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Convergence of formal solutions of fully nonlinear equations of Monge-Ampère type

MASAFUMI YOSHINO (吉野 正史)

Department of Mathematics, Nice UniveriSty
(Faculty of Economics, Chuo University)

Abstract. We present a sufficient condition for the convergence of all formal power series solutions of nonlinear equations whose linear part does not necessarily satisfy a so-called Poincaré condition. The condition is expressed in terms of a Riemann-Hilbert factorization condition. As an application, we shall show the solvability of Monge-Ampère equations in case it changes its type.

0. Introduction

In 1974, Kashiwara-Kawai-Sjöstrand showed the convergence of all formal power series solutions of the following linear partial differential equations of regular singular type

$$P \equiv \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)x^{\alpha}(\partial/\partial x)^{\beta} = f(x),$$

where m is an integer and $a_{\alpha\beta}(x)$ and $f(x)$ are analytic in some neighborhood of the origin of $x = (x_1, \dots, x_n) \in \mathbf{C}^n$. They gave a sufficient condition for the convergence of all formal power series solutions, a certain ellipticity condition of the equation. (cf. (0.2) in [5]). Their condition contains, as a special case, a so-called Poincaré condition. Concerning this, in the preceeding papers [7] and [8], we gave a necessary and sufficient condition for the convergence of all formal solutions of linear irregular singular type equations with two independent variables.

In the case of nonlinear equations there are corresponding counterparts under the Poincaré condition. (cf. [4]). Nevertheless, little results are known without a Poincaré condition. In the actual applications one often encounters equations without a Poincaré condition. For example, let us consider Monge-Ampère equations $M(u) = f$. Let u_0 and f_0 satisfy that $M(u_0) = f_0$. We want to find a solution of the equation $M(u) = f_0+g$ for analytic g with order greater than that of f_0. Here the order of a formal power series is defined as the smallest degree of its constituent monomials, that is, the smallest integer k such that $\partial_x^{\alpha} f_0(0) \neq 0$ for some $|\alpha| = k$ and $\partial_x^{\beta} f_0(0) = 0$ for all $|\beta| \leq k-1$. A

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curvature function f_0 may vanish or change its sign near the origin. It follows that the equation may be either degenerate elliptic or degenerate hyperbolic or even mixed type at $u = u_0$. We note that the Poincaré condition is not satisfied in general. (cf. Example 1.5 which follows.)

In view of this, we shall consider a certain class of nonlinear equations including Monge-Ampère equations without assuming a Poincaré condition. As we have seen in the above example, the type of the equations may change at the origin. For such operators, even a Fredholm solvability in a formal power series are open questions in general. Moreover, the inverse of such a operator, if exists, often has loss of derivatives. If this happens for linearized operators of our nonlinear equations, it causes the divergence of formal solutions.

In order to cope with these problems, we employ the method of Riemann-Hilbert factorization and a rapidly convergent iteration method. The former method enables us to obtain a sufficiently good estimate for the linearized operators which are either degenerate elliptic, degenerate hyperbolic or mixed type. The latter one ensures us to show the convergence of formal solutions in case there are loss of derivatives in the linearized equations.

1. Notations and results

1.1. Statement of results. Let $x = (x_1, x_2) \in \mathbf{R}^2$ and let M be a nonlinear operator of Monge-Ampère type

$$M(u) := \sum_{\nu,\mu=0}^{m} \sum_{|\alpha|=\nu,|\beta|=\mu} a_{\alpha\beta}(\partial^\alpha u)(\partial^\beta u) + \sum_{|\alpha|=|\beta|\le m} A_{\alpha\beta}(x)\partial^\alpha(x^\beta u), \tag{1.1}$$

where $a_{\alpha\beta} \in \mathbf{C}$, $m \ge 1$ is an integer, $A_{\alpha\beta}(x)$ is analytic at the origin, and where $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}$, $\alpha = (\alpha_1, \alpha_2)$ and so on.

Let $u_0(x)$ be a polynomial such that $\partial^\alpha u_0(0) = 0$ for all multiindices $\alpha, |\alpha| < 2m$. We set $f_0(x) = M(u_0)$. For g analytic at the origin such that $\partial^\alpha g(0) = 0$ for all $\alpha, |\alpha| \le 2m$ we are interested in the convergence of all formal power series solutions of the equation

$$M(u_0 + w) = f_0(x) + g(x). \tag{1.2}$$

Let M_{u_0} be the linearized operator of M at $u = u_0$;
(1.3)

$$M_{u_0}v := \sum_{\nu,\mu=0}^{m} \sum_{|\alpha|=\nu,|\beta|=\mu} a_{\alpha\beta}\left((\partial^\alpha u_0)(\partial^\beta v) + (\partial^\alpha v)(\partial^\beta u_0)\right) + \sum_{|\alpha|=|\beta|\le m} A_{\alpha\beta}(x)\partial^\alpha(x^\beta u).$$

Let $\tilde{u}_0$ be the homogeneous part of degree $2m$ of u_0. Let us define the operator P by

$$P \equiv P(x,\partial) := \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\left(\partial^\alpha \tilde{u}_0 \partial^\beta + \partial^\beta \tilde{u}_0 \partial^\alpha\right) + \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(0)\partial^\alpha(x^\beta u). \tag{1.4}$$

We denote by $p_m(x, \xi)$ the principal symbol of P, where $\xi = (\xi_1, \xi_2)$ is the covariable of x.

We define the two dimensional torus $\mathbf{T}^2$ by $\mathbf{T}^2 := \{z = (z_1, z_2) \in \mathbf{C}^2; |z_1| = 1, |z_2| = 1\}$ and we set $\mathbf{R}^2_+ = \{(\eta_1, \eta_2) \in \mathbf{R}^2; \eta_1 \geq 0, \eta_2 \geq 0\}$. Then the Toeplitz symbol $\sigma_{u_0}(z, \eta)$, $(z \in \mathbf{T}^2,\ \eta \in \mathbf{R}^2_+)$ at u_0 is defined by

$$\sigma_{u_0}(z, \eta) = p_m(z_1\eta_1, z_2\eta_2; z_1^{-1}, z_2^{-1}), \tag{1.5}$$

namely, we set $x = (z_1\eta_1, z_2\eta_2)$ and $\xi = (z_1^{-1}, z_2^{-1})$ in $p_m(x, \xi)$.

We assume the following conditions

$$\sigma_{u_0}(z, \eta) \neq 0 \quad \text{for all } z \in \mathbf{T}^2,\ \eta \in \mathbf{R}^2_+. \tag{A.1}$$

$$\mathrm{ind}_1\sigma_{u_0} = \mathrm{ind}_2\sigma_{u_0} = 0. \tag{A.2}$$

Here $\mathrm{ind}_1\,\sigma_{u_0}$ (resp. $\mathrm{ind}_2\,\sigma_{u_0}$) is defined by

$$\mathrm{ind}_1\sigma_{u_0} = \frac{1}{2\pi i}\oint_{|\zeta_1|=1} d_{z_1} \log \sigma_{u_0}(\zeta_1, z_2, \xi). \tag{1.6}$$

Remark. The conditions (A.1) and (A.2) are called as the Riemann-Hilbert factorization condition for $\sigma_{u_0}(z, \eta)$.

Theorem 1.1. *Suppose (A.1) and (A.2). Then there exist an integer $N \geq 2m$ depending only on u_0 such that, for any analytic g such that $\partial_x^\alpha g(0) = 0$ for all $|\alpha| < N$ the equation* (1.2) *has a unique analytic solution w in some neighborhood of the origin such that $\partial_x^\alpha w(0) = 0$ for all $|\alpha| < N$.*

The following theorem gives a generalization of K-K-S in [5] to Monge-Ampère type equations

Corollary 1.2. *Suppose (A.1) and (A.2). Then, for every g being analytic at the origin such that $\partial_x^\alpha g(0) = 0$ for all $|\alpha| < 2m$ all formal solutions of the equation* (1.2) *converge in some neighborhood of the origin.*

Remark. We note that P in (1.4) maps every set of homogeneous polynomials to itself. If P is injective on every set of homogeneous polynomials degree $k \geq N$, the integer k_0 in Theorem 1.1 satisfies $k_0 \leq N$.

Example 1.3. In the following, we shall give three examples which illustrate our results in case M is a Monge-Ampère equation. For the sake of simplicity, we write $x_1 = x$, $x_2 = y$. Let us consider the equation

$$M(u) := u_{xx}u_{yy} - u_{xy}^2 + c(x, y)u_{xy} = f(x, y), \tag{1.7}$$

where $c(x, y)$ and $f(x, y)$ are given functions and we abbreviate $u_{xx} = \partial_x^2 u$, $u_{yy} = \partial_y^2 u$, and so on. In what follows we assume that $u_0(x, y)$ is a homogeneous polynomial of degree 4 and $c(x, y)$ is a homogeneous polynomial of degree 2. We define $f_0(x, y)$ by $f_0(x, y) = M(u_0)$.

For g analytic in some neighborhood of the origin such that $\partial^\alpha g(0,0)=0$ for any $|\alpha|\le 4$ we are interested in the solvability of the problem

$$M(u_0+w)=f_0(x,y)+g(x,y). \tag{1.8}$$

Let $P=M_{u_0}$ be the linearized operator of M at $u=u_0$.

$$P:=(u_0)_{yy}\partial_x^2+(u_0)_{xx}\partial_y^2+(c(x,y)-2(u_0)_{xy})\partial_x\partial_y. \tag{1.9}$$

We first find a formal power series solution u of (1.8) in the form $u=u_0+v$, $v=\sum_{j=5}^\infty v_j$, $k=deg\, u_0$, where v_j are homogeneous polynomials of degree j. For simplicity, let us assume that $c(x)\equiv 0$. Note that f_0 is homogeneous of degree 4 and the Taylor expansion of g has powers greater than 5. Let $g=\sum_{j=5}^\infty g_j(x,y)$ be the expansion of g with g_j being homogeneous polynomial of degree j. By substituting the expansions of u and g into (1.8) we see, from the condition $f_0=M(u_0)$ that the homogeneous part of degree 4 vanishes. Hence, by comparing the homogeneous part of degree 5 we have the relation

$$Pv_5=g_5. \tag{1.10}$$

Similarly, by comparing the terms of homogeneous degree ν ($\nu\ge 5$) we have

$$Pv_\nu=g_\nu+\sum_{i+j=\nu+4,i>4,j>4}((v_i)_{xx}(v_j)_{yy}+(v_i)_{xy}(v_j)_{xy}). \tag{1.11}$$

We note that in the second term of the right-hand side of (1.11) there appears no v_ν. Hence we can determine v_ν formally if P is surjective.

If we denote the set of homogeneous polynomials of degree n by S_n, P maps S_n into S_n. Hence the surjectivity of P on S_n follows from the injectivity of P on S_n. The integer k_0 in Theorem 1.1 could be the smallest integer m such that $P:S_n\to S_n$ is injective for $n\ge m$. The convergence of all formal solutions and the solvability follows if we assume (A.1) and (A.2). For more detail, we consider three cases.

Example 1.4. Let $u_0(x,y)=x^2y^2$, $c(x,y)=kxy$ and $f_0(x,y)=4(k-3)x^2y^2$ in (1.7), where k is a real constant. Then, the linearized operator P, (1.9) is given by

$$P=2x^2\partial_x^2+2y^2\partial_y^2+(k-8)xy\partial_x\partial_y.$$

The Toeplitz symbol is given by

$$\sigma_{u_0}(z,\eta)=2(\eta_1^2+\eta_2^2)+(k-8)\eta_1\eta_2, \tag{1.12}$$

and the condition (A.1) is equivalent to

$$2+(k-8)\eta_1\eta_2\neq 0 \qquad \forall\eta\in\mathbf{R}_+^2,\ |\eta|=1. \tag{1.13}$$

Because $0\le\eta_1\eta_2\le 1/2$, it follows that (A.1) is equivalent to $k>4$. It is not difficult to see that (A.2) is automatically satisfied under this condition. Clearly, P satisfies a Poincaré condition if (A.1) and (A.2) are satisfied.

Now we study the type of our Monge-Ampère operator at $u = u_0$. By simple computations of the characteristic polynomials, we see that the Monge-Ampère operator at $u = u_0$ is an elliptic operator outside the set $xy = 0$ if and only if $4 < k < 12$. Note that if $k > 12$ the equation is degenerate hyperbolic. In any case, the degeneracy occurs on the line $x = 0$ or $y = 0$. Hence the type does not change at the origin. We will see in the following example that the type may change at the origin in general if Poincaré condition is not satisfied.

Next we want to estimate the integer k_0 in Theorem 1.1. For this purpose, we study the injectivity P on the sets of homogeneous polynomials when $k > 4$. Because P preserves homogeneous polynomials we may consider P on the set of homogeneous polynomials of degree greater than 5. By definition a monomial $x^\nu y^\mu$ $(\nu + \mu \geq 5)$ is in the kernel of P if and only if

$$2\nu(\nu-1) + 2\mu(\mu-1) + (k-8)\nu\mu \neq 0. \tag{1.14}$$

Since $k > 4$, we easily see that (1.14) is equivalent to that $k \neq 16/3, 9/2$ if $\nu = 2$ or $\mu = 2$. If $\nu = \mu = 3$ it follows from (1.14) that $k \neq 16/3$. Similarly, if $\nu = 3, \mu = 4$ or $\nu = 4, \mu = 3$ we get $k \neq 5$. More generally, if $\nu + \mu = n$ $(n \geq 5)$ the condition (1.14) is equivalent to $k \neq 8 - 2(n^2 - n)/(\nu\mu) \leq 4 + 8/n$. The equality holds when $\nu = \mu = n/2$.

Therefore, the injectivity of P on S_n holds in each of the following cases: a) $k > 16/3$, $n \geq 5$, b) $k > 5$, $n \geq 7$, c) $k > 4 + 8/n$, $n \geq 8$.

Example 1.5. We consider the case $u_0(x,y) = x^4 + kx^2y^2 + y^4$ and $c(x,y) \equiv 0$, where k is a real number. $f_0 = M(u_0)$ is given by $f_0(x,y) = 12(2kx^4 + 2ky^4 + (12 - k^2)x^2y^2)$.

The linearized operator P, (1.9) is given by

$$P = 12y^2\partial_x^2 + 12x^2\partial_y^2 + 2k(y^2\partial_y^2 + x^2\partial_x^2) - 8xy\partial_x\partial_y. \tag{1.15}$$

The Toeplitz symbol is given by

$$\sigma_P(z,\eta) = 2k(\eta_1^2 + \eta_2^2) - 8\eta_1\eta_2 + 12(z_1^2 z_2^{-2}\eta_1^2 + z_2^2 z_1^{-2}\eta_2^2). \tag{1.16}$$

Clearly, Poincarćondition is not satisfied in this case. The (A.1) is equivalent to

$$k - 4\eta_1\eta_2 + 6(\eta_1^2 t^2 + \eta_2^2 t^{-2}) \neq 0 \qquad \forall t \in \mathbf{C},\ |t| = 1\ \forall \eta \in \mathbf{R}_+^2,\ |\eta| = 1. \tag{1.17}$$

Suppose that $\eta_1 = \eta_2$. It follows from $|\eta| = 1$ that $\eta_1 = \eta_2 = 1/\sqrt{2}$. Hence (1.17) implies that $k \notin [-4, 8]$. On the other hand, if $\eta_1 \neq \eta_2$ we have that $2i\,Im\,(\eta_1^2 t^2 + \eta_2^2 t^{-2}) = (\eta_1^2 - \eta_2^2)(t^2 - t^{-2})$. This quantity vanishes only if $t^2 = \pm 1$. Because k is real (1.17) is verified if $t^2 \neq \pm 1$. On the other hand, if $t^2 = \pm 1$, (1.16) is written in $k \neq 4\eta_1\eta_2 \pm 6$, which is equivalent to $k \notin [-6, -4]$ and $k \notin [6, 8]$ since $0 \leq \eta_1\eta_2 \leq 1/2$. Therefore condition (A.1) is equivalent to $k < -6$ or $k > 8$.

In order to see (A.2) we take $\eta_1 = 0$, $\eta_2 = 1$. Then it follows that $\sigma_P(z,\eta) = k + 6t^{-2}$. Hence (A.2) is equivalent to $k > 6$ or $k < -6$. Therefore, the conditions (A.1) and (A.2) are equivalent to $k < -6$ or $k > 8$. If the condition holds and if the degree of the perturbation $g(x,y)$ is sufficiently large, then (1.8) has a solution by Theorem 1.1.

Let us see how the type of our Monge-Ampère equation changes in this case. It is easy to see that if $k > 8$ the zero set of f_0, $f_0(x, y) = 0$ consists of four lines in $\mathbf{R}^2$ which intersect at the origin. It follows that f_0 changes its sign if one crosses one of these lines. This implies that the equation is mixed, hyperbolic - elliptic in some neighborhood of the origin. On the other hand, if $k < -6$ we see that $x = y = 0$ is the only zero of f_0 in $\mathbf{R}^2$. Hence the linearized operator M_{u_0} and (1.9) may be degenerate hyperbolic at the origin. Note that without a Poincaré condition, the equation may be of degenerate mixed type at the origin.

Example 1.6. We consider $u_0(x, y) = x^4 + bx^2y^2$, $c(x, y) = cxy$, where $b > 6$ and c is a constant chosen later. By the same argument as above the linearized operator is given by

$$2bx^2\partial_x^2 + 2(6x^2 + by^2)\partial_y^2 + (c - 8b)xy\partial_x\partial_y. \tag{1.18}$$

We note that in this case the equation is degenerate elliptic operator which degenerates on the line $x = 0$. The Toeplitz symbol is given by $\sigma_P(z, \eta) = 2b + 12\eta_1^2 z_1^2 z_2^{-2} + (c - 8b)\eta_1\eta_2$. The condition (A.1) is given by

$$2b + 12\eta_1^2 t^2 + (c - 8b)\eta_1\eta_2 \neq 0, \qquad \forall t, |t| = 1,\ \forall \eta \in \mathbf{R}_+^2,\ |\eta| = 1. \tag{1.19}$$

The condition holds if $\eta_1 = 0$ or t^2 is not real. Hence we may restrict ourselves to the case $t = \pm 1$. Because $b > 6$ and $0 \leq \eta_1\eta_2 \leq 1/2$ it follows that (1.19) is satisfied if $c - 8b$ is sufficiently large positive number. On the other hand, the condition (A.2) for $\eta_1 = 0, \eta_2 = 1$ is trivially satisfied by definition. By Toeplitz operator method we can show the Fredholmness of the linearized operator P on the set of analytic functions. Moreover, it follows from Theorem 1.1 that (1.8) has a solution if the order of g is sufficiently large.

The proof of these results will be published elsewhere.

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Current adress:
Université de Nice Sophia-Antipolis
Mathematiques UMR 6621, Parc Valrose
06108 Nice France
e-mail: yoshinom@math.unice.fr
Permanent adress:
Faculty of Economics, Chuo University,
742-1, Higashinakano, Hachioji, Tokyo Japan
e-mail: yoshinom@tamacc.chuo-u.ac.jp