field $k$ and $\mathcal{F}$ be a smooth $\ell$-adic sheaf on $U$ tamely ramified along the boundary satisfying the simplifying assumption above. Then

$$\det(Fr_k : R\Gamma_c(U_k, \mathcal{F})) = \det(Fr_k : R\Gamma_c(U_k, \mathbb{Q}_\ell))^{\text{rank } \mathcal{F}} \times J_{X^\times} \times \det(c_{X,D}).$$

Here $\chi_{X^\times}$ is the family of $M = \text{deg } D \times \text{rank } \mathcal{F}$ characters $(\chi_{i,j})_{x \in D, 1 \leq i \leq \text{rank } \mathcal{F}}$ of $k^\times$ and $J_{X^\times}$ denotes the Jacobi sum

$$J_{X^\times} = (-1)^M \sum_{(a_{i,j}) \in V} \prod_{i,j} \chi_{i,j}(a_{i,j})$$

where $V = \{(a_{i,j}) \in \mathbb{P}^{M-1} | \sum a_{i,j} = 0, \forall a_{i,j} \neq 0\}$. The relative canonical class $c_{X,D}$ denotes the class

$$- \sum_{x \in U} \text{deg } \omega \cdot [x] \in \bigoplus_{x \in U} \mathbb{Z}/\{a \in K^\times | a \equiv 1 \mod D\} = CH^1(X, D)$$
where $\omega$ is a rational section of $\Omega^1_X(\log D)$ satisfying $\text{ord}_x \omega = 0, \text{res}_x = 1$ for $x \in D$ and $\det F(c_{X,D})$ denotes the value of the character of $\pi_1(U)^{\text{ab,tame}}$ corresponding to the rank 1 sheaf $\det F$ evaluated at the image of $c_{X,D}$ by the reciprocity map $CH^1(X, D) \to \pi_1(U)^{\text{ab,tame}}$ of the class field theory.

Our main result in higher dimension is formally the same as in the case of curve.

**Theorem 2.** Let $X$ be a projective smooth variety over a finite field $k$ and $U$ be an open subscheme such that the complement $D = X - U$ is a divisor with simple normal crossings. Let $F$ be a smooth $\ell$-adic sheaf on $U$ tamely ramified along the boundary $D$. Then

$$\det(F_{r_k} : R\Gamma_c(U_k, F)) = \det(F_{r_k} : R\Gamma_c(U_k, Q_\ell))^{\text{rank} F} \times J_{X,F} \times \det F(c_{X,D})$$

where $J_{X,F}$ and $\det F(c_{X,D})$ are defined below.

The proof is analogous to the constant coefficient case and is done by the induction on dimension by taking a Lefschetz pencil and by computing the vanishing cycle.

In the rest of this report, I explain the idea of the definition of the terms in the right hand side of Theorem 2. The definition of the Jacobi sum is easier. Let $X$ be a smooth compactification of $U$ such that the complement $D = X - U$ is a divisor with simple normal crossings. For simplicity, we assume the constant fields of the components $D_i$ of $D$ are $k$ and the Euler numbers $\chi_i = \sum_q (-1)^q \dim H^q(D_i^*, k)$ of $D_i^* = D_i \setminus \bigcup_{j \neq i} D_j$ are bigger or equal to 0. We also assume for simplicity that for each irreducible component $D_i$, the representation of the inertia group $I_i$ on the stalk of $F$ is the direct sum $\mathcal{F}|_{I_i} \simeq \bigoplus \chi_{i,j}$ of characters of the quotient $I_i \to k^\times : \sigma \mapsto \sigma^{(i-1)/i^{i-1}}$ where $\pi_i$ is a uniformizer of the divisor $D_i$. Under the above simplifying assumption, we define the Jacobi sum $J_{X,F}$ by

$$J_{X,F} = (-1)^M \sum_{(a_{i,j,k}) \in V(k)} \prod_{i,j,k} \chi_{i,j}(a_{i,j,k})$$

where $i$ runs the indices of the irreducible components of $D$, $1 \leq j \leq \text{rank} F$, $1 \leq k \leq \chi_i$, $M = \text{rank} F \times \sum_i \chi_i$ and $V = \{(a_{i,j,k}) \in F^{M-1} | \sum a_{i,j,k} = 0, \prod a_{i,j,k} \neq 0\}$.

Finally I explain the idea of the definition of the relative canonical class $c_{X,D}$ in higher dimension. Note that in the case of curve, the residue $\text{res}_x : \Omega^1_X(\log D) \otimes \kappa(x) \to \kappa(x)$ at $x \in D$ defines a trivialization of the invertible sheaf $\Omega^1_X(\log D)$ at $x$. In general case, for each irreducible component $D_i$ of the complement $D = X - U$, the residue $\text{res}_i : \Omega^1_X(\log D) \otimes \mathcal{O}_{D_i} \to \mathcal{O}_{D_i}$ defines a partial trivialization of the locally free sheaf $\Omega^1_X(\log D)$ of rank $n$. This family of partial trivializations enables us to define a refined chern class $c_n(\Omega^1_X(\log D), \text{res})$ as follows. Let's briefly recall a definition of the top chern class $c_n(\mathcal{E})$ of a locally free sheaf $\mathcal{E}$ of rank $n$ on a smooth variety $X$. It is the image of 1 by the composition map

$$Z \simeq H^0(V, \mathcal{K}_n) \to H^n(V, \mathcal{K}_n) \simeq H^n(X, \mathcal{K}_n) = CH_n(X)$$

where $V$ denotes the vector bundle associated to $\mathcal{E}$, $\mathcal{K}_n$ is the Zariski sheaf associated to Quillen's K-group and the last equality is a consequence of the Gersten resolution.
To define the refined chern class \( c_n(\Omega^1_X(\log D), \text{res}) \), let \( V \) be the vector bundle associated to \( \Omega^1_X(\log D) \) and we consider complexes

\[
\mathcal{K}_{n,X,D} = [\mathcal{K}_{n,X} \rightarrow \bigoplus_i \mathcal{K}_{n,D_i}], \quad \mathcal{K}_{n,V,\Delta} = [\mathcal{K}_{n,V} \rightarrow \bigoplus_i \mathcal{K}_{n,\Delta_i}]
\]

where \( \Delta_i \subset V_{D_i} \) is the inverse image of the 1-section by \( \text{res}_i : V_{D_i} \rightarrow \mathbb{A}_{D_i}^1 \). We define the class as the image of 1 by the composition map

\[
Z \simeq H^n_{\{0\}}(V, \mathcal{K}_{n,V,\Delta}) \to H^n(V, \mathcal{K}_{n,V,\Delta}) \simeq H^n(X, \mathcal{K}_{n,X,D}) = CH^n(X, D).
\]

Here the first isomorphism is by the fact \( \Delta \cap \{0\} = \emptyset \), the second isomorphism is by the homotopy property of \( K \)-cohomology and the equality is the definition. Thus \( c_{X,D} = (-1)^n c_n(\Omega^1_X(\log D), \text{res}) \in CH^n(X, D) \) is defined. By the reciprocity map \( CH^n(X, D) \to \pi_1(U)^{\text{ab}, \text{tame}} \), the value \( \det \mathcal{F}(c_{X,D}) \) of the character of \( \pi_1(U)^{\text{ab}, \text{tame}} \) corresponding to \( \det \mathcal{F} \) evaluated at the image of \( c_{X,D} \) is defined. This is the idea of the definition.

More detail will be found in [S1], [S2].

References


More references will be found in the lists in [S1], [S2].