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ON AUTOMORPHISMS OF GENERALIZED CUNTZ ALGEBRAS

YOSHIKAZU KATAYAMA AND HIROAKI TAKEHANA

1. X-APERIODICITY

Definition 1.1. Let X be a full right Hilbert B -bimodule of finite type. The C^* -algebra B is called to be X -aperiodic if for a non-zero positive element b of B , there exists $\{x_{i,j}\} \subset X, j = 1, 2, \dots, m_i, i = 1, 2, \dots, l$ such that

$$\sum_{i(1), \dots, i(l)}^{m_1, \dots, m_l} \langle x_{i(l), l}, \dots, \langle x_{i(2), 2} \langle x_{i(1), 1}, b x_{i(1), 1} \rangle_B x_{i(2), 2} \rangle_B \dots x_{i(l), l} \rangle_B \quad (1.1)$$

is invertible.

Note that, by functional calculus, the above equations (1.1) may be equal to an identity operator for the definition of X -aperiodicity. It is defined in [3] that B is X -simple if any non-zero X -invariant ideal J of B (i.e. $\langle x, Jy \rangle_B \subset J$ for $x, y \in X$) must be the whole space B . It is clear that X -aperiodicity implies X -simplicity. Let α be an automorphism of B and its associated imprimitivity Hilbert B -bimodule ${}_{\alpha}B$ is B as a vector space with

$$a \cdot x \cdot b = \alpha(a)xb, \quad {}_B \langle x, y \rangle = \alpha^{-1}(xy^*), \quad \langle x, y \rangle_B = x^*y \quad (1.2)$$

for $a, b \in B$ and $x, y \in {}_{\alpha}B$. We note that the unital C^* -algebra B is ${}_{\alpha}B$ -aperiodic if and only if B is simple (see Theorem 1.3). The notion of X -simple is related with its irreducible adjacent matrix in the case that B is finite abelian. The one of X -aperiodic is just related with its aperiodic adjacent matrix as follows.

Let X be full right Hilbert B -bimodule with finite dimensional abelian C^* -algebra B . Let Σ be a finite set such that $C(\Sigma) = B$ and $\{p_{\tau}\}_{\tau \in \Sigma}$ be all minimal projections of $C(\Sigma)$. as in [5]. We denote a matrix M by $(a_{\sigma, \tau})_{\sigma, \tau \in \Sigma}$ where $a_{\sigma, \tau} = \dim_{\mathbb{C}} p_{\sigma} X p_{\tau}$. Let $\{\xi_{\sigma, \tau, l} \in X : \sigma, \tau \in \Sigma \text{ with } a_{\sigma, \tau} \geq 1, 1 \leq l \leq a_{\sigma, \tau}\}$ be a basis of vector space X :

$$\begin{cases} p_{\sigma'} \xi_{\sigma, \tau, l} = \delta_{\sigma', \sigma} \xi_{\sigma, \tau, l}, \\ \xi_{\sigma, \tau, l} p_{\tau'} = \delta_{\tau, \tau'} \xi_{\sigma, \tau, l}, \\ \langle \xi_{\sigma, \tau, l}, \xi_{\sigma', \tau', l'} \rangle_B = \delta_{\sigma', \sigma} \delta_{\tau, \tau'} \delta_{l, l'} p_{\tau}. \end{cases} \quad (1.3)$$

We note that $\{\xi_{\sigma, \tau, l}\}_{\sigma, \tau, l}$ is right B -basis. We set $\xi_{\sigma} := \sum_{\tau, l} \xi_{\sigma, \tau, l}$, then

$$\langle \xi_{\sigma}, \xi_{\sigma'} \rangle_B = \delta_{\sigma, \sigma'} \sum_{\tau} a_{\sigma, \tau} p_{\tau}. \quad (1.4)$$

Therefore we have

$$\sum_{\sigma(1)} \langle \xi_{\sigma(1)}, p_{\sigma} \xi_{\sigma(1)} \rangle_B = \sum_{\tau} a_{\sigma, \tau} p_{\tau}. \quad (1.5)$$

Proposition 1.2. *Let X be as above. The finite dimensional abelian C^* -algebra B is X -aperiodic if and only if the matrix M is aperiodic (i.e. there exists integer m such that $M^m(\sigma, \tau) > 0$ for all $\sigma, \tau \in \Sigma$ where $M^m(\sigma, \tau)$ is (σ, τ) -component of the matrix M^m).*

Proof. By (1.5), we have

$$\begin{aligned} & \sum_{\sigma(1), \dots, \sigma(m)} \langle \xi_{\sigma(m)}, \dots \langle \xi_{\sigma(2)}, \langle \xi_{\sigma(1)}, p_{\sigma} \xi_{\sigma(1)} \rangle_B \xi_{\sigma(2)} \rangle_B \dots \xi_{\sigma(m)} \rangle_B \\ &= \sum_{\tau} M^m(\sigma, \tau) p_{\tau}. \end{aligned}$$

If M is aperiodic, then

$$\sum_{\sigma(1), \dots, \sigma(m)} \langle \xi_{\sigma(m)}, \dots \langle \xi_{\sigma(2)}, \langle \xi_{\sigma(1)}, p_{\sigma} \xi_{\sigma(1)} \rangle_B \xi_{\sigma(2)} \rangle_B \dots \xi_{\sigma(m)} \rangle_B$$

is invertible. Since B is finite dimensional, the C^* -algebra B is X -aperiodic.

Conversely for $x = \sum_{\sigma, \tau, l} c_{\sigma, \tau, l} \xi_{\sigma, \tau, l} \in B$, $c_{\sigma, \tau, l} \in \mathbb{C}$, by (1.3) we have

$$\langle x, p_{\sigma} x \rangle_B = \sum_{\tau, l, a_{\sigma, \tau} \neq 0} |c_{\sigma, \tau, l}|^2 p_{\tau}.$$

If the equation (1.1) holds, for $\sigma, \tau \in \Sigma$, there exists $\{\tau(i)\}_{i=1}^m \subset \Sigma$ such that $a_{\tau(i), \tau(i+1)} \neq 0$ for $i = 1, 2, \dots, m$, $\tau(1) = \sigma, \tau(m) = \tau$. Therefore $M^m(\sigma, \tau) > 0$ which implies that M is aperiodic. $\square$

Let $\mathcal{F}_m(X)$ be a relative tensor product $\overbrace{X \otimes_B X \otimes_B \dots \otimes_B X}^{m\text{-times}}$ for a full right Hilbert B -bimodule X and $\mathcal{F}_m$ is a C^* -subalgebra of $\mathcal{O}_X$ generated by

$$\{S_{x_1 \otimes x_2 \dots \otimes x_m} S_{y_1 \otimes y_2 \dots \otimes y_m}^* : x_1 \otimes x_2 \dots \otimes x_m, y_1 \otimes y_2 \dots \otimes y_m \in \mathcal{F}_m(X)\}.$$

There exists a unital isomorphism $\psi_m : K_B(\mathcal{F}_m(X)_B) \rightarrow \mathcal{F}_m$ such that:

$$\psi_m(\theta_{x_1 \otimes x_2 \dots \otimes x_m, y_1 \otimes y_2 \dots \otimes y_m}) = S_{x_1 \otimes x_2 \dots \otimes x_m} S_{y_1 \otimes y_2 \dots \otimes y_m}^*$$

for finite rank operators $\theta_{x_1 \otimes x_2 \dots \otimes x_m, y_1 \otimes y_2 \dots \otimes y_m} \in K_B(\mathcal{F}_m(X)_B)$. Since X is of finite type, we have

$$\sum_{i=1}^n S_{u_i} S_{u_i}^* = 1 \quad \text{and} \quad \mathcal{F}_m \subset \mathcal{F}_{m+1}.$$

We set $\mathcal{F}_X := \overline{\bigcup_{m=1}^{\infty} \mathcal{F}_m}$. Moreover $\mathcal{F}_X$ is the fixed point algebra $\mathcal{O}_X^T$ for the gauge action. We define a complete positive map $\sigma : \mathcal{O}_X \rightarrow \mathcal{O}_X$ by

$$\sigma(T) = \sum_{i=1}^n S_{u_i} T S_{u_i}^* \tag{1.6}$$

for $T \in \mathcal{O}_X$. In [3] Lemma 7.8, it is proved that the restriction of σ on $B' \cap \mathcal{O}_X$ is a unital isometric $*$ -homomorphism and it does not depend on the choice of B -basis. Moreover $\sigma^m(T)$ commutes with $\mathcal{F}_m$ for $T \in B' \cap \mathcal{O}_X$. There is an isomorphism $\pi_m : \mathcal{F}_m \rightarrow (B \otimes M_n)_{P_m}$ such that, for $x \in \mathcal{F}_m$,

$$\begin{cases} x = \sum_{\substack{i(1), \dots, i(m) \\ j(1), \dots, j(m)}} S_{u_{i(1)} \otimes u_{i(2)} \cdots \otimes u_{i(m)}} b_{i(1), \dots, i(m), j(1), \dots, j(m)} S_{u_{j(1)} \otimes u_{j(2)} \cdots \otimes u_{j(m)}}^* \\ \pi_m(x) = (b_{i(1), \dots, i(m), j(1), \dots, j(m)}) \in (B \otimes M_n)_{P_m} \end{cases} \quad (1.7)$$

where the projection P_m is

$$(\langle u_{i(1)} \otimes u_{i(2)} \cdots \otimes u_{i(m)}, u_{j(1)} \otimes u_{j(2)} \cdots \otimes u_{j(m)} \rangle_B).$$

We note that if X is a Hilbert B -bimodule ([4]Definition 1.3), there exists a conditional expectation E_m from $\mathcal{F}_X$ onto $\mathcal{F}_m$ such that

$$\begin{cases} E_m = \lim_{k \rightarrow \infty} E_m^{m+k} \\ E_m^{m+k}(\theta_{x_1 \otimes y_1, x_2 \otimes y_2}) = \theta_{x_1 B \langle y_1, y_2 \rangle, x_2} \end{cases} \quad (1.8)$$

for $x_1, x_2 \in \mathcal{F}_m(X), y_1, y_2 \in \mathcal{F}_k(X)$ ([4]Lemma 3.24, 3.25).

Theorem 1.3. *Let X be a full right Hilbert B -bimodule. The C^* -algebra $\mathcal{F}_X$ is simple if and only if B is X -aperiodic.*

Proof. Let J be a non-zero closed ideal of $\mathcal{F}_X$. Set $J_m := \mathcal{F}_m \cap J$ and $J = \overline{\bigcup_{m=1}^{\infty} J_m}$. Then for a non-zero element $x \in J_m$ for some m , there exists an element

$$(b_{i(1), \dots, i(m), j(1), \dots, j(m)}) \in (B \otimes M_n)_{P_m}$$

satisfying the relation (1.7). If necessary, consider x^*x instead of x , and we may assume that there is $(k(1), \dots, k(m))$ such that

$$b := b_{k(1), \dots, k(m), k(1), \dots, k(m)}$$

is a non-zero positive element of B . Suppose that B is X -aperiodic, and we choose the elements $\{x_{i,j}\}$ of X satisfying the relation (1.1) for b . We take $y_{i(1), \dots, i(l), s(1), \dots, s(m+l)}$ of $\mathcal{F}_{m+l}$:

$$\begin{cases} y_{i(1), \dots, i(l), s(1), \dots, s(m+l)} =: \\ \sum_{k(1), \dots, k(m)} S_{u_{s(1)} \otimes u_{s(2)} \cdots \otimes u_{s(m+l)}} S_{x_{i(1), 1} \otimes \cdots \otimes x_{i(l), l}}^* S_{u_{k(1)} \otimes u_{k(2)} \cdots \otimes u_{k(m)}}^* \end{cases} \quad (1.9)$$

Since $\pi_m(x)P_m = P_m\pi_m(x) = \pi_m(x)$, we compute

$$\begin{aligned} & y_{i(1), \dots, i(l), s(1), \dots, s(m+l)} b y_{i(1), \dots, i(l), s(1), \dots, s(m+l)}^* \\ &= S_{u_{s(1)} \otimes u_{s(2)} \cdots \otimes u_{s(m+l)}} \langle x_{i(1), 1} \otimes \cdots \otimes x_{i(l), l}, b x_{i(1), 1} \otimes \cdots \otimes x_{i(l), l} \rangle_B S_{u_{s(1)} \otimes u_{s(2)} \cdots \otimes u_{s(m+l)}}^* \end{aligned}$$

and

$$\begin{aligned} & \sum_{i(1), \dots, i(l)} y_{i(1), \dots, i(l), s(1), \dots, s(m+l)} b y_{i(1), \dots, i(l), s(1), \dots, s(m+l)}^* \\ &= S_{u_{s(1)} \otimes u_{s(2)} \cdots \otimes u_{s(m+l)}} \\ & \quad \times \sum_{i(1), \dots, i(l)} \langle x_{i(1), 1} \otimes \cdots \otimes x_{i(l), l}, b x_{i(1), 1} \otimes \cdots \otimes x_{i(l), l} \rangle_B S_{u_{s(1)} \otimes u_{s(2)} \cdots \otimes u_{s(m+l)}}^*. \end{aligned}$$

Since there is a positive number $\lambda \in \mathbb{R}$ such that

$$\sum_{i(1), \dots, i(l)} \langle x_{i(1), 1} \otimes \cdots \otimes x_{i(l), l}, b x_{i(1), 1} \otimes \cdots \otimes x_{i(l), l} \rangle_B \geq \lambda I,$$

we have

$$\left\{ \begin{array}{l} \sum_{\substack{i(1), \dots, i(l) \\ s(1), \dots, s(m+l)}} y_{i(1), \dots, i(l), s(1), \dots, s(m+l)} b y_{i(1), \dots, i(l), s(1), \dots, s(m+l)}^* \\ \geq \lambda \sum_{s(1), \dots, s(m+l)} S_{u_{s(1)} \otimes u_{s(2)} \cdots \otimes u_{s(m+l)}} S_{u_{s(1)} \otimes u_{s(2)} \cdots \otimes u_{s(m+l)}}^* = \lambda I \end{array} \right. \quad (1.10)$$

Thus J_{l+m} contains the above invertible element. We conclude that the ideal J is B .

Convesely we assume that $\mathcal{F}_X$ is simple. Since B is unital and X is full, by [3]Proposition15, there is a finite set $\{x_i\} \subset X$ such that

$$\sum_i \langle x_i, x_i \rangle_B = I. \quad (1.11)$$

For a non-zero element $b \in B$, we consider a closed ideal $\overline{\cup_{m=1}^{\infty} \mathcal{F}_m b \mathcal{F}_m}$ of $\mathcal{F}_X$. we can choose a finite subset $\{f_i\}_{i=1}^l \subset \mathcal{F}_m$ with $\sum_{i=1}^l f_i^* b f_i = I$ for some m . The element f_i of $\mathcal{F}_m$ is of the form:

$$f_i = \sum_{k=1}^{nm} S_{z_{1,k}^i \otimes \cdots \otimes z_{m,k}^i} S_{y_{1,k}^i \otimes \cdots \otimes y_{m,k}^i}^*.$$

Since an operator inequality :

$$\left(\sum_{i=1}^l T_i \right)^* S \left(\sum_{i=1}^l T_i \right) \leq l \left(\sum_{i=1}^l T_i^* S T_i \right)$$

holds, we obtain

$$\begin{aligned} I &= \sum_{i=1}^l f_i^* b f_i \\ &\leq nm \sum_{i,k} S_{y_{1,k}^i \otimes \cdots \otimes y_{m,k}^i} \langle z_{1,k}^i \otimes \cdots \otimes z_{m,k}^i, b z_{1,k}^i \otimes \cdots \otimes z_{m,k}^i \rangle_B S_{y_{1,k}^i \otimes \cdots \otimes y_{m,k}^i}^*. \end{aligned}$$

By (1.11), we have

$$\begin{aligned} I &= \sum_{i(1), \dots, i(m)} S_{x_{i(1)} \otimes \cdots \otimes x_{i(m)}}^* I S_{x_{i(1)} \otimes \cdots \otimes x_{i(m)}} \\ &\leq nm \sum_{i(1), \dots, i(m)} \sum_{i,k} \langle x_{i(1)} \otimes \cdots \otimes x_{i(m)}, y_{1,k}^i \otimes \cdots \otimes y_{m,k}^i \rangle_B \\ &\quad \times \langle z_{1,k}^i \otimes \cdots \otimes z_{m,k}^i, b z_{1,k}^i \otimes \cdots \otimes z_{m,k}^i \rangle_B \\ &\quad \times \langle y_{1,k}^i \otimes \cdots \otimes y_{m,k}^i, x_{i(1)} \otimes \cdots \otimes x_{i(m)} \rangle_B, \end{aligned}$$

which implies that B is X -aperiodic. $\square$

2. AUTOMORPHISMS OF $\mathcal{O}_X$

Let θ be an automorphism of B and U be an invertible $\mathbb{C}$ -linear map on the right Hilbert B -bimodule X satisfying

$$\langle Ux, Uy \rangle_B = \theta(\langle x, y \rangle_B), \quad U(bxb') = \theta(b)(Ux)\theta(b') \quad (2.1)$$

for $x, y \in X$ and $b, b' \in B$. This invertible operator U induces an automorphism α_U of $\mathcal{O}_X$ such that

$$\alpha_U(S_x) = S_{Ux}$$

for $x \in X$. We note that if the right Hilbert B -bimodule X is ${}_B\mathbb{C}^n$, then the U is a unitary operator on $\mathbb{C}^n$ and the automorphism α_U is the same as defined in [2]. It is remarked that the U is a unitary operator in ${}_B\mathcal{L}_B(X_B)$ if θ is trivial. At first, we give some results related with problems whether the restriction $\alpha_U|_{\mathcal{F}_X}$ on $\mathcal{F}_X$ for α_U is inner or not.

Proposition 2.1. *Let X be a right Hilbert B -bimodule of finite type and U be as (2.1). If the automorphism $\alpha_U|_{\mathcal{F}_X}$ is inner, then the restricted automorphism $\alpha_U|_{B' \cap \mathcal{F}_X}$ on the relative commutant $B' \cap \mathcal{F}_X$ must be trivial.*

Proof. Let $\alpha_U|_{\mathcal{F}_X}$ be of the form:

$$\alpha_U|_{\mathcal{F}_X} = \text{Ad}V$$

for some $V \in \mathcal{F}_X$. For a right B -basis $\{u_i\}$ and $x \in X$, we get

$$\begin{aligned} \sum_i (Uu_i) \langle Uu_i, x \rangle_B &= \sum_i (Uu_i) \theta(\langle u_i, U^{-1}x \rangle_B) \\ &= \sum_i U(u_i \langle u_i, U^{-1}x \rangle_B) = UU^{-1}x = x. \end{aligned}$$

Hence $\{Uu_i\}$ is also right B -basis. Since σ on $B' \cap \mathcal{F}_X$ does not depend on the choice of B -basis, we have

$$\alpha_U \sigma|_{B' \cap \mathcal{F}_X} = \sigma \alpha_U|_{B' \cap \mathcal{F}_X}. \quad (2.2)$$

Since $\sigma^m(T)$ for $T \in B' \cap \mathcal{F}_X$ commutes with $\mathcal{F}_m$ and σ is isometric $*$ -homomorphism, we get

$$\begin{aligned} &\|\sigma_U(T) - T\| \\ &= \lim_{m \rightarrow \infty} \|\sigma^m \alpha_U(T) - \sigma^m(T)\| \\ &= \lim_{m \rightarrow \infty} \|\alpha_U \sigma^m(T) - \sigma^m(T)\| \\ &= \lim_{m \rightarrow \infty} \|V \sigma^m(T) V^* - \sigma^m(T)\| = 0. \end{aligned}$$

We conclude that $\alpha_U(T) = T$ for $T \in B' \cap \mathcal{F}_X$. □

Next under the some restricted condition, we shall prove that α_U is inner on $\mathcal{F}_X$ if and only $Ux = \lambda uxu^*$ for some unitary u of B , $\lambda \in \mathbb{T}$ and all $x \in X$.

Lemma 2.2. *Let X be a full Hilbert B -bimodule with $Z(B) = \mathbb{C}$ and U be the invertible operator in (2.1). If the automorphism α_U is of the form:*

$$\alpha_U(T) = \text{Ad}V(T)$$

for some $V \in \mathcal{F}_X$ and all $T \in \mathcal{F}_X$, then the automorphism θ of B is inner, i.e. $\theta = \text{Ad}u$ for a unitary u in B .

Proof. Let E_m be an expectation as in (1.8). Then, for sufficient large m , the invertible $E_m(V)$ satisfies

$$\alpha_U(T)E_m(V) = E_m(V)T$$

for $T \in \mathcal{F}_m$. By [3] Lemma 1.6, this operator $E_m(V)$ is scalar multiple of a unitary $V_m \in \mathcal{F}_m$ such that $\alpha_U(T) = \text{Ad}V_m(T)$ for $T \in \mathcal{F}_X$. We compute, for $T = S_{U^{-1}x_1 \otimes \dots \otimes U^{-1}x_m} b S_{U^{-1}y_1 \otimes \dots \otimes U^{-1}y_m}^*$,

$$\begin{aligned} & S_{x_1 \otimes \dots \otimes x_m}^* \alpha_U(T) S_{y_1 \otimes \dots \otimes y_m} \\ &= S_{x_1 \otimes \dots \otimes x_m}^* S_{x_1 \otimes \dots \otimes x_m} \theta(b) S_{y_1 \otimes \dots \otimes y_m}^* S_{y_1 \otimes \dots \otimes y_m} \\ &= \langle x_1 \otimes \dots \otimes x_m, x_1 \otimes \dots \otimes x_m \rangle_B \theta(b) \langle y_1 \otimes \dots \otimes y_m, y_1 \otimes \dots \otimes y_m \rangle_B \end{aligned}$$

and

$$\begin{aligned} & S_{x_1 \otimes \dots \otimes x_m}^* (V_m T V_m^*) S_{y_1 \otimes \dots \otimes y_m} \\ &= \{ S_{x_1 \otimes \dots \otimes x_m}^* V_m S_{x_1 \otimes \dots \otimes x_m} \} b \{ S_{y_1 \otimes \dots \otimes y_m}^* V_m^* S_{y_1 \otimes \dots \otimes y_m} \}. \end{aligned}$$

Since $\{ S_{x_1 \otimes \dots \otimes x_m}^* V_m S_{x_1 \otimes \dots \otimes x_m} \}$ is an element of B , denoted by $d(x_1 \otimes \dots \otimes x_m)$, we get

$$\begin{aligned} & \langle x_1 \otimes \dots \otimes x_m, x_1 \otimes \dots \otimes x_m \rangle_B \theta(b) \langle y_1 \otimes \dots \otimes y_m, y_1 \otimes \dots \otimes y_m \rangle_B^* \\ &= d(x_1 \otimes \dots \otimes x_m) b d(y_1 \otimes \dots \otimes y_m)^*. \end{aligned}$$

Since X is full, there exists a finite subset $\{z_i\}$ in X such that

$$\sum_i \langle z_i, z_i \rangle_B = I.$$

Thus we get, for all $b \in B$,

$$\begin{aligned} \theta(b) &= \sum_{\substack{i(1), \dots, i(m) \\ j(1), \dots, j(m)}} \langle z_{i(1)} \otimes \dots \otimes z_{i(m)}, z_{i(1)} \otimes \dots \otimes z_{i(m)} \rangle_B \theta(b) \\ &\quad \times \langle z_{j(1)} \otimes \dots \otimes z_{j(m)}, z_{j(1)} \otimes \dots \otimes z_{j(m)} \rangle_B \\ &= u b u^* \end{aligned}$$

where $u = \sum_{i(1), \dots, i(m)} d(z_{i(1)} \otimes \dots \otimes z_{i(m)})$. Therefore we conclude that the automorphism θ is implemented by the unitary u . $\square$

If α_U is inner on $\mathcal{F}_X$, then by considering a perturbed operator U' on X by the unitary u such that $U'x = u^*(Ux)u$ for $x \in X$, we may assume that the invertible operator U is a unitary of ${}_B\mathcal{L}_B(x_B)$ and θ is trivial. The idea of the following lemma is borrowed from Cuntz [1]

Lemma 2.3. *Let U be a unitary of ${}_B\mathcal{L}_B(x_B)$. Then an operator W defined by:*

$$W = \sum_{i=1}^n S_{U u_i} S_i^* \quad (2.3)$$

satisfies the statements:

1. W is independent of the choice for right B -basis $\{u_i\}$
2. W is a unitary operator of $B' \cap \mathcal{F}_1$ such that $\text{Ad}W = \alpha_U$ on $\mathcal{F}_1$.

Moreover set $W_m := W\sigma(W)\dots\sigma^{m-1}(W)$ and the W_m is a unitary operator of $B' \cap \mathcal{F}_m$ such that $\text{Ad}W_m = \alpha_U$ on $\mathcal{F}_m$.

Proof. Let $\{v_j\}$ be another right B -basis for X . Then we have

$$u_i = \sum_j v_j \langle v_j, u_i \rangle_B$$

and

$$\begin{aligned} W &= \sum_i S_{(U \sum_j v_j \langle v_j, u_i \rangle_B)} S_{u_i}^* \\ &= \sum_{i,j} S_{U v_j \langle v_j, u_i \rangle_B} S_{u_i}^* \\ &= \sum_j S_{U v_j} S_{v_j}^* \end{aligned}$$

Hence the operator W in $\mathcal{F}_1$ is independent of the choice for right B -basis. To show the unitarity of W , we compute

$$\begin{aligned} W^*W &= \sum_{i,j} S_{u_i} S_{U u_i}^* S_{U u_j} S_{u_j}^* \\ &= \sum_{i,j} S_{u_i \langle U u_i, U u_j \rangle_B} S_{u_j}^* \\ &= \sum_{i,j} S_{u_i \langle u_i, u_j \rangle_B} S_{u_j}^* = I \end{aligned}$$

and similarly we have

$$WW^* = \sum_i S_{U u_i} S_{U u_i}^* = I.$$

For $b \in B$, we calculate

$$\begin{aligned} bW &= \sum_i S_{U b u_i} S_{u_i}^* \\ &= \sum_{i,j} S_{U u_j \langle u_j, b u_i \rangle_B} S_{u_i}^* \\ &= \sum_j S_{U u_j} S_{\sum_i u_i \langle u_i, b^* u_j \rangle_B}^* \\ &= \sum_j S_{U u_j} S_{b^* u_j}^* = Wb. \end{aligned}$$

Therefore W is an element of $B' \cap \mathcal{F}_1$. Since

$$\begin{aligned} WS_x &= \sum_i S_{U u_i} S_{u_i}^* S_x \\ &= \sum_i S_{U u_i \langle u_i, x \rangle_B} \\ &= S_{U x} = \alpha_U(S_x) \end{aligned}$$

and $\mathcal{F}_1$ is generated by $\{S_x S_y^* : x, y \in X\}$, we obtain $\alpha_U = \text{Ad}W$ on $\mathcal{F}_1$. Finally it is clear that W_m is a unitary of $B' \cap \mathcal{F}_m$ by the definition of W_m . Since $\{u_{i(1)} \otimes$

$\cdots \otimes u_{i(m)}\}$ is a right basis for $\overbrace{X \otimes_B \cdots \otimes_B X}^{m\text{-times}}$ and

$$W_m = \sum_{i(1), \dots, i(m)} S_{U u_{i(1)} \otimes \cdots \otimes U u_{i(m)}} S_{u_{i(1)} \otimes \cdots \otimes u_{i(m)}}^*,$$

it follows from (2) that $\alpha_U = \text{Ad}W_m$ on $\mathcal{F}_m$. $\square$

Proposition 2.4. *Let X be a full Hilbert B -bimodule of finite type with a left inner product ${}_B \langle \cdot, \cdot \rangle$ and the center $Z(B) = \mathbb{C}$. Then the automorphism $\alpha_U|_{\mathcal{F}_X}$ is inner if and only if $Ux = \lambda uxu^*$ for some unitary u of B some $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and all $x \in X$ and the automorphism θ is of the form: $\theta = \text{Ad}u$.*

Proof. The part of "if" is trivial.

The automorphism $\alpha_U|_{\mathcal{F}_X}$ is of the form: $\alpha_U|_{\mathcal{F}_X} = \text{Ad}V$ for some unitary V in $\mathcal{F}_X$. By Lemma 2.2, we may assume that θ is trivial and U is a unitary of ${}_B\mathcal{L}_B(X_B)$ by perturbing U by a unitary u in B . It follows from Lemma 2.3 and $\alpha_U(\mathcal{F}_m) = \mathcal{F}_m$ that

$$\begin{aligned} E_m(V)T &= E_m(VT) = E_m(\alpha_U(T)V) \\ &= \alpha_U(T)E_m(V) = W_m T W_m^* E_m(V) \end{aligned}$$

for $T \in \mathcal{F}_m$ where E_m is the expectation in (1.8). By $Z(\mathcal{F}_m) \simeq Z(B)$ in [3] Lemma 16, the element $W_m^* E_m(V) \in Z(\mathcal{F}_m)$ is scalar λ_m . Since $\lim_{m \rightarrow \infty} E_m(T) = T$ for $T \in \mathcal{F}_X$, we have $\lim_{m \rightarrow \infty} |\lambda_m| = 1$ and

$$\begin{aligned} &\lim_{m \rightarrow \infty} \|\lambda_{m+1}^{-1} \lambda_m - W\| \\ &= \lim_{m \rightarrow \infty} \|\lambda_{m+1}^{-1} \lambda_m - \sigma^{m+1}(W)\| \\ &= \lim_{m \rightarrow \infty} \|\lambda_{m+1}^{-1} \lambda_m - W_m^* W_{m+1}\| \\ &= \lim_{m \rightarrow \infty} \|\lambda_m W_m - \lambda_{m+1} W_{m+1}\| = 0. \end{aligned}$$

Hence there exists $\lambda \in \mathbb{C}$ such that $W = \lambda I$. For $x \in X$, we obtain

$$\begin{aligned} \lambda S_x &= W S_x = \sum_i S_{U u_i} S_{u_i}^* S_x \\ &= \sum_i S_{U u_i} \langle u_i, x \rangle_B = S_{U x}. \end{aligned}$$

We conclude that $Ux = \lambda x$ for $x \in X$. $\square$

Next we give some results related with problems whether the automorphism α_U on $\mathcal{O}_X$ is inner or not. The X -aperiodicity of B plays a crucial role in proving the outerness of its automorphism.

Theorem 2.5. *Let X be a full right Hilbert B -bimodule of finite type and C^* -algebra B is X -aperiodic. The automorphism α_U of $\mathcal{O}_X$ induced by the invertible operator U satisfying (2.1) is not inner if $B' \cap \mathcal{F}_X$ is not trivial and the restricted automorphism $\alpha_U|_{B' \cap \mathcal{F}_X}$ on $B' \cap \mathcal{F}_X$ is not trivial.*

Proof. Suppose that there is a unitary V in $\mathcal{O}_X$ such that

$$\alpha_U(T)V = VT$$

for $T \in \mathcal{O}_X$. By taking a consideration of a Fourier expansion $\{V_m\}_{m \in \mathbb{Z}}$ of V with respect to the gauge action, we have

$$\alpha_U(T)V_m = V_m T \quad (2.4)$$

for $T \in \mathcal{F}_X$ and $\alpha_t(V_m) = e^{-imt}V_m$. Its proof is divided into three cases:

- (i) there is a positive integer m with $V_m \neq 0$
- (ii) there is a negative integer $-m$ with $V_{-m} \neq 0$
- (iii) $V_m = 0$ for all m except $m = 0$.

In the case (i), $V_m^*V_m$ and $V_mV_m^*$ are non-zero elements of $Z(\mathcal{F}_X)$. Since $\mathcal{F}_X$ is simple by Theorem 1.3, $V_m^*V_m$ and $V_mV_m^*$ must be non-zero scalars. Hence we may assume that V_m is a unitary. The unitary V_m is of the form:

$$V_m = \sum_{i(1), \dots, i(m)} S_{u_{i(1)} \otimes u_{i(2)} \cdots \otimes u_{i(m)}} \left\{ S_{u_{i(1)} \otimes u_{i(2)} \cdots \otimes u_{i(m)}}^* V_m \right\} \\ \in \sum_{i(1), \dots, i(m)} S_{u_{i(1)} \otimes u_{i(2)} \cdots \otimes u_{i(m)}} \mathcal{F}_X.$$

Since $\sigma^m(T)$ for $T \in B' \cap \mathcal{F}_X$ commutes with $\mathcal{F}_m$ and $\mathcal{F}_X = \overline{\bigcup_{m=1}^{\infty} \mathcal{F}_m}$, for $\varepsilon > 0$, there is an integer $l_0 \in \mathbb{N}$ such that for $l > l_0$

$$\|V_m \sigma^l(T) - \sigma^{l+m}(T)V_m\| \\ = \|V_m \sigma^l(T) - \sum_{i(1), \dots, i(m)} S_{u_{i(1)} \otimes u_{i(2)} \cdots \otimes u_{i(m)}} \sigma^l(T) S_{u_{i(1)} \otimes u_{i(2)} \cdots \otimes u_{i(m)}}^* V_m\| \leq \varepsilon$$

for $T \in B' \cap \mathcal{F}_X$. By (2.2), we have for $l > l_0$

$$\begin{aligned} & \|\alpha_U(T) - \sigma^m(T)\| \\ &= \|\sigma^l(\alpha_U(T)) - \sigma^{l+m}(T)\| \\ &= \|\alpha_U(\sigma^l(T))V_m - \sigma^{l+m}(T)V_m\| \\ &= \|V_m \sigma^l(T) - \sigma^{l+m}(T)V_m\| \leq \varepsilon. \end{aligned}$$

Therefore we obtain $\alpha_U = \sigma^m$ on $B' \cap \mathcal{F}_X$. By the assumption: $B' \cap \mathcal{F}_X \neq \mathbb{C}$, take a non-scalar $T_0 \in B' \cap \mathcal{F}_X$. Since

$$T_0 = \alpha_U^{-1} \alpha_U(T_0) = \alpha_U^{-1} \sigma^m(T_0),$$

the operator T_0 commutes with $\mathcal{F}_m$. By an iteration, the operator T_0 is an element of $Z(\mathcal{F}_X)$. Since $\mathcal{F}_X$ is simple, the operator T_0 must be a scalar, which is a contradiction.

In the case (ii), the relation $\alpha_U(T)V_{-m} = V_{-m}T$ in (2.4) is equivalent to $\alpha_U^{-1}(T)V_{-m}^* = V_{-m}^*T$. Hence, by the same way as in the case (i), we get $\alpha_U^{-1} = \sigma^m$ on $B' \cap \mathcal{F}_X$ and we get a contradiction similarly.

In the case (iii), It follows from Proposition 2.1 that α_U is not inner. $\square$

We apply Theorem 2.5 to Cuntz-Krieger algebras.

Proposition 2.6 (Cuntz-Krieger Algebra). *Let X be a full right Hilbert B -bimodule of finite type ($\dim_{\mathbb{C}} X > 1$) and the finite dimensional abelian C^* -algebra B is X -aperiodic. The invertible operator U on X and the automorphism θ of B satisfy the relation (2.1). Then α_U is inner if and only if the operator U is of the form:*

$$Ux = uxu^* \quad (x \in X) \quad (2.5)$$

for some $u \in B$ and the automorphism θ is trivial.

Proof. The part of "if" is clear.

We suppose that $\alpha_U = \text{Ad}V$ for some unitary $V \in \mathcal{O}_X$. Since $B' \cap \mathcal{F}_X$ is always not trivial, by Theorem 2.5 and its proof, we may assume that the unitary V is an element of $\mathcal{F}_X$ and $\alpha_U|_{B' \cap \mathcal{F}_X}$ is trivial. It can be shown that the operator U in (2.1) is of the form:

$$U\xi_{\sigma,\tau,l} = \sum_k c_{\sigma,\tau}(l,k)\xi_{\theta(\sigma),\theta(\tau),k} \quad (2.6)$$

where $C_{\sigma,\tau} := (c_{\sigma,\tau}(l,k))_{l,k}$ are unitary matrices and $\{\xi_{\sigma,\tau,l}\}$ is the basis for X . Moreover the automorphism θ satisfies a relation:

$$a_{\sigma,\tau} = a_{\theta(\sigma),\theta(\tau)}$$

where $a_{\sigma,\tau}$ is the entries of the matrix M above (1.3). By Lemma 2.2, the automorphism θ must be trivial. By considering element of $B' \cap \mathcal{F}_1$:

$$S_{\xi_{\sigma,\tau,l}} p_\tau S_{\xi_{\sigma,\tau,k}}^*$$

for $\sigma, \tau \in \Sigma$ and $1 \leq l, k \leq a_{\sigma,\tau}$, we have

$$\begin{aligned} & S_{\xi_{\sigma,\tau,l}} p_\tau S_{\xi_{\sigma,\tau,k}}^* \\ &= \alpha_U(S_{\xi_{\sigma,\tau,l}} p_\tau S_{\xi_{\sigma,\tau,k}}^*) \\ &= S_{U\xi_{\sigma,\tau,l}} p_\tau S_{U\xi_{\sigma,\tau,k}}^* \\ &= \sum_{l',k'} c_{\sigma,\tau}(l,l') \overline{c_{\sigma,\tau}(k,k')} S_{\xi_{\sigma,\tau,l'}} p_\tau S_{\xi_{\sigma,\tau,k'}}^*. \end{aligned}$$

Hence a relation:

$$c_{\sigma,\tau}(l,l') \overline{c_{\sigma,\tau}(k,k')} = \delta(l,l') \delta(k,k') 1.$$

holds for all $1 \leq l, l', k, k' \leq a_{\sigma,\tau}$. This implies that the matrices $C_{\sigma,\tau}$ are scalar. Those scalar is denoted by $C_{\sigma,\tau}$ and $|C_{\sigma,\tau}| = 1$. Take elements of $B' \cap \mathcal{F}_m$:

$$S_{\xi_{\sigma,\sigma(1),l(1)} \otimes \xi_{\sigma(1),\sigma(2),l(2)} \otimes \cdots \otimes \xi_{\sigma(m-1),\tau,l(m)}} p_\tau S_{\xi_{\sigma,\tau(1),l(1)} \otimes \xi_{\tau(1),\tau(2),l(2)} \otimes \cdots \otimes \xi_{\tau(m-1),\tau,l(m)}}^*$$

for the two paths $\sigma\sigma(1)\sigma(2)\dots\sigma(m-1)\tau$ and $\sigma\tau(1)\tau(2)\dots\tau(m-1)\tau$ between σ and τ , and we get

$$\begin{aligned} & S_{\xi_{\sigma,\sigma(1),l(1)} \otimes \xi_{\sigma(1),\sigma(2),l(2)} \otimes \cdots \otimes \xi_{\sigma(m-1),\tau,l(m)}} p_\tau S_{\xi_{\sigma,\tau(1),l(1)} \otimes \xi_{\tau(1),\tau(2),l(2)} \otimes \cdots \otimes \xi_{\tau(m-1),\tau,l(m)}}^* \\ &= \alpha_U(S_{\xi_{\sigma,\sigma(1),l(1)} \otimes \xi_{\sigma(1),\sigma(2),l(2)} \otimes \cdots \otimes \xi_{\sigma(m-1),\tau,l(m)}} p_\tau S_{\xi_{\sigma,\tau(1),l(1)} \otimes \xi_{\tau(1),\tau(2),l(2)} \otimes \cdots \otimes \xi_{\tau(m-1),\tau,l(m)}}^*) \\ &= C_{\sigma,\sigma(1)} C_{\sigma(1),\sigma(2)} \cdots C_{\sigma(m-1),\tau} \overline{C_{\sigma,\tau(1)} C_{\tau(1),\tau(2)} \cdots C_{\tau(m-1),\tau}} \\ & \quad \times S_{\xi_{\sigma,\sigma(1),l(1)} \otimes \xi_{\sigma(1),\sigma(2),l(2)} \otimes \cdots \otimes \xi_{\sigma(m-1),\tau,l(m)}} p_\tau S_{\xi_{\sigma,\tau(1),l(1)} \otimes \xi_{\tau(1),\tau(2),l(2)} \otimes \cdots \otimes \xi_{\tau(m-1),\tau,l(m)}}^*. \end{aligned}$$

Therefore we have, for all $m \in \mathbb{N}$,

$$C_{\sigma,\sigma(1)} C_{\sigma(1),\sigma(2)} \cdots C_{\sigma(m-1),\tau} = C_{\sigma,\tau(1)} C_{\tau(1),\tau(2)} \cdots C_{\tau(m-1),\tau}. \quad (2.7)$$

Since the value of $C_{\sigma,\sigma(1)} C_{\sigma(1),\sigma(2)} \cdots C_{\sigma(m-1),\tau}$ depends only on the two end points σ, τ , it is denoted by $D^m(\sigma, \tau)$. Since B is X -aperiodic, there is a integer $m \in \mathbb{N}$ such that, for all $\sigma, \tau \in \Sigma$, a path of m -length connecting σ and τ exists. Fix $\tau_0 \in \Sigma$ and we have, by (2.7),

$$D^m(\sigma, \tau) D^m(\tau, \tau_0) = D^m(\sigma, \tau_0) D^m(\tau_0, \tau_0).$$

Set $d(\sigma) := D^m(\sigma, \tau_0)$ and $D^m(\sigma, \tau)$ is equal to $d(\sigma)d(\tau_0)\overline{d(\tau)}$. Then we compute, for two paths $\sigma\sigma(1)\dots\sigma(m-1)\tau$ and $\rho\tau(1)\dots\tau(m-1)\tau$,

$$\begin{aligned} & \alpha_U(S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}}p_\tau S_{\xi_{\rho,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}^*) \\ &= D^m(\sigma, \tau)\overline{D^m(\rho, \tau)}S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}}p_\tau S_{\xi_{\rho,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}^* \\ &= d(\sigma)\overline{d(\rho)}S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}}p_\tau S_{\xi_{\rho,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}^*. \end{aligned}$$

We set a unitary $u \in B$:

$$u := \sum_{\sigma} d(\sigma)p_{\sigma}.$$

Then all for two paths $\sigma\sigma(1)\dots\sigma(m-1)\tau$ and $\rho\tau(1)\dots\tau(m-1)\tau$,

$$\begin{aligned} & \alpha_U(S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}}p_\tau S_{\xi_{\rho,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}^*) \\ &= \text{Adu}(S_{\xi_{\sigma,\sigma(1),l(1)}\otimes\xi_{\sigma(1),\sigma(2),l(2)}\otimes\cdots\otimes\xi_{\sigma(m-1),\tau,l(m)}}p_\tau S_{\xi_{\rho,\tau(1),l(1)}\otimes\xi_{\tau(1),\tau(2),l(2)}\otimes\cdots\otimes\xi_{\tau(m-1),\tau,l(m)}}^*). \end{aligned}$$

Hence the automorphism α_U satisfies $\alpha_U(T) = \text{Adu}(T)$ for $T \in \mathcal{F}_m$. Since

$$D^{km}(\sigma, \tau) = D^m(\sigma, \sigma(1)) \dots D^m(\sigma(k-1), \tau) = d(\sigma)d(\tau_0)^k\overline{d(\tau)},$$

by the same argument as above, we get

$$\alpha_U(T) = \text{Adu}(T)$$

for $T \in \mathcal{F}_{km}$. Then $\alpha_U = \text{Adu}$ on $\mathcal{F}_X$. On the other hand, $\alpha_U = \text{Ad}V$ on $\mathcal{O}_X$ for $V \in \mathcal{F}_X$. Since $\mathcal{F}_X$ is simple, we conclude that $V = \lambda u$ for a scalar $\lambda, |\lambda| = 1$. We compute

$$\begin{aligned} C_{\sigma,\tau}S_{\xi_{\sigma,\tau,l}} &= \alpha_U(S_{\xi_{\sigma,\tau,l}}) \\ &= uS_{\xi_{\sigma,\tau,l}}u^* \\ &= d(\sigma)\overline{d(\tau)}S_{\xi_{\sigma,\tau,l}}. \end{aligned}$$

Finally we get $C_{\sigma,\tau} = d(\sigma)\overline{d(\tau)}$, and

$$U\xi_{\sigma,\tau,l} = u\xi_{\sigma,\tau,l}u^*$$

for all σ, τ, l . We conclude that $Ux = uxu^*$ for $x \in X$. $\square$

When we consider the imprimitivity bimodule ${}_aB$ defined in (1.2), The C^* -algebras $\mathcal{F}_aB$ and $\mathcal{O}_aB$ are isomorphic to B and the crossed product $B \rtimes_a \mathbb{Z}$ respectively. Let U be an invertible operator defined by

$$Ub = \alpha(b)$$

for $b \in {}_aB$. Then the automorphism α_U is inner in $\mathcal{O}_aB = B \rtimes_a \mathbb{Z}$ with $\alpha_U = \text{Ad}S_I^*$ where I is an identity of ${}_aB$. Therefore, for our purpose, we need the assumption that the Hilbert B -bimodule X is not an imprimitivity bimodule.

Theorem 2.7. *Let X be a full self conjugate Hilbert B -bimodule of finite type and X is not similar to an imprimitivity Hilbert B -bimodule. The C^* -algebra B is X -aperiodic with $Z(B) = \mathbb{C}$. Then the automorphism α_U is inner on $\mathcal{O}_X$ if and only if*

$$Ux = uxu^*$$

for some unitary u in B and all $x \in X$ and the automorphism θ is implemented by u .

Proof. Since X is a self conjugate Hilbert B -bimodule with its conjugate Hilbert B -bimodule $\bar{X}$, There exists Jones projection e_X in ${}_B\mathcal{L}_B(X \otimes_B \bar{X}) = {}_B\mathcal{L}_B(X \otimes_B X) \simeq B' \cap \mathcal{F}_2$ such that

$$e_X(x \otimes \bar{x}') = (\text{r-ind}[X])^{-1} \sum_i u_i \otimes \bar{u}_i \quad {}_B \langle x, x' \rangle \quad (2.8)$$

where $x \in X$ and $\bar{x}' \in \bar{X}$ and $\text{r-ind}[X]$ is a right index of X ([4]). Suppose that the projection e_X is an identity. The projection e_X induces the conditional expectation F from $\mathcal{L}_B(X_B)$ to B as follows:

$$F(T) = (\text{r-ind}[X])^{-1} \sum_i {}_B \langle T u_i, u_i \rangle \quad (2.9)$$

for $T \in \mathcal{L}_B(X_B)$ ([4] Proposition 3.2) and

$$e_X(T \otimes I)e_X = (F(T) \otimes I)e_X.$$

Therefore the fact $e_X = I$ leads us that the expectation $F = I$. Hence by (2.9), we have

$$\begin{aligned} x \langle y, z \rangle_B &= \theta_{x,y} z = F(\theta_{x,y}) z \\ &= (\text{r-ind}[X])^{-1} \sum_i {}_B \langle \theta_{x,y} u_i, u_i \rangle = (\text{r-ind}[X])^{-1} {}_B \langle x, y \rangle z. \end{aligned}$$

Defining a new left inner product ${}_B \langle x, y \rangle'$ on X by

$${}_B \langle x, y \rangle' = (\text{r-ind}[X])^{-1} {}_B \langle x, y \rangle,$$

the Hilbert B -bimodule X is similar to an imprimitivity Hilbert B -bimodule. This is a contradiction. Therefore $B' \cap \mathcal{F}_X$ contains non trivial projection $e_X \in B' \cap \mathcal{F}_2$. By the same proof as the cases (i) and (ii) in Theorem 2.5, we obtain that the automorphism α_U is of the form:

$$\alpha_U(T) = V T V^*$$

for some unitary $V \in \mathcal{F}_X$ and all $T \in \mathcal{O}_X$. By Proposition 2.4, we get

$$Ux = \lambda u x u^*$$

for some unitary $u \in B$ and $\lambda \in \mathbb{C}, |\lambda| = \aleph$. Since $\mathcal{F}_X$ is simple and

$$u^* V S_x V^* u = u^* \alpha_U(S_x) u = \lambda S_x$$

for $x \in X$, the element $u^* V$ in $\mathcal{F}_X$ is contained in the center $Z(\mathcal{F}_X) = \mathbb{C}$. Hence $V = \gamma u$ for some $\gamma \in \mathbb{C}, |\gamma| = \aleph$. We finally obtain that

$$S_{Ux} = \alpha_U(S_x) = V S_x V^* = u S_x u^* = S_{uxu^*}$$

and

$$Ux = uxu^*$$

for $x \in X$. □

Professor T. Kajiwara teaches us the existence of Jones projection e_X for a bimodule X .

Example 2.8. The Hilbert B -bimodule ${}_B A_B$, induced by a C^* -inclusion $(B \subset A, E)$ of finite index type with index $E > 1$ ([6]), is always full, self conjugate and not similar to an imprimitivity Hilbert B -bimodule. If the C^* -algebra B is simple, it is clear that $\mathcal{F}_{{}_B A_B}$ is simple. Therefore for $\varphi \in \text{Aut}(B, A) := \{\varphi \in \text{Aut}(A) :$

$\varphi(B) = B\}$, α_φ is inner if and only if $\varphi(a) = uau^*$ for $a \in A$ and some unitary $u \in B$.

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