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Kyoto University
DETERMINANT OF \(\ell\)-ADIC COHOMOLOGY

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We consider the following problem.

Let \(\mathcal{F}\) be a smooth \(\ell\)-adic sheaf on a smooth scheme \(U\) over a field \(k\) of characteristic \(\neq \ell\). Determine the 1-dimensional \(\ell\)-adic representation

\[
\det R\Gamma_c(U_{\overline{k}}, \mathcal{F}) = \bigotimes_q (\Lambda^\text{dim} H^q_c(U_{\overline{k}}, \mathcal{F})) \otimes (-1)^q
\]

of the absolute Galois group \(\text{Gal}(\overline{k}/k)\).

Under certain mild assumptions, the answers are roughly given as follows.

1. When the sheaf \(\mathcal{F}\) is constant, it is determined by the discriminant of the de Rham cohomology.
2. In general, it is the tensor product of the following 3 contributions.
   1. That for the constant sheaf raised to its rank-th power.
   2. The determinant \(\det \mathcal{F} = \Lambda^{\text{rank}}\mathcal{F}\) "evaluated" at the canonical cycle.
   3. The Jacobi sum Hecke character determined by the ramification at the boundary.

In this report, only basic ideas will be sketched because the papers [S1], [S2] are already published. There is a Hodge-de Rham version as announced in [S-T].

1. Basic examples.

   A. Fermat curve and Jacobi sum.

   We consider the \(\ell\)-adic sheaf \(\mathcal{F}\) on \(U = \mathbb{P}^1 - \{0, 1, \infty\} = \{(u : v : w) \in \mathbb{P}^2 | u + v + w = 0, uvw \neq 0\}\) over a field \(k\) containing a primitive \(m\)-th root of unity defined by the covering by a Fermat curve

   \[
   X = \{(x : y : z) \in \mathbb{P}^2 | x^m + y^m + z^m = 0\} \rightarrow \mathbb{P}^1 = \{(u : v : w) \in \mathbb{P}^2 | u + v + w = 0\}
   \]

   \[
   (x : y : z) \mapsto (x^m : y^m : z^m)
   \]

   unramified on \(U\). Let \(\mathfrak{a} = (a, b, c) \in \text{Ker}((\mathbb{Z}/m)^3 \xrightarrow{\text{sum}} \mathbb{Z}/m), a, b, c \neq 0\) be a character of \(\text{Gal}(X/\mathbb{P}^1) = \mu_m^3/\text{diag.}\) and let \(\mathcal{F}_\mathfrak{a}\) be the corresponding smooth \(\ell\)-adic sheaf of rank 1 on \(U\). Since \(H^q_c(U_{\overline{k}}, \mathcal{F}_\mathfrak{a}) = 0\) except for \(q = 1\) and is of dimension 1 for \(q = 1\), the determinant \(\det R\Gamma_c(U_{\overline{k}}, \mathcal{F}_\mathfrak{a})\) is the dual of \(H^1_c(U_{\overline{k}}, \mathcal{F}_\mathfrak{a})\). It is a well-known fact that the 1-dimensional \(\ell\)-adic representation \(H^1_c(U_{\overline{k}}, \mathcal{F}_\mathfrak{a})\) is that defined

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by the Jacobi sum Hecke character \( J_{a} \). When \( k = \mathbb{Q}(\zeta_{m}) \), for a finite place \( p \mid m \), the algebraic Hecke character \( J_{a} \) is defined by

\[
J_{a}(p) = - \sum_{(u:v:w) \in V(\kappa(p))} \left( \frac{u}{p} \right)^{a} \left( \frac{v}{p} \right)^{b} \left( \frac{w}{p} \right)^{c}
\]

where \( \left( \frac{p}{m} \right) \) denotes the \( m \)-th power residue symbol at \( v \) and \( V = \{(u : v : w) \in \mathbb{P}^{2} | u + v + w = 0, uvw \neq 0 \} \). The fact above is a consequence of the Grothendieck trace formula. Here we note that \( a, b, c \) appearing in the definition of \( J_{a} \) determine the restriction of the character \( a \) to the inertia groups \( \mu_{m} \subset \mu_{m}^{3}/\text{diag} \), at the points \( u = 0, v = 0, w = 0 \) respectively. As a conclusion, we see the contribution (iii) of the ramification at boundary in this case.

B. Unramified case (cf. [SS]).

In case A above, we only get the contribution of the ramification. However, in a general case, we have contributions (i) and (ii) of global invariants. This is found by Shuji Saito in the case where the base field is finite.

**Theorem.** (Shuji Saito) Let \( \mathcal{F} \) be a smooth \( \ell \)-adic sheaf on a projective smooth variety \( X \) over a finite field \( k \) of characteristic \( \neq \ell \). Then the action of the geometric Frobenius \( Fr_{k} \in \text{Gal}(k_{\text{sep}}/k) \) on \( \det R\Gamma(X_{\overline{k}}, \mathcal{F}) \) is given by

\[
\det(Fr_{k} : R\Gamma_{c}(X_{\overline{k}}, \mathcal{F})) = \det(Fr_{k} : R\Gamma_{c}(X_{\overline{k}}, \mathbb{Q}_{\ell}))^{\text{rank } \mathcal{F}} \times \det(\mathcal{F}(c_{X})).
\]

Here \( \det(\mathcal{F}(c_{X})) \) denotes the value of the \( \ell \)-adic character of the arithmetic fundamental group \( \pi_{1}(X)^{\text{ab}} \) corresponding to \( \mathcal{F} = \wedge^{\text{rank } \mathcal{F}} \mathcal{F} \) evaluated at the image of the canonical class \( c_{X} = (-1)^{n} c_{n}(\Omega_{X}^{1}) \in CH^{n}(X), n = \dim X \), by the reciprocity map \( CH^{n}(X) \rightarrow \pi_{1}(X)^{\text{ab}} \) of the class field theory.

2. Constant coefficient.

If we assume that our \( U \) admits a smooth compactification \( X \) such that the complement \( D = X - U \) is a divisor with simple normal crossings, then the determinant \( \det R\Gamma(U_{\overline{k}}, \mathbb{Q}_{\ell}) \) is the alternating product of \( \det R\Gamma(D_{J, \overline{k}}, \mathbb{Q}_{\ell}) \) where the intersections \( D_{J} = \bigcap_{i \in J} D_{i} \) of the irreducible components of \( D = \bigcup_{i \in I} D_{i} \) are proper and smooth. In the sequel, we consider the case where \( U = X \) is projective and smooth. By Poincaré duality, we see

\[
\det R\Gamma(X_{\overline{k}}, \mathbb{Q}_{\ell})^{\otimes 2} \simeq \mathbb{Q}_{\ell}(-n\chi)
\]

where \( n = \dim X \) and \( \chi \) is the Euler characteristic of \( X_{\overline{k}} \). Hence there is a character \( \epsilon \) of \( \text{Gal}(k_{\text{sep}}/k) \) of order 2 such that

\[
\det R\Gamma(X_{\overline{k}}, \mathbb{Q}_{\ell}) \simeq \epsilon(-\frac{n\chi}{2}).
\]

When the dimension \( n \) is odd, since the cup-product on \( H^{n} \) is a non-degenerate alternating form, the dimension of \( H^{n} \) and hence the Euler number \( \chi \) are even and \( \epsilon \) is trivial. Therefore the only non-trivial problem is to determine \( \epsilon \) when \( n \) is even. The answer is the following.
Theorem 1. Assume char $k \neq 2$ and let $X$ be a projective smooth variety over $k$ of even dimension $n = 2m$. Then the character $\epsilon$ corresponds to the square roots of

$$(-1)^{m}x^{b^{-}} \cdot \text{disc } H_{dR}^{n}$$

where $\chi$ is the Euler number, $b^{-} = \sum_{q<n} H_{dR}^{n}(X/k)$ and $\text{disc } H_{dR}^{n}$ is the discriminant of the cup-product of the de Rham cohomology of the middle degree.

Proof is done by taking a Lefschetz pencil and by computing the vanishing cycles by the Picard-Lefschetz formula.

3. With coefficient.

Our result gives an answer under the following rather mild assumption.

(1) The ramification of $\mathcal{F}$ along the boundary is tame. More precisely, we take a smooth compactification $X$ of $U$ such that the complement $D = X - U$ is a divisor with simple normal crossings and, at each irreducible component of $D$, the pro-$p$ Sylow subgroup of the inertia group acts trivially on the stalk of $\mathcal{F}$.

(2) There is a subring $A \subset k$ finitely generated over $\mathbb{Z}$ such that $\mathcal{F}$ is defined on a model of $U$ on $A$.

The condition (1) is satisfied if char $k = 0$ and (2) is satisfied if $\mathcal{F}$ is defined geometrically.

Under the hypothesis (2), by the Cebotarev density, the problem is reduced to the residue fields of the maximal ideals of $A$ and hence, for simplicity, we will assume $k$ is finite in the sequel.

First we describe the formula for curve. Let $U$ be a smooth curve over a finite field $k$ of order $q$ and $X$ be the smooth compactification. Let $\mathcal{F}$ be a smooth $\ell$-adic sheaf $(\ell \nmid q)$ on $U$ at most tamely ramified at the boundary $D = X - U$. For simplicity, we assume that the points $x_i \in D = \{x_i\}$ are rational over $k$ and that, for each $x_i$, the representation of the inertia group $I_i$ on the stalk of $\mathcal{F}$ is the direct sum $\mathcal{F}|_{I_i} \simeq \bigoplus_j \chi_{i,j}$ of characters of the quotient $I_i \to k^\times : \sigma \mapsto \sigma(\pi_i^{-1})/\pi_i^{-1}$ where $\pi_i$ is a uniformizer at $x_i$. In this case, the product formula [L] of Laumon gives us

Theorem. (Laumon) Let $U$ be a smooth curve over a finite field $k$ and $\mathcal{F}$ be a smooth $\ell$-adic sheaf on $U$ tamely ramified along the boundary satisfying the simplifying assumption above. Then

$$\det(Fr_k : R\Gamma_c(U_k, \mathcal{F})) = \det(Fr_k : R\Gamma_c(U_k, \mathbb{Q}_\ell))^{\text{rank } \mathcal{F}} \times J_{X, k} \times \det(\mathcal{F}(c_X, D)).$$

Here $\chi_{X, k}$ is the family of $M = \text{deg } D \times \text{rank } \mathcal{F}$ characters $(\chi_{i,j})_{x \in D, 1 \leq i \leq \text{rank } \mathcal{F}}$ of $k^\times$ and $J_{X, k}$ denotes the Jacobi sum

$$J_{X, k} = (-1)^M \sum_{(a_{i,j}) \in V(k)} \prod \chi_{i,j}(a_{i,j})$$

where $V = \{\{a_{i,j}\} \in \mathbb{P}^{M-1} | \sum a_{i,j} = 0, \prod a_{i,j} \neq 0\}$. The relative canonical class $c_X, D$ denotes the class

$$- \sum_{x \in U} \deg_x \omega \cdot [x] \in \bigoplus_{x \in U} \mathbb{Z}/\{a \in K^\times | a \equiv 1 \mod D\} = CH_1(X, D)$$
where $\omega$ is a rational section of $\Omega^1_X(\log D)$ satisfying $\text{ord}_x \omega = 0$, $\text{res}_x = 1$ for $x \in D$ and $\det F(c_{X,D})$ denotes the value of the character of $\pi_1(U)^{ab,tame}$ corresponding to the rank 1 sheaf $\det F$ evaluated at the image of $c_{X,D}$ by the reciprocity map $CH^1(X, D) \to \pi_1(U)^{ab,tame}$ of the class field theory.

Our main result in higher dimension is formally the same as in the case of curve.

**Theorem 2.** Let $X$ be a projective smooth variety over a finite field $k$ and $U$ be an open subscheme such that the complement $D = X - U$ is a divisor with simple normal crossings. Let $F$ be a smooth $\ell$-adic sheaf on $U$ tamely ramified along the boundary $D$. Then

$$\det(Fr_k : R\Gamma_c(U_k, F)) = \det(Fr_k : R\Gamma_c(U_k, \mathbb{Q}_\ell))^{\text{rank} F} \times J_{\chi_F} \times \det F(c_{X,D})$$

where $J_{\chi_F}$ and $\det F(c_{X,D})$ are defined below.

The proof is analogous to the constant coefficient case and is done by the induction on dimension by taking a Lefschetz pencil and by computing the vanishing cycle.

In the rest of this report, I explain the idea of the definition of the terms in the right hand side of Theorem 2. The definition of the Jacobi sum is easier. Let $X$ be a smooth compactification of $U$ such that the complement $D = X - U$ is a divisor with simple normal crossings. For simplicity, we assume the constant fields of the components $D_i$ of $D$ are $k$ and the Euler numbers $\chi_i = \sum_q (-1)^q \dim H^q(D_i^*, k)$ of $D_i^* = D_i - \bigcup_{j \neq i} D_j$ are bigger or equal to 0. We also assume for simplicity that for each irreducible component $D_i$, the representation of the inertia group $I_i$ on the stalk of $F$ is the direct sum $F|_{I_i} \simeq \bigoplus \chi_{i}$ of characters of the quotient $I_i \to k^\times : \sigma \mapsto \sigma(\pi_i^{1-q}/\pi_i^{q-1})$ where $\pi_i$ is a uniformizer of the divisor $D_i$. Under the above simplifying assumption, we define the Jacobi sum $J_{\chi_F}$ by

$$J_{\chi_F} = (-1)^M \sum_{(a_{i,j,k}) \in V(k)} \prod \chi_{i,j}(a_{i,j,k})$$

where $i$ runs the indices of the irreducible components of $D$, $1 \leq j \leq \text{rank} F$, $1 \leq k \leq \chi_i$, $M = \text{rank} F \times \sum \chi_i$ and $V = \{(a_{i,j,k}) \in \mathbb{P}^{M-1} \mid \sum a_{i,j,k} = 0, \prod a_{i,j,k} \neq 0\}$.

Finally I explain the idea of the definition of the relative canonical class $c_{X,D}$ in higher dimension. Note that in the case of curve, the residue $\text{res}_x : \Omega^1_X(\log D) \otimes \kappa(x) \to \kappa(x)$ at $x \in D$ defines a trivialization of the invertible sheaf $\Omega^1_X(\log D)$ at $x$. In general case, for each irreducible component $D_i$ of the complement $D = X - U$, the residue $\text{res}_i : \Omega^1_X(\log D) \otimes \mathcal{O}_{D_i} \to \mathcal{O}_{D_i}$ defines a partial trivialization of the locally free sheaf $\Omega^1_X(\log D)$ of rank $n$. This family of partial trivializations enables us to define a refined Chern class $c_n(\Omega^1_X(\log D), \text{res})$ as follows. Let’s briefly recall a definition of the top Chern class $c_n(\mathcal{E})$ of a locally free sheaf $\mathcal{E}$ of rank $n$ on a smooth variety $X$. It is the image of 1 by the composition map

$$Z \simeq H^n_{(0)}(V, \mathcal{K}_n) \to H^n(V, \mathcal{K}_n) \simeq H^n(X, \mathcal{K}_n) = CH_n(X)$$

where $V$ denotes the vector bundle associated to $\mathcal{E}$, $\mathcal{K}_n$ is the Zariski sheaf associated to Quillen’s $K$-group and the last equality is a consequence of the Gersten resolution.
To define the refined chern class $c_n(\Omega^1_X(\log D),\text{res})$, let $V$ be the vector bundle associated to $\Omega^1_X(\log D)$ and we consider complexes

$$\mathcal{K}_{n,X,D} = [\mathcal{K}_{n,X} \to \bigoplus_i \mathcal{K}_{n,D_i}], \mathcal{K}_{n,V,\Delta} = [\mathcal{K}_{n,V} \to \bigoplus_i \mathcal{K}_{n,\Delta_i}]$$

where $\Delta_i \subset V_{D_i}$ is the inverse image of the 1-section by $\text{res}_i : V_{D_i} \to A^1_{D_i}$. We define the class as the image of 1 by the composition map

$$Z \simeq H^n_{\text{et}}(V,\mathcal{K}_{n,V,\Delta}) \to H^n(V,\mathcal{K}_{n,V,\Delta}) \simeq H^n(X,\mathcal{K}_{n,X,D}) = CH^n(X, D).$$

Here the first isomorphism is by the fact $\Delta \cap \{0\} = \emptyset$, the second isomorphism is by the homotopy property of $K$-cohomology and the equality is the definition. Thus $c_{X,D} = (-1)^nc_n(\Omega^1_X(\log D),\text{res}) \in CH^n(X, D)$ is defined. By the reciprocity map $CH^n(X, D) \to \pi_1(U)^{ab,\text{tame}}$, the value $\det \mathcal{F}(c_{X,D})$ of the character of $\pi_1(U)^{ab,\text{tame}}$ corresponding to $\det \mathcal{F}$ evaluated at the image of $c_{X,D}$ is defined. This is the idea of the definition.

More detail will be found in [S1], [S2].

**References**


More references will be found in the lists in [S1], [S2].