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Kyoto University
EXponentially Nash Vector
Bundles With Group Action

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1. Introduction.

Let $G$ be an affine exponentially Nash group. In this note, we are concerned with fundamental properties of exponentially Nash $G$ vector bundles. Our results in the present note are an exposition of [6].

Nash vector bundles (resp. Nash manifolds) are bundles (resp. manifolds) intermediate between real algebraic ones and $C^\omega$ ones. It is known that there are some useful categories between Nash one and $C^\infty$ one (e.g. [3], [11], [12], [13], [14], [24], [25]). One of them is an exponentially Nash category.

Nash manifolds have been studied for a long time and there are many brilliant works (e.g. [1], [2], [10], [17], [18], [19], [20], [21]).

The semialgebraic subsets of $\mathbb{R}^n$ are just the subsets of $\mathbb{R}^n$ definable in the standard structure $\mathbb{R}_{stan} := (\mathbb{R}, <, +, \cdot, 0, 1)$ of the field $\mathbb{R}$ of real numbers [22]. It is known that there are only three useful collections of sets definable in $\mathbb{R}_{stan}$ [15]. These collections are the sets of semilinear sets, semibounded sets, and semialgebraic sets. However any non-polynomially bounded function is not definable in $\mathbb{R}_{stan}$, where a polynomially bounded function means a function $f: \mathbb{R} \rightarrow \mathbb{R}$ admitting an integer $N \in \mathbb{N}$ and a real number $x_0 \in \mathbb{R}$ with $|f(x)| \leq x^N$, $x > x_0$. C. Miller [16] proved that if there exists a non-polynomially bounded function definable in an o-minimal expansion $(\mathbb{R}, <, +, 0, 1, \ldots)$ of $\mathbb{R}_{stan}$, then the exponential function $exp: \mathbb{R} \rightarrow \mathbb{R}$ is definable in this structure. Hence $\mathbb{R}_{exp} := (\mathbb{R}, <, +, exp, 0, 1)$ is a natural expansion of $\mathbb{R}_{stan}$. There are a number of results on $\mathbb{R}_{exp}$ (e.g. [11], [12], [13], [14], [25]), in particular $\mathbb{R}_{exp}$ is o-minimal. Since $\mathbb{R}_{exp}$ does not have elimination of quantifiers, in $\mathbb{R}_{exp}$ Tarski-Seidenberg Theorem does not hold true. Remark that there are another expansions of $\mathbb{R}_{stan}$ with similar properties of $\mathbb{R}_{exp}$ ([3], [4], [25]).

We say that a $C^r$ manifold ($0 \leq r \leq \omega$) is an exponentially $C^r$ Nash manifold if it is definable in $\mathbb{R}_{exp}$ (See Definition 2.5). Equivariant such manifolds are defined in the similar way (See Definition 2.8). Equivaraint exponentially Nash vector bundles are defined as well as Nash ones (See Definition 2.11).

In this note, all exponentially Nash groups, all exponentially Nash $G$ manifolds and exponentially Nash $G$ vector bundles are of class $C^\omega$, and every manifold does not have boundary unless otherwise stated.

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Theorem [6]. Let \( G \) be a compact affine exponentially Nash group and let \( X \) be a compact affine exponentially Nash \( G \) manifold.

(1) For every \( C^\infty G \) vector bundle \( \eta \) over \( X \), there exists a strongly exponentially Nash \( G \) vector bundle (See Definition 2.13) \( \zeta \) which is \( C^\infty G \) vector bundle isomorphic to \( \eta \).

(2) For any two strongly exponentially Nash \( G \) vector bundles over \( X \), they are exponentially Nash \( G \) vector bundle isomorphic if and only if they are \( C^0 G \) vector bundle isomorphic.

(3) If \( \dim X \geq 1 \) and \( X \) has a 0-dimensional orbit, then for any \( C^\infty G \) vector bundle \( \eta' \) of positive rank over \( X \), there exists a non-strongly exponentially Nash \( G \) vector bundle \( \zeta' \) which is \( C^\infty G \) vector bundle isomorphic to \( \eta' \).

In the equivariant Nash category, a stronger version of Theorem (3) holds true [9]. Remark that Nash structures of \( C^\infty G \) manifolds and \( C^\infty G \) vector bundles are studied in [8] and [5], respectively.

2. Exponentially Nash \( G \) manifolds and exponentially Nash \( G \) vector bundles.

Recall the definition of exponentially Nash \( G \) manifolds and exponentially Nash \( G \) vector bundles [7] and basic facts [7].

Definition 2.1. (1) An \( \mathbb{R}_{exp} \)-term is a finite string of symbols obtained by repeated applications of the following two rules:

[1] Constants and variables are \( \mathbb{R}_{exp} \)-terms.

[2] If \( f \) is an \( m \)-place function symbol of \( \mathbb{R}_{exp} \) and \( t_1, \ldots, t_m \) are \( \mathbb{R}_{exp} \)-terms, then the concatenated string \( f(t_1, \ldots, t_m) \) is an \( \mathbb{R}_{exp} \)-term.

(2) An \( \mathbb{R}_{exp} \)-formula is a finite string of \( \mathbb{R}_{exp} \)-terms satisfying the following three rules:

[1] For any two \( \mathbb{R}_{exp} \)-terms \( t_1 \) and \( t_2 \), \( t_1 = t_2 \) and \( t_1 > t_2 \) are \( \mathbb{R}_{exp} \)-formulas.

[2] If \( \phi \) and \( \psi \) are \( \mathbb{R}_{exp} \)-formulas, then the negation \( \neg \phi \), the disjunction \( \phi \lor \psi \), and the conjunction \( \phi \land \psi \) are \( \mathbb{R}_{exp} \)-formulas.

[3] If \( \phi \) is an \( \mathbb{R}_{exp} \)-formula and \( v \) is a variable, then \( (\exists v)\phi \) and \( (\forall v)\phi \) are \( \mathbb{R}_{exp} \)-formulas.

(3) An exponentially definable set \( X \subset \mathbb{R}^n \) is the set defined by an \( \mathbb{R}_{exp} \)-formula (with parameters).

(4) Let \( X \subset \mathbb{R}^n \) and \( Y \subset \mathbb{R}^m \) be exponentially definable sets. A map \( f : X \longrightarrow Y \) is called exponentially definable if the graph of \( f \subset \mathbb{R}^n \times \mathbb{R}^m \) is exponentially definable.

On the other hand, using [12] any exponentially definable subset of \( \mathbb{R}^n \) is the image of an \( \mathcal{R}_{n+m} \)-semianalytic set by the natural projection \( \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \) for some \( m \). Here a subset \( X \) of \( \mathbb{R}^n \) is called \( \mathcal{R}_{n} \)-semianalytic if \( X \) is a finite union of sets of the following form:

\[ \{ x \in \mathbb{R}^n | f_i(x) = 0, g_j(x) > 0, 1 \leq i \leq k, 1 \leq j \leq l \}, \]

where \( f_i, g_j \in \mathbb{R}[x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)] \).

The following is a collections of properties of exponentially definable sets (cf. [7]).
Proposition 2.2 (cf. [7]). (1) Any exponentially definable set consists of only finitely many connected components.

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be exponentially definable sets.

(2) The closure $\text{Cl}(X)$ and the interior $\text{Int}(X)$ of $X$ in $\mathbb{R}^n$ are exponentially definable.

(3) The distance function $d(x, X)$ from $x$ to $X$ defined by $d(x, X) = \inf\{||x - y|||y \in X\}$ is a continuous exponentially definable function, where $||\cdot||$ denotes the standard norm of $\mathbb{R}^n$.

(4) Let $f : X \rightarrow Y$ be an exponentially definable map. If a subset $A$ of $X$ is exponentially definable then so is $f(A)$, and if $B \subset Y$ is exponentially definable then so is $f^{-1}(B)$.

(5) Let $Z \subset \mathbb{R}^l$ be an exponentially definable set and let $f : X \rightarrow Y$ and $h : Y \rightarrow Z$ be exponentially definable maps. Then the composition $h \circ f : X \rightarrow Z$ is also exponentially definable. In particular for any two polynomial functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$, the function $h : \mathbb{R} - \{f = 0\} \rightarrow \mathbb{R}$ defined by $h(x) = e^{g(x)/f(x)}$ is exponentially definable.

(6) The set of exponentially definable functions on $X$ forms a ring.

(7) Any two disjoint closed exponentially definable sets $X$ and $Y \subset \mathbb{R}^n$ can be separated by a continuous exponentially definable function. \[\square\]

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open exponentially definable sets. A $C^r$ ($0 \leq r \leq \omega$) map $f : U \rightarrow V$ is called an exponentially $C^r$ Nash map if it is exponentially definable. An exponentially $C^r$ Nash map $g : U \rightarrow V$ is called an exponentially $C^r$ Nash diffeomorphism if there exists an exponentially $C^r$ Nash map $h : V \rightarrow U$ such that $g \circ h = \text{id}$ and $h \circ g = \text{id}$. Remark that the graph of an exponentially $C^r$ Nash map may be defined by an $\mathcal{R}_{\exp}$-formula with quantifiers.

Theorem 2.3 [14]. Let $S_1, \ldots, S_k \subset \mathbb{R}^n$ be exponentially definable sets. Then there exists a finite family $\mathcal{W} = \{\Gamma^d_\alpha\}$ of subsets of $\mathbb{R}^n$ satisfying the following four conditions:

(1) $\Gamma^d_\alpha$ are disjoint, $\mathbb{R}^n = \bigcup_{\alpha, d} \Gamma^d_\alpha$ and $S_i = \bigcup\{\Gamma^d_\alpha|\Gamma^d_\alpha \cap S_i \neq \emptyset\}$ for $1 \leq i \leq k$.

(2) Each $\Gamma^d_\alpha$ is an analytic cell of dimension $d$.

(3) $\overline{\Gamma^d_\alpha} - \Gamma^d_\alpha$ is a union of some cells $\Gamma^e_\beta$ with $e < d$.

(4) If $\Gamma^d_\alpha, \Gamma^e_\beta \in \mathcal{W}$, $\Gamma^e_\beta \subset \overline{\Gamma^d_\alpha} - \Gamma^d_\alpha$ then $(\Gamma^d_\alpha, \Gamma^e_\beta)$ satisfies Whitney's conditions (a) and (b) at all points of $\Gamma^e_\beta$. \[\square\]

Theorem 2.3 allows us to define the dimension of an exponentially definable set $E$ by

$$\dim E = \max\{\dim \Gamma|\Gamma \text{ is an analytic submanifold contained in } E\}.$$ 

Example 2.4. (1) The $C^\infty$ function $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\lambda(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
 e^{(-1/x)} & \text{if } x > 0
\end{cases}$$

is an exponentially $C^\infty$ Nash map. This example shows that an exponentially definable $C^\infty$ map is not always analytic. This phenomenon does not occur in the usual Nash category. Notice that every $C^\infty$ Nash map is a $C^\infty$ Nash map.
The Zariski closure of the graph of the exponential function \( \exp : \mathbb{R} \to \mathbb{R} \) in \( \mathbb{R}^2 \) is the whole space \( \mathbb{R}^2 \). Hence the dimension of the graph of \( \exp \) is smaller than that of its Zariski closure.

The continuous function \( h : \mathbb{R} \to \mathbb{R} \) defined by

\[
h(x) = \begin{cases} 
  e^{x-n} & \text{if } n \leq x \leq n+1 \\
  e^{n+2-x} & \text{if } n+1 \leq x \leq n+2
\end{cases}
\]

for \( n \in 2\mathbb{Z} \),

is not exponentially definable, but the restriction of \( h \) on any bounded exponentially definable set is exponentially definable.

**Definition 2.5.** Let \( r \) be a non-negative integer, \( \infty \) or \( \omega \).

1. An exponentially \( C^r \) Nash manifold \( X \) of dimension \( d \) is a \( C^r \) manifold admitting a finite system of charts \( \{ \phi_i : U_i \to \mathbb{R}^d \} \) such that for each \( i \) and \( j \), \( \phi_i(U_i \cap U_j) \) is an open exponentially definable subset of \( \mathbb{R}^d \) and the map \( \phi_j \circ \phi_i^{-1} \mid \phi_i(U_i \cap U_j) \) is an exponentially \( C^r \) Nash diffeomorphism (an exponentially Nash homeomorphism if \( r = 0 \)). We call these atlas exponentially \( C^r \) Nash. Exponentially \( C^r \) Nash manifolds with compatible atlases are identified. A subset \( M \) of \( X \) is called exponentially definable if every \( \phi_i(U_i \cap M) \) is exponentially definable.

2. An exponentially definable subset \( X \) of \( \mathbb{R}^n \) is called a \( d \)-dimensional exponentially \( C^r \) Nash submanifold of \( \mathbb{R}^n \) if for any \( x \in X \) there exists an exponentially \( C^n \) Nash diffeomorphism \( \phi \) from some open exponentially definable neighborhood \( U \) of the origin in \( \mathbb{R}^n \) onto some open exponentially definable neighborhood \( V \) of \( x \) in \( \mathbb{R}^n \) such that \( \phi(0) = x, \phi(U \cap R^n) = X \cap V \). Here \( R^d \) denotes the subset of \( \mathbb{R}^n \) those which the last \( (n-d) \) components are zero. An exponentially \( C^n \) \( (r > 0) \) Nash submanifold is of course an exponentially \( C^n \) Nash manifold [7].

3. Let \( X \) (resp. \( Y \)) be an exponentially \( C^n \) Nash manifold with exponentially \( C^n \) Nash atlas \( \{ \phi_i : U_i \to \mathbb{R}^n \} \) (resp. \( \{ \psi_j : V_j \to \mathbb{R}^m \} \)). A \( C^r \) map \( f : X \to Y \) is said to be an exponentially \( C^r \) Nash map if for any \( i \) and \( j \), \( \phi_i(f^{-1}(V_j) \cap U_i) \) is open and exponentially definable in \( \mathbb{R}^n \), and that the map \( \psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \to \mathbb{R}^m \) is an exponentially \( C^r \) Nash map.

4. Let \( X \) and \( Y \) be exponentially \( C^n \) Nash manifolds. We say that \( X \) is exponentially \( C^n \) Nash diffeomorphic to \( Y \) if one can find exponentially \( C^n \) Nash maps \( f : X \to Y \) and \( h : Y \to X \) such that \( f \circ h = id \) and \( h \circ f = id \).

5. An exponentially \( C^n \) Nash manifold is said to be \( C^n \) affine if it is exponentially \( C^n \) Nash diffeomorphic to some exponentially \( C^n \) Nash submanifold of \( \mathbb{R}^d \). We simply write \( affine \) instead of \( C^n \) affine if \( r = \omega \).

Remark that any \( C^\omega \) Nash manifold is a \( C^\omega \) Nash manifold, but there exists an exponentially \( C^\omega \) Nash manifold which is not an exponentially \( C^\omega \) Nash manifold (See Example 2.4).

**Definition 2.6.** (1) A group \( G \) is called an exponentially \( C^r \) Nash group (resp. an affine exponentially \( C^r \) Nash group) if \( G \) is an exponentially Nash manifold (resp. an affine exponentially Nash manifold) and that the multiplication \( G \times G \to G \) and the inversion \( G \to G \) are exponentially Nash maps.

(2) Let \( G \) be an exponentially Nash group. A representation of \( G \) means a group homomorphism from \( G \) to some \( GL(\mathbb{R}^d) \) which is an exponentially Nash map. We use a representation as a representation space.
Example 2.7. Subgroups

\[ \{ \exp(t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) | t \in \mathbb{R} \} \text{ and } \{ \exp(t \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) | t \in \mathbb{R} \} \]

of \( GL_2(\mathbb{R}) \) are exponentially Nash groups but not Nash ones.

Definition 2.8. Let \( G \) be an exponentially Nash group and let \( r \) be a non-negative integer, \( \infty \) or \( \omega \).

(1) An exponentially \( C^r \) Nash submanifold in a representation of \( G \) is called an exponentially \( C^r \) Nash \( G \) submanifold if it is \( G \) invariant.

(2) An exponentially \( C^r \) Nash \( G \) manifold is a pair \((X,\theta)\) consisting of an exponentially \( C^r \) Nash manifold \( X \) and a group action \( \theta \) of \( G \) on \( X \), such that \( \theta : G \times X \rightarrow X \) is an exponentially \( C^r \) Nash map. For simplicity of notation, we write \( X \) instead of \((X,\theta)\).

(3) Let \( X \) and \( Y \) be exponentially \( C^r \) Nash \( G \) manifolds. An exponentially \( C^r \) Nash map \( f : X \rightarrow Y \) is called an exponentially \( C^r \) Nash \( G \) map if it is a \( G \) map. An exponentially \( C^r \) Nash \( G \) map \( g : X \rightarrow Y \) is said to be an exponentially \( C^r \) Nash \( G \) diffeomorphism if there exists an exponentially \( C^r \) Nash \( G \) map \( h : Y \rightarrow X \) such that \( g \circ h = \text{id} \) and \( h \circ g = \text{id} \).

(4) We say that an exponentially \( C^r \) Nash \( G \) manifold is \( C^r \) affine if it is exponentially \( C^r \) Nash \( G \) diffeomorphic to an exponentially \( C^r \) Nash \( G \) submanifold of some representation of \( G \). If \( r = \omega \), then we simply write affine instead of \( C^r \) affine.

We have the following implications on groups:

an algebraic group \( \Longrightarrow \) an affine Nash group \( \Longrightarrow \) an affine exponentially Nash group
\( \Longrightarrow \) an exponentially Nash group \( \Longrightarrow \) a Lie group.

Let \( G \) be an algebraic group. Then we obtain the following implications on \( G \) manifolds:

a nonsingular algebraic \( G \) set \( \Longrightarrow \) an affine Nash \( G \) manifold
\( \Longrightarrow \) an affine exponentially Nash \( G \) manifold \( \Longrightarrow \) an exponentially Nash \( G \) manifold \( \Longrightarrow \) a \( C^\infty \) \( G \) manifold.

Moreover, notice that a Nash \( G \) manifold is not always an affine exponentially Nash \( G \) manifold.

In the equivariant exponentially Nash category, the equivariant tubular neighborhood result holds true [7].

Proposition 2.9 [7]. Let \( G \) be a compact affine exponentially Nash group and let \( X \) be an affine exponentially Nash \( G \) submanifold possibly with boundary in a representation \( \Omega \) of \( G \). Then there exists an exponentially Nash \( G \) tubular neighborhood \((U, p)\) of \( X \) in \( \Omega \), namely \( U \) is an affine exponentially Nash \( G \) submanifold in \( \Omega \) and the orthogonal projection \( p : U \rightarrow X \) is an exponentially Nash \( G \) map. \( \square \)

The following lemma is useful to prove the existence of nonaffine exponentially Nash manifolds, which is a generalization of the usual Nash case (I.2.2.15 [21]).
Proposition 2.10 [7]. Let $M$ and $N$ be exponentially Nash manifolds and let $h : M \to N$ be a locally exponentially Nash map. If $N$ is affine then $h$ is an exponentially Nash map. Here we say that $h$ is locally exponentially Nash if for any $x \in M$ and $f(x) \in N$, there exist open exponentially definable neighborhoods $U$ of $x$ in $M$ and $V$ of $f(x)$ in $N$ such that $f(U) \subset V$ and $f[U : U \to V$ is an exponentially Nash map. \hfill \Box

Definition 2.11. Let $G$ be an exponentially Nash group and let $r$ be a non-negative integer, $\infty$ or $\omega$.

1. A $C^r G$ vector bundle $(E, p, X)$ of rank $k$ is said to be an exponentially $C^r$ Nash $G$ vector bundle if the following three conditions are satisfied:
   
   (a) The total space $E$ and the base space $X$ are exponentially $C^r$ Nash $G$ manifolds.
   
   (b) The projection $p$ is an exponentially $C^r$ Nash $G$ map.
   
   (c) There exists a family of finitely many local trivializations $\{U_i, \phi_i : U_i \times \mathbb{R}^k \to p^{-1}(U_i) \}$ such that $\{U_i\}$ is an open exponentially definable covering of $X$ and that for any $i$ and $j$ the map $\phi_i^{-1} \circ \phi_j(U_i \cap U_j) \times \mathbb{R}^k : (U_i \cap U_j) \times \mathbb{R}^k \to (U_i \cap U_j) \times \mathbb{R}^k$ is an exponentially $C^r$ Nash map.

We call these local trivializations exponentially $C^r$ Nash.

2. Let $\eta = (E, p, X)$ (resp. $\zeta = (F, q, X)$) be an exponentially $C^r$ Nash $G$ vector bundle of rank $n$ (resp. $m$). Let $\{U_i, \phi_i : U_i \times \mathbb{R}^n \to p^{-1}(U_i)\}$ (resp. $\{V_j, \psi_j : V_j \times \mathbb{R}^m \to q^{-1}(V_j)\}$) be exponentially $C^r$ Nash local trivializations of $\eta$ (resp. $\zeta$). A $C^r G$ vector bundle map $f : \eta \to \zeta$ is said to be an exponentially $C^r$ Nash $G$ vector bundle map if for any $i$ and $j$ the map $(\psi_j)^{-1} \circ f \circ \phi_i(U_i \cap U_j) \times \mathbb{R}^n : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^m$ is an exponentially $C^r$ Nash map. A $C^r G$ section $s$ of $\eta$ is called exponentially $C^r$ Nash if each $\phi_i^{-1} \circ s|U_i : U_i \to U_i \times \mathbb{R}^n$ is exponentially $C^r$ Nash.

3. Two exponentially $C^r$ Nash $G$ vector bundles $\eta$ and $\zeta$ are said to be exponentially $C^r$ Nash $G$ vector bundle isomorphic if there exist exponentially $C^r$ Nash $G$ vector bundle maps $f : \eta \to \zeta$ and $h : \zeta \to \eta$ such that $f \circ h = id$ and $h \circ f = id$.

Recall universal $G$ vector bundles (cf. [5]).

Definition 2.12. Let $G$ be a compact exponentially Nash group. Let $\Omega$ be an $n$-dimensional representation of $G$ and $B$ the representation map $G \to GL_n(\mathbb{R})$ of $\Omega$. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$-matrices with the action $(g, A) \in G \times M(\Omega) \to B(g)^{-1}AB(g) \in M(\Omega)$. For any positive integer $k$, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:

$$G(\Omega, k) = \{A \in M(\Omega)|A^2 = A, A = A', Tr A = k\},$$

$$E(\Omega, k) = \{(A, v) \in G(\Omega, k) \times \Omega|Av = v\},$$

$$u : E(\Omega, k) \to G(\Omega, k) : u((A, v)) = A,$$

where $A'$ denotes the transposed matrix of $A$ and $Tr A$ stands for the trace of $A$. Then $\gamma(\Omega, k)$ is an algebraic set. Since the action on $\gamma(\Omega, k)$ is algebraic, it is an algebraic $G$ vector bundle. We call it the universal $G$ vector bundle associated with $\Omega$ and $k$. Since $G(\Omega, k)$ and $E(\Omega, k)$ are nonsingular, $\gamma(\Omega, k)$ is a Nash $G$ vector bundle, hence it is an exponentially Nash one.
Definition 2.13. Let $G$ be a compact exponentially Nash group and let $X$ be an exponentially $C^r$ Nash $G$ manifold. An exponentially $C^r$ Nash $G$ vector bundle $\eta = (E, p, X)$ of rank $k$ is said to be strongly exponentially $C^r$ Nash if the base space $X$ is $C^r$ affine and that there exist some representation $\Omega$ of $G$ and an exponentially $C^r$ Nash $G$ map $f : X \to G(\Omega, k)$ such that $\eta$ is exponentially $C^r$ Nash $G$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$. If $r = \omega$, then strongly exponentially $C^r$ Nash is abbreviated to strongly exponentially Nash.

Let $G$ be a compact Nash group. Then we have the following implications on $G$ vector bundles over an affine Nash $G$ manifold:

a Nash $G$ vector bundle $\implies$ an exponentially Nash $G$ vector bundle $\implies$ a $C^\omega$ vector bundle, and

a strongly Nash $G$ vector bundle $\implies$ a strongly exponentially Nash $G$ vector bundle $\implies$ an exponentially Nash $G$ vector bundle.

3. Sketch of proof.

Sketch of proof of Theorem (1) and (2). We now give a sketch of proof of (1). Since $G$ and $X$ are compact, there exist a representation $\Omega$ of $G$ and a $C^\infty G$ map $f : X \to G(\Omega, k)$ such that $\eta$ is $C^\infty G$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where $k$ denotes the rank of $\eta$. Averaging a polynomial approximation of $f$ and by Proposition 2.9, we have an exponentially Nash $G$ map $h : X \to \gamma(\Omega, k)$ which approximates $f$. By [24], $\zeta := h^*(\gamma(\Omega, k))$ is the required one.

We now sketch the proof of (2). Let $\zeta_1$ and $\zeta_2$ be two strongly exponentially Nash $G$ vector bundles over $X$. Then $\text{Hom}(\zeta_1, \zeta_2)$ is a strongly exponentially Nash $G$ vector bundle. By the assumption, there exists an element $F$ in $\text{Iso}(\zeta_1, \zeta_2)$. Approximating $F$ by an exponentially Nash $G$ section of $\text{Hom}(\zeta_1, \zeta_2)$, we have the desired isomorphism because $\text{Iso}(\zeta_1, \zeta_2)$ is open in $\text{Hom}(\zeta_1, \zeta_2)$. □

We prepare the following result to prove Theorem (3).

Proposition 3.1 [7]. Let $G$ be a compact affine exponentially Nash group and let $\eta = (E, p, Y)$ be an exponentially Nash $G$ vector bundle of rank $k$ over an affine exponentially Nash $G$ manifold $Y$. Then $\eta$ is strongly exponentially Nash if and only if $E$ is affine. □

Sketch of proof of Theorem (3). By Theorem (1) we may assume that $\eta'$ is exponentially Nash $G$ vector bundle. Since $X$ has a 0-dimensional orbit $G(x)$ and by Proposition 2.9, one can find an open $G$ invariant exponentially definable neighborhood $U$ of $G(x)$ such that $\eta' U$ is exponentially Nash $G$ vector bundle isomorphic to $U \times \Xi$ for some representation $\Xi$. Using Proposition 2.9, we can construct three open $G$ invariant exponentially definable subsets of $U$ which cover $U$. We paste their overlaps with a collection of exponentially Nash $G$ diffeomorphisms. By Proposition 2.10, we can show that the total space of the resulting exponentially Nash $G$ vector bundle $\zeta'$ is nonaffine. Therefore we have (3) by Proposition 3.1. □

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