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Kyoto University
Conserved Quantities of "Random-Time Toda Equation"

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ABSTRACT: "Random-time Toda equation" is obtained by replacing the time-interval of the discrete-time Toda equation by random variables. The random-time Toda equation has higher-order conserved quantities in spite of the randomness introduced to the equation. Also obtained are the higher-order conserved quantities of a class of "Random-time soliton equations" which are related to the random-time Toda equation via Miura transformations.

KEYWORDS: Toda equation, randomness, conserved quantities, Miura transformation

In this letter we present "Random-time Toda equation" where the time-interval of the discrete-time Toda equation are replaced by random variables, and show a Lax pair of the
random-time Toda equation which gives higher-order conserved quantities in spite of the randomness introduced to the equation.

We have the Toda equation of the form

\[
\frac{d}{dt} J_n = V_{n-1} - V_n, \tag{1}
\]
\[
\frac{d}{dt} \log V_n = J_n - J_{n+1}, \tag{2}
\]

which we discretized in a previous paper \(^1\) in the following form

\[
J_n^{m+1} - \delta V_{n-1}^{m+1} = J_n^m - \delta V_n^m, \tag{3}
\]
\[
V_n^{m+1}(1 - \delta J_n^{m+1}) = V_n^m(1 - \delta J_n^{m+1}), \tag{4}
\]

where \(\delta\) is the time-interval and \(t = m\delta\) for integers \(m\). We called a couple of equations (3) and (4) "Discrete-time Toda equation".

Now we replace the time-interval \(\delta\) in Eqs.(3) and (4) by random variables \(\delta^n\) in the following way

\[
J_n^{m+1} - \delta^{m+1} V_{n-1}^{m+1} = J_n^m - \delta V_n^m, \tag{5}
\]
\[
V_n^{m+1}(1 - \delta^{m+1} J_n^{m+1}) = V_n^m(1 - \delta^{m} J_n^{m+1}), \tag{6}
\]

which we call "Random-time Toda Equation".

Let us introduce new dependent variables \(x_n, \hat{x}_n, y_n, \hat{y}_n\), by the following relations:

\[
x_n^m = J_n^m - \delta^m V_{n-1}^m, \tag{7}
\]
\[
\hat{x}_n^m = J_n^m - \delta^m V_n^m, \quad (8)
\]
\[
y_n^m = V_n^m (1 - \delta^m J_n^m), \quad (9)
\]
\[
\hat{y}_n^m = V_n^m (1 - \delta^m J_{n+1}^m). \quad (10)
\]

Then the random-time Toda equation is written in a simple form:

\[
x_{n}^{m+1} = \hat{x}_{n}^m, \quad (11)
\]
\[
y_{n}^{m+1} = \hat{y}_{n}^m. \quad (12)
\]

Then, a Lax pair \{L, A\} of the random-time Toda equation under the periodic boundary conditions:

\[
V_{N+1}^m = V_1^m, \quad (13)
\]
\[
\xi_{N+1}^m = \xi_1^m \quad (14)
\]

is expressed as follows.

\[
L^m = \begin{pmatrix}
1 - c^m x_1^m & c^m & 0 & \cdots & 0 & c^m y_N^m \\
c^m y_1^m & 1 - c^m x_2^m & c^m & \cdots & 0 & 0 \\
0 & c^m y_2^m & 1 - c^m x_3^m & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 1 - c^m x_{N-1}^m & c^m \\
c^m & 0 & 0 & \cdots & c^m y_{N-1}^m & 1 - c^m x_N^m
\end{pmatrix}
\]
$A^m$ are arbitrary constants.

It is easy to see that a commutation relation:

$$A^m L^{m+1} - L^m A^m = 0$$

(16)

gives the random-time Toda equation under the periodic boundary condition.

Eq. (16) gives higher conserved quantities $H_n = \text{Trace}[L^m]^n$, $n = 1, 2, 3, \ldots$ of the random-time Toda equation because that

$$\text{Trace}[L^{m+1}]^n = \text{Trace}[L^m]^n.$$  

(17)

We have shown in a previous paper 2) that Miura transformations generate higher-order conserved quantities of a class of discrete soliton equations which are related to the discrete-time Toda equation. Similarly we obtain in the present paper higher-order conserved quantities of a class of "Random-time soliton equations" which are related to the random-time Toda equation via Miura transformations.
We have the Random-time Toda equation

\[ J_{n}^{m+1} - \delta^{m+1} V_{n-1}^{m+1} = J_{n}^{m} - \delta^{m} V_{n}^{m}, \]  

(18)

\[ V_{n}^{m+1}(1 - \delta^{m+1} J_{n}^{m+1}) = V_{n}^{m}(1 - \delta^{m} J_{n+1}^{m}), \]  

(19)

which is related to "Random-time Lotka-Volterra equation of type I"

\[ v_{n}^{m+1}(1 - \delta^{m+1} v_{n-1}^{m+1}) = v_{n}^{m}(1 - \delta^{m} v_{n+1}^{m}) \]  

(20)

via the Miura transformation:

\[ V_{n}^{m} = v_{2n+1}^{m}, \]  

(21)

\[ J_{n}^{m} = v_{2n-1}^{m} + v_{2n}^{m} - \delta^{m} v_{2n-1}^{m} v_{2n}^{m}. \]  

(22)

The random-time Lotka-Volterra equation of type I" is related to "Random-time Lotka-Volterra equation of type II"

\[ \frac{w_{n}^{m+1}}{(1 + \delta^{m+1} w_{n-1}^{m+1})(1 + \delta^{m+1} w_{n+1}^{m+1})} = \frac{w_{n}^{m}}{(1 + \delta^{m} w_{n}^{m})(1 + \delta^{m} w_{n+1}^{m})} \]  

(23)

via the Miura transformation:

\[ v_{n}^{m} = \frac{w_{n}^{m}}{1 + \delta^{m} w_{n}^{m}}. \]  

(24)

The random-time Lotka-Volterra equation of type II" is related to "Random-time KdV equation"

\[ \frac{1}{u_{n}^{m+1}} + \delta^{m+1} \frac{1}{u_{n-1}^{m+1}} = \frac{1}{u_{n}^{m}} + \delta^{m} \frac{1}{u_{n+1}^{m}} \]  

(25)
via the Miura transformation:

\[ w_n^m = u_n^m u_{n+1}^m. \]  

Following the same procedure as one developed in the previous paper \(^2\), higher order conserved quantities of these equations are expressed by using the higher order conserved quantities of the random-time Toda equation \( H_n \).

**References**
