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Kyoto University
THE MODULI SPACE OF ONCE PUNCTURED ELLIPTIC CURVES
WITH LAGRANGIAN SUBLATTICES

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1. INTRODUCTION

We introduced a new method of constructing once punctured Riemann surfaces in [H-O1] (see also [H-O2]). In our construction we use line segments in the complex plane $\mathbb{C}$ and parallel transformations: For a pair of disjoint parallel line segments with the same length in $\mathbb{C}$, we first cut $\mathbb{C}$ along the segments and paste each side of one segment and the opposite side of the other segment by a parallel transformation obtaining a once punctured elliptic curve. The puncture is at infinity. We shall call such a pair an Igeta. (Igeta is a Japanese word coming from a technical term “Igeta-kuzushi” used in a Japanese martial art.) Putting $g$ disjoint pieces of Igeta on $\mathbb{C}$, we obtain a once punctured Riemann surface of genus $g$ in the same way. (See Figure 1. The numbers (1),..., (6) in Figure 1 indicate where to paste.) We denote a set of $g$ disjoint Igeta by $\Gamma$ and the resulting once punctured Riemann surface by $(R(\Gamma), p_\infty)$. Moreover when we move the position of $g$ Igeta, there appears a family of once punctured Riemann surfaces of genus $g$. All the possible configurations of $g$ disjoint Igeta up to the affine automorphisms of $\mathbb{C}$ form a $3g - 2$-dimensional complex $\Gamma$-manifold and this dimension is the same as the dimension of the moduli space $M_{g,1}$ of once punctured Riemann surfaces of genus $g$. We thus expect to have a visual image of the moduli space by using this construction.

Let $I_g\eta_0$ be the collection of those $\Gamma$ having $[0,1]$ as one of its $2g$ line segments. $I_g\eta_0$ turns out to be a $3g - 2$-dimensional complex manifold. We showed in [H-O1] that the Kodaira-Spencer map

$$\rho_{\Gamma,0} : T(I_g\eta_0)_{\Gamma} \rightarrow H^1(R(\Gamma);\Theta(-p_\infty))$$

is an isomorphism for any $\Gamma \in I_g\eta_0$, where $T(I_g\eta_0)_{\Gamma}$ is the holomorphic tangent space of $I_g\eta_0$ at $\Gamma$ and $\Theta(-p_\infty)$ is the sheaf of germs of holomorphic vector fields on $R(\Gamma)$ having zero at $p_\infty$. This implies that the family of once punctured Riemann surfaces of genus $g$ by Igeta-construction is complete and effectively parametrized at any point for each $g$.

We also showed that any once punctured Riemann surface $(R,p)$ can be obtained from $\mathbb{C}$ by cutting along line segments and pasting by parallel transformations. (Note that Igeta-construction is a special way of cutting and pasting.) This result is obtained by considering a Lagrangian sublattice $\Lambda$ of $R$, a subgroup of $H_1(R;\mathbb{Z})$ which coincides its orthogonal complemet with respect to the intersection form on $H_1(R;\mathbb{Z})$.

When we construct a once punctured Riemann surface $(R(\Gamma), p_\infty)$ from $\Gamma$, $R(\Gamma)$ has a natural Lagrangian sublattice $\Lambda_{\Gamma}$. Igeta-construction leads us to consider the moduli space of once punctured Riemann surfaces with Lagrangian sublattices.

In this paper we consider the case of genus 1, and describe the moduli space using a natural extension of Igeta-construction, that is, we make a complete list of once punctured elliptic curves with Lagrangian sublattices (see §2):
Theorem 1. For any once punctured elliptic curve with a Lagrangian sublattice $(E, p, \Lambda)$, there exists one and only one $(E(a, b, x), p_{\infty}, \Lambda_0)$ isomorphic to $(E, p, \Lambda)$.

For any once punctured Riemann surface $(R, p)$, a Lagrangian sublattice $\Lambda$ and the puncture $p$ determine a certain Abelian differential $\omega_{\Lambda}$ of the second kind up to scalars. To prove Theorem 1 we study the geometry of geodesics on once punctured elliptic curves having metrics with conical singularities induced by the Abelian differentials $\omega_{\Lambda}$. For each $(E(a, b, x), p_{\infty}, \Lambda_0)$, all the closed geodesics can be described visually by using our construction.

In §3 we consider the complex structure of the moduli space of once punctured elliptic curves with Lagrangian sublattices by using the description given in §2.

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2. Elliptic curves with Lagrangian sublattices

In this section, we give a description of all the once punctured elliptic curves with Lagrangian sublattices by using a natural extension of Igeta-construction.

We prepare some notations and terms about the metrics induced by Abelian differentials. For an Abelian differential $\omega$ on a closed Riemann surface $R$, we denote by $\omega^{-1}(0)$...
(resp. $\omega^{-1}(\infty)$) the set of zeros (resp. poles) of $\omega$. We call a simple path or simple loop $\gamma : [0, 1] \to R$ \textit{\omega-line-segment} if its image contains no poles of $\omega$ and the integral

$$\int_{\gamma(0)}^{\gamma(1)} \omega$$

depends on $t \in [0, 1]$ linearly. We also call its image \textit{\omega-line-segment}. Let us denote by $g_\omega$ the flat metric on $R - \omega^{-1}(\infty)$ induced by the 2-form $\frac{i}{2} \omega \wedge \overline{\omega}$, which has conical singularities at $\omega^{-1}(0)$. Then \omega-line-segments are geodesics for $g_\omega$. Let us call a \omega-line-segment $\gamma : [0, 1] \to R - \omega^{-1}(\infty)$ \textit{\omega-edge} if $\gamma([0, 1]) \cap \omega^{-1}(0) = \{\gamma(0), \gamma(1)\}$. Then closed geodesics with respect to $g_\omega$ which contain some zeros of $\omega$ consist of $\omega$-edges because the integral of $\omega$ gives rise to a local isometry between $R - (\omega^{-1}(\infty) \cup \omega^{-1}(0))$ and $\mathbb{C}$.

An Abelian differential $\omega$ of the second kind on a closed Riemann surface $R$ induces an element of $H^1(R; \mathbb{C})$, and further an element denoted by PD$[\omega]$ of $H_1(R; \mathbb{C})$ via the Poincaré duality. It holds that

$$\int_\sigma \omega = \alpha \cdot \text{PD}[\omega] \quad \text{for any } \alpha \in H_1(R; \mathbb{Z}).$$

We look for a singular 1-cycle $\sigma$ representing PD$[\omega]$ such that

$$\sigma = \sum_{k=1}^{N} c_k \gamma_k, \quad c_k \in \mathbb{C}$$

where $\gamma_1, \ldots, \gamma_N$ are $\omega$-edges. (We can find it if $\omega^{-1}(0)$ is nonempty.)

Let $(R, p)$ be a once punctured Riemann surface of genus $g$ and $\Lambda$ a Lagrangian sublattice of $H_1(R; \mathbb{Z})$. The kernel $Z_\Lambda$ of the homomorphism given by Abelian integrals

$$H^0(R; \Omega^1(2p)) \longrightarrow \text{Hom}(\Lambda, \mathbb{C}) \cong \mathbb{C}^g$$

is always one-dimensional because it holds that

$$\dim H^0(R; \Omega^1(2p)) = g + 1$$

from the Riemann-Roch formula and the surjectivity is implied by the bilinear relations of Riemann. Accordingly, a Lagrangian sublattice and a point on the surface determine an Abelian differential up to scalars. (Note that $\Lambda \cong \mathbb{Z}^g$.)

From now on we consider the case of genus one.

Figure 2 is a list of some once punctured elliptic curves with Lagrangian sublattices denoted by $\tilde{E}(a, b, x)$ (or $(E(a, b, x), p_\infty, \Lambda_0)$) constructed from $\mathbb{C}$ by cutting along the line segments and pasting by parallel transformations. (The numbers $(1), (2), \ldots$ in each figure indicate pasting data or where to paste.) They split into the following four types, and each one punctured elliptic curve $(E(a, b, x), p_\infty)$ is constructed as follows:

**I:** (The case where $a = 0, b = 1$, and $x$ is a complex number in the upper half plane $\mathbb{H}$ or a real number in the interval $(0, 1)$.) Cut the complex plane along

$$x([0, 1]) \cup (x([0, 1]) + 1),$$

and paste each side of $x([0, 1])$ and the opposite side of $(x([0, 1]) + 1)$ by a parallel transformation. (Igeta-construction)

**II:** (The case where $a$ and $b$ are relatively prime positive integers, and $x$ is a complex number in the upper half plane $\mathbb{H}$.) Cut the complex plane along

$$x([0, 1]) \cup [0, a + b] \cup (x([0, 1]) + a + b),$$
I \((E(0,1,x), p_{\infty})\) where \(x \in \mathbb{H} \cup (0,1)\).

II \((E(a,b,x), p_{\infty})\) where \(a,b \in \mathbb{Z}^+, (a,b)=1, x \in \mathbb{H}\).

III \((E(a,b,x), p_{\infty})\) where \(a,b \in \mathbb{Z}^+, a \neq b, (a,b)=1, x \in (\max(a,b), a+b)\).

IV \((E(a,b,0), p_{\infty})\) where \(a,b \in \mathbb{Z}^+, (a,b)=1\).

**Figure 2.** The list of \(\tilde{E}(a,b,x)\)'s
and paste each side of $x([0,1])$ and the opposite side of $(x([0,1])+a+b)$ by a parallel transformation and paste the upper side of $[0,a]$ (resp. $[a,a+b]$) and the lower side of $[b,a+b]$ (resp. $[0,b]$) by a parallel transformation.

III: (The case where $a$ and $b$ are distinct relatively prime positive integers, and $x$ is a real number such that $x \in (\max(a,b), a+b)$.) Cut the complex plane along $[0,x]$, and paste the lower side of $[0,x-a]$ (resp. $[x-a,b]$), $[b,x]$ and the upper side of $[a,x]$ (resp. $[x-b,a]$), $[0,x-b]$ by a parallel transformation.

IV: (The case where $a$ and $b$ are relatively prime positive integers, and $x = 0$.) Cut the complex plane along $[0,a+b]$ and paste the upper side of $[0,a]$ (resp. $[a,a+b]$) and the lower side of $[b,a+b]$ (resp. $[0,b]$) by a parallel transformation.

Note that each elliptic curve constructed in this way has a natural Abelian differential $\omega_0$ induced by the differential $d\zeta$ of the standard coordinate $\zeta$ of $\mathbb{C}$ and that the Lagrangian sublattice $\Lambda_0$ of $\tilde{E}(a,b,x)$ is characterized as the kernel of the period map of $\omega_0$ from $H_1(E(a,b,x);\mathbb{Z})$ to $\mathbb{C}$. Further, the primitive period of $\omega_0$ is equal to 1 because $a$ and $b$ are relatively prime. When $\tilde{E}(a,b,x)$ is of type I, II, or III, the set $\omega_0^{-1}(0)$ consists of the two points $p_0$ and $p_1$ coming from the origin and the point $x$ of $\mathbb{C}$, and when $\tilde{E}(a,b,x)$ is of type IV, the set $\omega_0^{-1}(0)$ consists of the one point $p_0$ coming from the origin in $\mathbb{C}$. (See Figure 2.)

Our goal in this section is to prove

**Theorem 1.** For any once punctured elliptic curve with a Lagrangian sublattice $(E,p,\Lambda)$, there exists one and only one $\tilde{E}(a,b,x)$ isomorphic to $(E,p,\Lambda)$.

We call two once punctured Riemann surfaces with Lagrangian sublattices $(R,p,\Lambda)$ and $(R',p',\Lambda')$ isomorphic, if there exists a biholomorphic map from $R$ to $R'$ transforming $p$ to $p'$ and $\Lambda$ into $\Lambda'$.

In order to prove Theorem 1 we investigate the geometry of geodesics on once punctured elliptic curves having metrics with conical singularities induced by Abelian differentials.

For a once punctured elliptic curve with a Lagrangian sublattice $(E,p,\Lambda)$, there exists an Abelian differential $\omega_\Lambda$, unique up to sign, in the kernel $Z_\Lambda$ (see [H-O1] or [H-O2]) such that the primitive period of $\omega_\Lambda$ is equal to 1, because $Z_\Lambda$ is one-dimensional and any non-trivial element in $Z_\Lambda$ has non-trivial periods. Then $\text{PD}[\omega_\Lambda]$ is an integral homology class. Since the metric $g_\omega$ on $E-p$ depends only on $\Lambda$ in this case, we shall shortly denote this metric by $g_\Lambda$. For the same reason we shall also use the term $\Lambda$-edge instead of $\omega_\Lambda$-edge.

We first show that $\tilde{E}(a,b,x)$'s are not isomorphic to one another by studying $\Lambda_0$-edges.

For each $\tilde{E}(a,b,x)$, the Abelian differential $\omega_{\Lambda_0}$ coincides with $\omega_0$ up to sign. So $\Lambda_0$-edges are line segments in $\mathbb{C}$ between the points of $\omega_0^{-1}(0)$ in each case of Figure 2. Accordingly we can easily obtain the following proposition.

**Proposition 1.** All the $\Lambda_0$-edges can be described as in Figure 3 for each $\tilde{E}(a,b,x)$.

Proposition 1 implies that all the closed geodesics containing some zeros of $\omega_0$, which consist of $\Lambda_0$-edges, can be described visually for each $\tilde{E}(a,b,x)$. This proposition yields the following corollary.

**Corollary 1.** If $(a,b,x) \neq (a',b',x')$, then $\tilde{E}(a,b,x)$ is not isomorphic to $\tilde{E}(a',b',x')$.

**Proof.** Suppose $(E,p_\infty,\Lambda_0)$ is isomorphic to $\tilde{E}(a,b,x)$. We shall recover the data $(a,b,x)$ from $(E,p_\infty,\Lambda_0)$. 


Figure 3. $\Lambda_0$-edges
We distinguish the type IV from others by the degeneracy of zeros of $\omega_{\Lambda_0}$, and distinguish the types I, II and III from each other by the properties of $\Lambda_0$-edges on $E$. Let $\{q_0, q_1\}$ be the set of singular points of $g_{\Lambda_0}$. (When $(E, p_\infty, \Lambda_0)$ is of type IV, let $q_0$ be the unique singular point of $g_{\Lambda_0}$.)

When we describe all the directions of $\Lambda_0$-edges around $q_0$, we obtain Figure 4 from Proposition 1. We can specify the $\Lambda_0$-edges $\alpha_0$, $\alpha_1$ and $\alpha_2$ on $E$ in Figure 4.

Note that these figures do not depend on the choice of $q_0$, and that there exists a $\Lambda_0$-edge which has the opposite direction of $\alpha_0$ in the case where $(E, p_\infty, \Lambda_0)$ is of type I, however in the case where $(E, p_\infty, \Lambda_0)$ is of type II there exists no $\Lambda$-edge of this kind. We thus specify the type of $(E, p_\infty, \Lambda)$. We shall orient the $\Lambda_0$-edges $\alpha_0$, $\alpha_1$ and $\alpha_2$ in Figure 4, which are also in Figure 3, from $q_0$ to $q_1$ in the case of type I, II or III.

When $(E, p_\infty, \Lambda_0)$ is of type I or II, we choose the sign of $\omega_{\Lambda_0}$ such that $\text{Im} \int_{\alpha_0} \omega_{\Lambda_0} > 0$. Then we obtain

$$x = \int_{\alpha_0} \omega_{\Lambda_0}.$$
In case \((E, p_{\infty}, \Lambda_{0})\) is of type II, we also obtain the following: (See Figure 3)

\[
\begin{align*}
a &= \int_{\alpha_{0} - \alpha_{2}} \omega_{\Lambda_{0}}, \\
b &= \int_{\alpha_{1} - \alpha_{0}} \omega_{\Lambda_{0}}.
\end{align*}
\]

When \((E, p_{\infty}, \Lambda_{0})\) is of type III, we choose the sign of \(\omega_{\Lambda_{0}}\) such that \(\int_{\alpha_{0}} \omega_{\Lambda_{0}} > 0\). Then we obtain the following: (See Figure 3)

\[
\begin{align*}
a &= \int_{\alpha_{0} - \alpha_{2}} \omega_{\Lambda_{0}}, \\
b &= \int_{\alpha_{1} - \alpha_{0}} \omega_{\Lambda_{0}}, \\
x &= \int_{\alpha_{0} + \alpha_{1} - \alpha_{2}} \omega_{\Lambda_{0}}.
\end{align*}
\]

When \((E, p_{\infty}, \Lambda_{0})\) is of type IV, we obtain the following: (See Figure 3)

\[
\begin{align*}
a &= |\int_{\alpha_{0}} \omega_{\Lambda_{0}}|, \\
b &= |\int_{\alpha_{1}} \omega_{\Lambda_{0}}|.
\end{align*}
\]

We thereby recover the data \((a, b, x)\) from \((E, p_{\infty}, \Lambda_{0})\) in any case. Hence this corollary follows.

Corollary 1 implies the uniqueness in Theorem 1. In order to finish our proof of Theorem 1, we next show the completeness; for a given once punctured elliptic curve with a Lagrangian sublattice \((E, p, \Lambda)\), there exists an \(E(a, b, x)\) which is isomorphic to \((E, p, \Lambda)\).

Let \(\gamma\) be an oriented loop or map from \(S^{1}\) to \(E\) representing a generator of \(\Lambda\). The Abelian differential \(\omega_{\Lambda}\) is exact on \(E - \gamma(S^{1})\) because the integral of \(\omega_{\Lambda}\) along a loop \(\alpha\) equals zero if and only if \(\alpha\) represents a homology class whose intersection number with the homology class \([\gamma]\) equals zero. When we denote by \(||\gamma||\) the length of \(\gamma\) with respect to the metric \(g_{\Lambda}\), it is verified from the following two facts (1), (2) that there exists a loop representing a generator of \(\Lambda\) which has the smallest length among such loops, and that the loop consists of \(\Lambda\)-edges. We shall call a loop consisting of \(\Lambda\)-edges a \(\Lambda\)-edge-loop.

1. For any loop \(\alpha\) on \(E\), there exists a \(\Lambda\)-edge-loop \(\alpha'\) consisting of \(\Lambda\)-edges such that \(\alpha'\) is homotopic to \(\alpha\) and \(||\alpha'|| \leq ||\alpha||\).

2. For any positive number \(r\), there exist at most finitely many \(\Lambda\)-edge-loops of length smaller than \(r\).

We can show that fact (1) follows this way: we first approximate \(\alpha\) by a polygonal loop, and then we reduce the number of vertices which do not lie on \(\omega_{\Lambda}(0)\). On the other hand, for a positive number \(r\), it is obvious that there exist at most finitely many \(\Lambda\)-edges with length smaller than \(r\), and then the fact (2) follows.

**Lemma 1.** The image of an oriented \(\Lambda\)-edge-loop \(\gamma\) representing a generator of \(\Lambda\) contains \(\omega_{\Lambda}^{-1}(0)\).
Proof. The Abelian differential $\omega_{\Lambda}$ has one point as zero of order 2, or has two points as zeros of order 1. We show the lemma in the second case. (In the other case the statement is trivial.) We set $\omega_{\Lambda}^{-1}(0) = \{q_{0}, q_{1}\}$.

Suppose the image of $\gamma$ contains $q_{0}$ and does not contain $q_{1}$. (The image of $\gamma$ consists of only $\Lambda$-edges from $q_{0}$ to $q_{0}$ itself.) We cut along $\gamma$ with $E$ and denote by $\overline{E}$ the resulting Riemann polygon or Riemann surface with line-segment-boundary. (See [H-O1] or [H-O2].) Let $\tilde{q}_{0}$ be a point in $\overline{E}$ corresponding to $q_{0}$. The holomorphic map

$$\Phi(q) = \int_{q_{0}}^{q} \omega_{\Lambda}$$

from $E - (\gamma(S^{1}) \cup \{p\})$ to $\mathbb{C}$ extends to one from $\overline{E}$ to $\mathbb{C}P_{1}(= \mathbb{C} \cup \{p_{\infty}\})$, and $\Phi$ maps a local neighborhood of any point which is not in $\omega_{\Lambda}^{-1}(0)$ to a subset of $\mathbb{C}P_{1}$ biholomorphically. Moreover, $\Phi$ maps the boundary $\partial \overline{E}$ of $\overline{E}$ into the real axis in $\mathbb{C}P_{1}$, because $\omega_{\Lambda}$ only has real periods.

Then we shall find two paths from $q_{1}$ to $p$

$$\alpha_{1}, \alpha_{2} : [0, 1] \longrightarrow \overline{E}$$

such that

$$\alpha_{1}([0, 1]) \cap \alpha_{2}([0, 1]) = \{q_{1}, p\},$$

$$\Phi \circ \alpha_{1}([0, 1]) = (\Phi \circ \alpha_{2})([0, 1]).$$

We fix a path $\beta$ on $\mathbb{C}P_{1}$ from $\Phi(q_{1})$ to $p_{\infty}$ such that $\beta([0, 1]) \cap \Phi(\partial \overline{E}) \subseteq \{\Phi(q_{1})\}$. (See Figure 5.) Around $q_{1}$ we can take two distinct paths

$$\alpha_{1}', \alpha_{2}' : [0, \epsilon] \longrightarrow \overline{E}$$

such that

$$\alpha_{1}'([0, 1]) \cap \alpha_{2}'([0, 1]) = \{q_{1}\},$$

$$\Phi \circ \alpha_{1}'(t) = (\Phi \circ \alpha_{2}')(t) = \beta(t) \quad (t \in [0, \epsilon]).$$
Since \( \Phi \) is a local biholomorphic map except \( q_0, q_1 \), we obtain the desired paths \( \alpha_1, \alpha_2 \) as the pull-back of \( \beta \) by \( \Phi \) which coincide near \( q_1 \) with \( \alpha_1', \alpha_2' \) respectively. Meanwhile, the map \( \Phi \) is biholomorphic around \( p \). This is a contradiction. \( \square \)

We recall three well-known facts about simple loops on elliptic curves.

**Fact (1):** The homology class represented by a simple loop is trivial or primitive.

**Fact (2):** If \( \gamma \) and \( \gamma' \) are simple loops such that their homology classes \( [\gamma], [\gamma'] \) are primitive and their intersection number \( [\gamma] \cdot [\gamma'] \) is equal to zero, then \( [\gamma] = \pm [\gamma'] \).

**Fact (3):** If \( \gamma \) and \( \gamma' \) are simple loops such that their intersection number is equal to \( \pm 1 \), then the pair \(([\gamma], [\gamma'])\) is a basis of \( H_1(E; \mathbb{Z}) \).

Since the integral value of \( \omega_{\Lambda} \) along any \( \Lambda \)-edge does not vanish, the following lemma follows from Fact (2) and the definition of \( \Lambda \)-edge immediately, and we omit its proof. (Note that two distinct parallel \( \Lambda \)-edges have no common points except the points in \( \omega_{\Lambda}^{-1}(0) \).)

**Lemma 2.** Suppose \( \omega_{\Lambda}^{-1}(0) \) consists of two distinct points \( q_0, q_1 \). If \( \gamma_1 \) (resp. \( \gamma_2 \)) is a \( \Lambda \)-edge from \( q_0 \) to \( q_0 \) itself (resp. \( q_1 \) to \( q_1 \) itself), then \( [\gamma_1] = \pm [\gamma_2] \in H_1(E; \mathbb{Z}) \).

Let \( \gamma(0) \) be an oriented \( \Lambda \)-edge-loop representing a generator of \( \Lambda \) and having the smallest length. When we consider \( \gamma(0) \) as an ordered set of oriented \( \Lambda \)-edges, we may obtain another loop \( \gamma'(0) \) representing the same homology class by reordering the oriented \( \Lambda \)-edges suitably. (We assume that each \( \Lambda \)-edge inherits its orientation from \( \gamma(0) \).) We obtain the following lemma about \( \gamma(0) \).

**Lemma 3.** The image of \( \gamma(0) \) consists of \( \Lambda \)-edges which have no intersection with one another except \( \omega^{-1}(0) \).

**Proof.** When \( \omega_{\Lambda} \) has a zero of order 2, the lemma is immediate because all the \( \Lambda \)-edges are parallel. We thus consider the case where \( \omega_{\Lambda} \) has two points \( q_0, q_1 \) as zeros.

We consider \( \gamma(0) \) as an ordered set \((\gamma_1, \gamma_2, \ldots, \gamma_l)\) of oriented \( \Lambda \)-edges. Suppose \( \Lambda \)-edges \( \gamma_i \) and \( \gamma_j \) (\( 1 \leq i \leq j \leq l \)) have an intersection at a point \( q \not\in \omega_{\Lambda}^{-1}(0) \). Then both \( \gamma_i \) and \( \gamma_j \) are \( \Lambda \)-edges between \( q_0 \) and \( q_1 \), and they intersect each other transversely.

We prepare paths \( \gamma'_i \) and \( \gamma'_j \) modifying \( \gamma_i \) and \( \gamma_j \) around \( q \) as in Figure 6.

We replace \( \gamma_i, \gamma_j \) with \( \gamma'_i, \gamma'_j \) respectively, and set
\[
\gamma'_0 = (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_j, \ldots, \gamma_l).
\]
Then the cycle \( \gamma'_0 \) represents the homology class \( [\gamma(0)] \).

In the case (1) or (2), we can consider \( \gamma'_0 \) as a loop, and the length \( ||\gamma'_0|| \) is smaller than \( ||\gamma(0)|| \). This contradicts the definition of \( \gamma(0) \).

In the case (3) or (4), it is necessary to reorder the elements of \( \gamma'_0 \). If there still exist \( \Lambda \)-edges between \( q_0 \) and \( q_1 \) in \( \gamma'_0 \), then it is easy to obtain a loop by reordering the paths and loops \( \{\gamma_1, \ldots, \gamma'_1, \ldots, \gamma'_j, \ldots, \gamma_k\} \) whose length is smaller than \( ||\gamma(0)|| \). This also contradicts the definition of \( \gamma(0) \). If there exist no more \( \Lambda \)-edges between \( q_0 \) and \( q_1 \) in \( \gamma'_0 \), we further replace the loop \( \gamma'_1 \) by \( \Lambda \)-edges from \( q_0 \) to \( q_0 \) itself, the loop \( \gamma'_j \) by \( \Lambda \)-edges from \( q_1 \) to \( q_1 \) itself preserving the homology class. Moreover, by using Lemma 2, we can replace all the \( \Lambda \)-edges from \( q_1 \) to \( q_1 \) itself by \( \Lambda \)-edges from \( q_0 \) to \( q_0 \) preserving the homology class.
Then we obtain a cycle representing the cohomology class \([\gamma_{(0)}]\), and consisting of only \(\Lambda\)-edges from \(q_0\) to \(q_0\) itself. This cycle can be considered as a loop, and this contradicts Lemma 1.

**Proof of Theorem 3.** Let \(\gamma_{(0)}\) be an oriented \(\Lambda\)-edge-loop representing a generator of \(\Lambda\) and having the smallest length as before. We describe \(\gamma_{(0)}\) as an ordered set of oriented \(\Lambda\)-edges:

\[
\gamma_{(0)} = (\gamma_1, \ldots, \gamma_l).
\]

We first consider the case where \(\omega_{\Lambda}^{-1}(0)\) consists of only one point \(q_0\). In this case, each element \(\gamma_k\) of \(\gamma_{(0)}\) is an oriented simple loop on \(E\), and the integral of \(\omega_{\Lambda}\) along \(\gamma_k\) is a non-zero integer. Hence it follows from Fact (2) that \(\gamma_k\) represents a primitive class in \(H_1(E; \mathbb{Z})\). We shall choose the sign of \(\omega_{\Lambda}\) such that

\[
\int_{\gamma_1} \omega_{\Lambda} > 0.
\]

Since the integral of \(\omega_{\Lambda}\) along \(\gamma_{(0)}\) is equal to zero, we may further assume by reordering

\[
\int_{\gamma_2} \omega_{\Lambda} < 0.
\]

The loops \(\gamma_1\) and \(-\gamma_2\) represent different homology classes, because \(\gamma_{(0)}\) has the smallest length. On the other hand, they intersect each other only at \(q_0\). Hence from Facts (1) and (2) it follows that their intersection number at \(q_0\) is equal to 1 or \(-1\) and that the pair \(\{[\gamma_1], [\gamma_2]\}\) is a basis of \(H_1(E; \mathbb{Z})\).

Therefore we can describe the direction as in (i) or (ii) in Figure 7.
Cut $E$ along $\gamma_1$ and $\gamma_2$, and denote by $\tilde{E}$ the resulting Riemann polygon. In each case in Figure 7, by choosing a point $\tilde{q}_0$ in $\partial\tilde{E}$ suitably (see Figure 7), we obtain a holomorphic map $\Phi$ from $\tilde{E}$ to $\mathbb{C}P_1$:

$$\Phi(q) = \int_{\tilde{q}_0}^{q} \omega_{\Lambda}.$$ 

In case (i), set

$$a = \int_{\gamma_1} \omega_{\Lambda},$$
$$b = -\int_{\gamma_2} \omega_{\Lambda}.$$ 

In case (ii), set

$$a = -\int_{\gamma_2} \omega_{\Lambda},$$
$$b = \int_{\gamma_1} \omega_{\Lambda}.$$ 

Then the integers $a$ and $b$ are relatively prime because the pair $([\gamma_1], [\gamma_2])$ is a basis of $H_1(E; \mathbb{Z})$, and then $\Phi$ gives rise to an isomorphism between $(E, p, \Lambda)$ and $\tilde{E}(a, b, 0)$.

We next consider the other case: $\omega_{\Lambda}^{-1}(0)$ consists of two distinct points $q_0$ and $q_1$. In this case, from Lemma 1 there exists at least one $\Lambda$-edge from $q_0$ to $q_1$ and at least one $\Lambda$-edge from $q_1$ to $q_0$ in $\gamma(0)$. We may assume that $\gamma_1$ is a $\Lambda$-edge from $q_0$ to $q_1$. We may further assume from Lemma 2 that $\gamma(0)$ does not contain $\Lambda$-edges from $q_1$ to $q_1$ itself. Since $\gamma_1$ is a $\Lambda$-edge from $q_0$ to $q_1$, the $\Lambda$-edge $\gamma_2$ is from $q_1$ to $q_0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Figure 7}
\end{figure}
Suppose there exists a $\Lambda$-edge $\gamma_i$ from $q_1$ to $q_0$ in $\gamma_{(0)}$ such that

$$\int_{\gamma_i} \omega_\Lambda = - \int_{\gamma_1} \omega_\Lambda.$$  

Because $\gamma_{(0)}$ has the smallest length, the simple loop $(\gamma_1, \gamma_i)$ is non-trivial, and hence primitive (Fact 1). Therefore we obtain $\gamma_{(0)} = (\gamma_1, \gamma_i)$. (Note that a loop $\gamma$ representing a generator of $\Lambda$ is characterized by two properties; one is that $\gamma$ represents a primitive homology class, and the other is that the integral of $\omega_\Lambda$ along $\gamma$ vanishes.)

Cut $E$ along $\gamma_1$ and $\gamma_i$, denote by $\overline{E}$ the resulting Riemann polygon, and set the signature of $\omega_\Lambda$ such that

$$\text{Im}(\int_{\gamma_1} \omega_\Lambda) > 0$$

or

$$\int_{\gamma_1} \omega_\Lambda > 0.$$  

By choosing a point $\tilde{q}_0$ in $\partial \overline{E}$ suitably, we obtain a holomorphic map $\Phi$ from $\overline{E}$ to $\mathbb{C}P_1$:

$$\Phi(q) = \int_{r}^{q} \omega_\Lambda.$$  

If we set

$$x = \int_{\gamma_1} \omega_\Lambda,$$

then $\Phi$ gives rise to an isomorphism between $(E, p, \Lambda)$ and $\overline{E}(0, 1, x)$. (Igeta-construction)

(ii) Suppose

$$\int_{\gamma_i} \omega_\Lambda \neq - \int_{\gamma_1} \omega_\Lambda$$

for any $\gamma_i$, $\Lambda$-edge from $q_1$ to $q_0$ in $\gamma_{(0)}$.

(ii-a) When the integral of $\omega_\Lambda$ along $\gamma_1$ is not a real number, we choose the sign of $\omega_\Lambda$ such that

$$\text{Im}(\int_{\gamma_1} \omega_\Lambda) > 0.$$  

We may assume without loss of generality that there exists a $\Lambda$-edge $\gamma_j$ from $q_1$ to $q_0$ in $\gamma_{(0)}$ such that the integral of $\omega_\Lambda$ along $\gamma_1 + \gamma_i$ is a positive integer. We may further assume by reordering that the integral of $\omega_\Lambda$ along $\gamma_1 + \gamma_2$ is minimal among such $\gamma_i$'s.

$$\int_{\gamma_1 + \gamma_i} \omega_\Lambda \geq \int_{\gamma_1 + \gamma_2} \omega_\Lambda \geq 0$$

Since the integral of $\omega_\Lambda$ along $\gamma_{(0)}$ vanishes, we see from the inequality above that there occur the following two cases:

1: There exists in $\gamma_{(0)}$ a $\Lambda$-edge $\gamma_j$ from $q_0$ to $q_0$ such that

$$\int_{\gamma_j} \omega_\Lambda < 0.$$  

2: There exist in $\gamma_{(0)}$ $\Lambda$-edges $\gamma_\mu$ and $\gamma_\nu$ ($\gamma_\mu$ is from $q_0$ to $q_1$, $\gamma_\nu$ from $q_1$ to $q_0$) such that

$$\int_{\gamma_\mu + \gamma_\nu} \omega_\Lambda < 0.$$
Suppose the following two inequalities hold for any $\gamma_{\mu}$ from $q_{0}$ to $q_{1}$ and for any $\gamma_{\nu}$ from $q_{1}$ to $q_{0}$.

\[
\int_{\gamma_{j}+\gamma_{\mu}}\omega_{\Lambda} \geq 0, \quad \int_{\gamma_{1}+\gamma_{\nu}}\omega_{\Lambda} \geq 0
\]

Then we obtain the following inequality from the second inequality above and the choice of $\gamma_{2}$.

\[
\int_{\gamma_{1}+\gamma_{\nu}}\omega_{\Lambda} \geq \int_{\gamma_{1}+\gamma_{2}}\omega_{\Lambda}
\]

Furthermore we obtain the following inequality, and this is a contradiction.

\[
\int_{\gamma_{\mu}+\gamma_{\nu}}\omega_{\Lambda} = \int_{\gamma_{\mu}+\gamma_{2}}\omega_{\Lambda} - \int_{\gamma_{2}+\gamma_{1}}\omega_{\Lambda} + \int_{\gamma_{1}+\gamma_{\nu}}\omega_{\Lambda} \\
\geq \int_{\gamma_{\mu}+\gamma_{2}}\omega_{\Lambda} \\
\geq 0.
\]

Therefore the following three cases occur:

(1) there is $\gamma_{j}$ from $q_{0}$ to $q_{1}$ such that

\[
\int_{\gamma_{2}+\gamma_{j}}\omega_{\Lambda} < 0.
\]
(2) there is $\gamma_j$ from $q_1$ to $q_0$ such that
\[ \int_{\gamma_1+\gamma_j} \omega_{\Lambda} < 0. \]

(3) there is $\gamma_j$ from $q_0$ to $q_0$ such that
\[ \int_{\gamma_j} \omega_{\Lambda} < 0. \]

The figures on the left-hand side of Figure 8 indicate the images of $\gamma_1$, $\gamma_2$, and $\gamma_j$ by the integral of $\omega_{\Lambda}$ in each case above.

In the case (1), we shall show that the homology classes represented by the simple loops $(\gamma_1, \gamma_2)$ and $(\gamma_j, \gamma_2)$ form a basis of $H_1(E; \mathbb{Z})$; If the homology classes $[(\gamma_1, \gamma_2)]$ and $-[(\gamma_j, \gamma_2)]$ represent the same homology class, then we get a loop $\gamma'_0$ representing a generator of $\Lambda$ by removing $\gamma_1$, $\gamma_2$, and $\gamma_j$ from $\gamma_0$ and placing $-\gamma_2$ instead, and $\gamma'_0$ is shorter than $\gamma_0$. This is a contradiction. Hence from Facts (1) and (2) it follows that the intersection number $[(\gamma_1, \gamma_2)] \cdot [(\gamma_j, \gamma_2)]$ is not equal to zero. On the other hand, $(\gamma_1, \gamma_2)$ intersects $(\gamma_j, \gamma_2)$ only at $\gamma_2$. Therefore the pair $([(\gamma_1, \gamma_2)], [(\gamma_j, \gamma_2)])$ is a basis of $H_1(E; \mathbb{Z})$. In the same way, we obtain the fact that the pair $([(\gamma_1, \gamma_2)], [(\gamma_1, \gamma_j)])$ is a basis of $H_1(E; \mathbb{Z})$ in the case (2), and that the pair $([(\gamma_1, \gamma_2)], [(\gamma_j)])$ is a basis of $H_1(E; \mathbb{Z})$ in the case (3).

Cut $E$ along $\gamma_1$, $\gamma_2$, and $\gamma_j$ in each case, and then denote by $\tilde{E}$ the resulting Riemann polygon. We obtain a holomorphic map $\Phi$ from $\tilde{E}$ to $\mathbb{C}P_1$
\[ \Phi(q) = \int_{q_0}^q \omega_{\Lambda} \]
when we choose a base point $q_0$ in $\tilde{E}$, because we obtain a basis of $H_1(E; \mathbb{Z})$ from $\gamma_1$, $\gamma_2$, and $\gamma_j$.

In the case (1), let $q_0$ be the point in the boundary $\partial \tilde{E}$ which is the initial point of $\gamma_1$ and is also the end point of $\gamma_2$. Then we get the image $\Phi(\tilde{E})$ as in the middle figure in Figure 8. It is now easy to modify $\tilde{E}$ by cutting and pasting from $\Phi(\tilde{E})$ such that we can obtain a Riemann polygon of type II. (See Figure 8.) Note that $a'$ and $b'$ are relatively prime integers where $b'$ is the integral of $\omega_{\Lambda}$ along $(\gamma_j, \gamma_2)$.

In the similar way, we obtain a Riemann polygon of type II in the cases (2) and (3), and omit the explanation.

(ii-b) When the integral of $\omega_{\Lambda}$ along $\gamma_1$ is a real number, there also exists an element $\gamma_j$ in $\gamma_0$ different from $\gamma_1$ and $\gamma_2$. We obtain the fact that $\gamma_j$ joins $q_0$ with $q_1$ as follows: Suppose $\gamma_j$ is a $\Lambda$-edge from $q_0$ to $q_0$ itself. We may assume
\[ (\int_{\gamma_1+\gamma_2} \omega_{\Lambda}) \cdot (\int_{\gamma_j} \omega_{\Lambda}) < 0. \]
The simple loop $\gamma_j$ intersects the simple loop $(\gamma_1, \gamma_2)$ at $q_0$ with intersection number zero because the directions of $\gamma_1$, $\gamma_2$, and $\gamma_j$ can be described as in the following figure. Hence it follows from Fact (2) that the homology class $[\gamma_1 + \gamma_2 + \gamma_j]$ vanishes. This contradicts the definition of $\gamma_j$.

Now we may assume that $\gamma_j$ is an edge from $q_0$ to $q_1$. Then the two homology classes $[(\gamma_1, \gamma_2)]$ and $[(\gamma_j, \gamma_2)]$ form a basis of $H_1(E; \mathbb{Z})$. We can also describe the directions of the three $\Lambda$-edges $\gamma_1$, $\gamma_2$, and $\gamma_j$ around $q_0$ and $q_1$ as in Figure 9.
Figure 9

Cut $E$ along $\gamma_1$, $\gamma_2$, and $\gamma_j$, and denote by $\tilde{E}$ the resulting Riemann polygon. We choose the point $\tilde{q}_0$ at the vertex on $\partial \tilde{E}$ where the boundary is smooth. (See Figure 9.) We choose the sign of $\omega_\Lambda$ such that $\Phi$ maps each point to a positive number around $\tilde{q}_0$. Then the map $\Phi$ gives rise to an isomorphism between $(E,p,\Lambda)$ and $\tilde{E}(a,b,x)$ of type III for some $(a,b,x)$. We have completed the proof of Theorem 1.

We denote by $\mathcal{ML}_{1,1}$ the set of once punctured elliptic curves with Lagrangian sublattices:

$$\mathcal{ML}_{1,1} = \{ \tilde{E}(a,b,x) \}.$$  

We will consider the complex structure of $\mathcal{ML}_{1,1}$ in the next section.

3. Complex structure of $\mathcal{ML}_{1,1}$

When we consider the forgetful map from the set $\mathcal{ML}_{1,1}$ of once punctured elliptic curves with Lagrangian sublattices to the moduli space $M_{1,1}$ of once punctured elliptic curves, it should be a holomorphic map; $\mathcal{ML}_{1,1}$ should be a complex V-manifold whose complex structure is induced from $M_{1,1}$ by the forgetful map.

We consider the complex structure of $\mathcal{ML}_{1,1}$ by using the description in Theorem 1: We give a local coordinate of $\mathcal{ML}_{1,1}$ around each once punctured elliptic curve with
a Lagrangian sublattice $\tilde{E}(a, b, x)$. For this purpose we first prepare two methods of deforming complex structures of Riemann surfaces.

Let $t$ be a complex number and let $s_t$ be the line segment between $\sqrt{t}$ and $-\sqrt{t}$ in $\mathbb{C}$. Set

$$U_{|t|} := \{z \in \mathbb{C}; |z| > |\sqrt{t}|\}.$$ 

Then the following map $J_t$ is biholomorphic:

$$J_t : U_{|t|} \ni z \mapsto \frac{1}{2}(z + \frac{t}{z}) \in \mathbb{C} - s_t.$$ 

Hence we obtain a biholomorphic map $J_t \circ J_{-t}^{-1}$ from $\mathbb{C} - s_{-t}$ to $\mathbb{C} - s_t$. Note that the map $J_t \circ J_{-t}^{-1}$ is parametrized by $t$ holomorphically. (See Figure 10.)

On the other hand, let $s_t'$ be the union of two line segments; the line segment from 0 to 1 and the line segment from 0 to $e^{\frac{3}{2} \pi i}$. We denote by $K$ the biholomorphic map from $U_1 = \{z \in \mathbb{C}; |z| > 1\}$ to $\mathbb{C} - s_t'$ such that $K(1) = 0$. (There exists uniquely such a biholomorphic map due to Riemann’s mapping theorem.) Let $t$ be a complex number as above. We also denote by $t$ the automorphism of $\mathbb{C}$ defined by multiplication by $t$. Set

$$s_t' = t(s_t').$$ 

Then $K_t := t \circ K \circ t^{-1}$ is a biholomorphic map from $U_{|t|}$ to $\mathbb{C} - s_t'$. The map $K_t$ is parametrized by $t$ holomorphically even at $t = 0$. We consider the biholomorphic map $K_t \circ J_{t^2}^{-1}$ from $\mathbb{C} - s_t$ to $\mathbb{C} - s_t'$, which is also parametrized by $t$ holomorphically. (See Figure 11.)

Let $p$ be a point on a Riemann surface $R$, and fix a local coordinate $z : \mathcal{U}_p \to \mathbb{C}$ ($z(p) = 0$) around $p$. The subset $\mathcal{U}_p$ of $R$ is also regarded as a subset of $\mathbb{C}$, and then both sets $s_t$ and $s_t'$ are also subsets of $\mathcal{U}_p$ if $t$ is sufficiently small. We can choose a sufficiently
small positive number $\epsilon$ such that
\[
(J_t \circ J_{-t}^{-1})(\{z \in \mathbb{C}; |z| < \epsilon\} - s_{-1}) \subset \mathcal{U}_p,
\]
\[
(K_t \circ J_{t}^{-1})(\{z \in \mathbb{C}; |z| < \epsilon\} - s_{2}) \subset \mathcal{U}_p.
\]
for any $t \in U_p$.

\begin{itemize}
\item \textbf{\pi-deformation of $(R, (U_p, z))$:} For a complex number $t \in D_\epsilon = \{t \in \mathbb{C}; |t| < \epsilon\}$ we paste $R - s_t$ and $\{z \in \mathbb{C}; |z| < \epsilon\}$ by the attaching map
\[
J_t \circ J_{-t}^{-1} : \{z \in \mathbb{C}; |z| < \epsilon\} - s_{-1} \rightarrow \mathcal{U}_p - s_t \subset R - s_t.
\]
Then we obtain a holomorphic family of Riemann surfaces on $D_t$. Note that the fiber on the origin is $R$, and that the fiber $R_t$ on $t$ is obtained as follows: cut $R$ along $s_t$ and paste by identifying $k\sqrt{t}$ and $-k\sqrt{t}$ on one side of $s_t$ and identifying $\ell\sqrt{t}$ and $-\ell\sqrt{t}$ on the other side of $s_t$ ($0 < k, \ell \leq 1$).

\item \textbf{$\frac{2}{3}\pi$-deformation of $(R, (U_p, z))$:} For a complex number $t \in D_\epsilon = \{t \in \mathbb{C}; |t| < \epsilon\}$ we paste $R - s'_t$ and $\{z \in \mathbb{C}; |z| < \epsilon\}$ by the attaching map
\[
K_t \circ J_{t}^{-1} : \{z \in \mathbb{C}; |z| < \epsilon\} - s_{2} \rightarrow \mathcal{U}_p - s'_t \subset R - s'_t.
\]
Then we obtain a holomorphic family of Riemann surfaces on $D_t$. Note that the fiber on the origin is $R$, and that the fiber $R_t$ on $t$ is obtained as follows: cut $R$ along $s'_t$ and paste by identifying $kt$ and $kte^{\frac{2}{3}\pi i}$ on one side of $s'_t$ and identifying $\ell t$ and $-\ell t e^{\frac{2}{3}\pi i}$ on the other side of $s'_t$ ($0 < k, \ell \leq 1$).
\end{itemize}

We apply a $\pi$-deformation to each $\tilde{E}(a, b, x)$ of type I, II or III.

Let $p_1$ be the point on $\tilde{E}(0, 1, x)$ ($x \in \mathbb{H}$) coming from $x$ and $x + 1$, and fix a local coordinate $z : \mathcal{U}_{p_1} \rightarrow \mathbb{C}$ around $p_1$ such that $\zeta = z^2 + x$ or $\zeta = z^2 + x + 1$, where $\zeta$ is the global coordinate of $\mathbb{C}$ as before. When we consider the coordinate $(\mathcal{U}_{p_1}, z)$, the line segment $s_t$ is transferred onto two parallel line segments with the same length $t$ in
ζ-plane: one is from $x$ to $x + t$ and the other is from $x + 1$ to $x + 1 + t$. The fiber $F_t$ on $t \in D_\epsilon$ is obtained from $\tilde{E}(0, 1, x)$ by cutting and pasting along the two line segments; $F_t$ is equivalent to $\tilde{E}(0, 1, x + t)$. (See Figure 12. The numbers (1), ... , (4) indicate where to paste.) Accordingly we can regard $x \in \mathbb{H}$ as a local coordinate when $a = 0$ and $b = 1$.

Applying a $\pi$-deformation to $\tilde{E}(a, b, x)$ of type II in the same way, we see that $x$ is also a local coordinate if we fix $a$ and $b$. (See Figure 13.)
We apply a $\pi$-deformation to $\tilde{E}(0, 1, x \in (0, 1))$ of type I in the same way, and it follows that the fiber $F_t$ is equivalent to $\tilde{E}(0, 1, x+t)$ when $0 \leq \arg t \leq \pi$, and that $F_t$ is equivalent to $\tilde{E}(0, 1, -x - t)$ when $\pi < \arg t < 2\pi$ (we use the rotation of angle $\pi$). (See Figure 14.)

We also apply a $\pi$-deformation to $\tilde{E}(a, b, x)$ of type III in the same way: Fix a local coordinate $z : U_{p_1} \to \mathbb{C}$ around $p_1$ such that

$$\zeta = \begin{cases} 
  z^2 + x & -\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi, \\
  z^2 + x - a & \frac{1}{2}\pi < \arg z < \pi, \\
  z^2 + x - b & \pi < \arg z < \frac{3}{2}\pi.
\end{cases}$$

We consider the case when $a$ is less than $b$. It is easy to see that the fiber $F_t$ of this deformation is equivalent to $\tilde{E}(a, b, x + t)$ when $\arg t = 0$ or $\pi$. When $0 < \arg t < \pi$, we see by cutting and pasting that $F_t$ is equivalent to $\tilde{E}(a - (b/a), (b/a), -x + a - t)$ where $(b/a)$ is a non-negative integer such that $0 \leq b/a < a$ and $(b/a) \equiv b$ mod $a$. Note that $(b/a)$ is equal to $0$ if and only if $a$ is equal to $1$, and then $F_t$ is equivalent to $\tilde{E}(0, 1, -x + 1 - t)$ of type I. (See Figure 15.) Accordingly we obtain the following injective map $f_{\tilde{E}(a,b,x)} : D_\epsilon \to \mathcal{ML}_{1,1}$ when $a$ is less than $b$, and the inverse mapping of $f_{\tilde{E}(a,b,x)}$ is a local coordinate of $\mathcal{ML}_{1,1}$ around $\tilde{E}(a, b, x)$.

$$f_{\tilde{E}(a,b,x)}(t) = \begin{cases} 
  \tilde{E}(a, b, x + t) & \text{arg } t = 0 \text{ or } \pi, \\
  \tilde{E}(a - (b/a), (b/a), -x + a - t) & \pi < \arg t < 2\pi \ (a \neq 1), \\
  \tilde{E}(0, 1, -x + 1 - t) & \pi < \arg t < 2\pi \ (a = 1).
\end{cases}$$
Similarly we obtain the following injective map $f_{\tilde{E}(a,b,x)} : D_{\epsilon} \to \mathcal{ML}_{1,1}$ when $a$ is greater than $b$, and the inverse mapping of $f_{\tilde{E}(a,b,x)}$ is a local coordinate of $\mathcal{ML}_{1,1}$ around $\tilde{E}(a,b,x)$.

$$f_{\tilde{E}(a,b,x)}(t) = \begin{cases} \tilde{E}(a,b,x+t) & \text{arg} t = 0 \text{ or } \pi, \\ \tilde{E}(a-b,b,-x+a-t) & \pi < \text{arg} t < 2\pi, \\ \tilde{E}((a/b),b-(a/b),x-b+t) & \pi < \text{arg} t < 2\pi. \end{cases}$$

To $\tilde{E}(a,b,0)$ of type IV we apply a $\frac{2}{3}\pi$-deformation: Fix a local coordinate $z : U_{p_0} \to \mathbb{C}$ around $p_0$, the unique zero of $\omega_0$, such that

$$\zeta = \begin{cases} z^3 & 0 < \text{arg} z < \frac{2}{3}\pi, \\ z^3 + a & \frac{2}{3}\pi < \text{arg} z < \pi, \\ z^3 + a + b & \pi < \text{arg} z < \frac{5}{3}\pi, \\ z^3 + b & \frac{5}{3}\pi < \text{arg} z < 2\pi. \end{cases}$$

Then the segment $s'_t$ is transferred to two parallel line segments with the same length $t^3$ in $\zeta$-plane.

By cutting and pasting we obtain the following map $f_{\tilde{E}(a,b,0)} : D_{\epsilon} \to \mathcal{ML}_{1,1}$:
Figure 16. $\frac{2}{3}\pi$-deformation of $\tilde{E}(a, b, 0)$ of type IV ($a < b$ and $a \neq 1$)
When $a < b$ and $a \neq 1$ (Figure 16),

\[
f_{E(a,b,0)}(t) = \begin{cases} 
\tilde{E}(a, b, a + b - t^3) & \text{arg } t = 0, \\
\tilde{E}(a - (b/a), (b/a), t^3 - b) & 0 < \text{arg } t < \frac{1}{3}\pi, \\
\tilde{E}(a, a + b, a + b - t^3) & \text{arg } t = \frac{1}{3}\pi, \\
\tilde{E}(a, b, a - t^3) & \frac{1}{3}\pi < \text{arg } t < \frac{2}{3}\pi, \\
\tilde{E}(a + b, b, a + b - t^3) & \text{arg } t = \frac{2}{3}\pi, \\
\tilde{E}(a, a + b, a + b + t^3) & \frac{2}{3}\pi < \text{arg } t < \pi, \\
\tilde{E}(a, b, a + b + t^3) & \text{arg } t = \pi, \\
\tilde{E}(a - (b/a), (b/a), -t^3 - b) & \pi < \text{arg } t < \frac{4}{3}\pi, \\
\tilde{E}(a, a + b, a + b + t^3) & \text{arg } t = \frac{4}{3}\pi, \\
\tilde{E}(a, b, t^3) & \frac{4}{3}\pi < \text{arg } t < \frac{5}{3}\pi, \\
\tilde{E}(a + b, b, a + b - t^3) & \text{arg } t = \frac{5}{3}\pi, \\
\tilde{E}(a, b - a, a - t^3) & \frac{5}{3}\pi < \text{arg } t < 2\pi.
\end{cases}
\]

When $a = 1$ and $a < b$,

\[
f_{E(1,b,0)}(t) = \begin{cases} 
\tilde{E}(1, b, 1 + b - t^3) & \text{arg } t = 0, \\
\tilde{E}(0, 1, t^3 - b) & 0 < \text{arg } t < \frac{1}{3}\pi, \\
\tilde{E}(1, 1 + b, 1 + b - t^3) & \text{arg } t = \frac{1}{3}\pi, \\
\tilde{E}(1, b, -t^3) & \frac{1}{3}\pi < \text{arg } t < \frac{2}{3}\pi, \\
\tilde{E}(1 + b, b, 1 + b + t^3) & \text{arg } t = \frac{2}{3}\pi, \\
\tilde{E}(1, b - 1, 1 + t^3) & \frac{2}{3}\pi < \text{arg } t < \pi, \\
\tilde{E}(1, b, 1 + b + t^3) & \text{arg } t = \pi, \\
\tilde{E}(0, 1, -t^3 - b) & \pi < \text{arg } t < \frac{4}{3}\pi, \\
\tilde{E}(1, 1 + b, 1 + b + t^3) & \text{arg } t = \frac{4}{3}\pi, \\
\tilde{E}(1, b, t^3) & \frac{4}{3}\pi < \text{arg } t < \frac{5}{3}\pi, \\
\tilde{E}(1 + b, b, 1 + b - t^3) & \text{arg } t = \frac{5}{3}\pi, \\
\tilde{E}(1, b - 1, 1 - t^3) & \frac{5}{3}\pi < \text{arg } t < 2\pi.
\end{cases}
\]
When \( a = b = 1 \),

\[
f_{E(1,1,0)}(t) = \begin{cases} 
\tilde{E}(0,1,1-t^3) & \text{arg } t = 0, \\
\tilde{E}(0,1,t^3-1) & 0 < \text{arg } t < \frac{1}{3}\pi, \\
\tilde{E}(1,2,2-t^3) & \text{arg } t = \frac{1}{3}\pi, \\
\tilde{E}(1,1,-t^3) & \frac{1}{3}\pi < \text{arg } t < \frac{2}{3}\pi, \\
\tilde{E}(2,1,2+t^3) & \text{arg } t = \frac{2}{3}\pi, \\
\tilde{E}(0,1,1+t^3) & \frac{2}{3}\pi < \text{arg } t < \pi, \\
\tilde{E}(0,1,1+t^3) & \text{arg } t = \pi, \\
\tilde{E}(0,1,-t^3-1) & \pi < \text{arg } t < \frac{4}{3}\pi, \\
\tilde{E}(1,2,2+t^3) & \text{arg } t = \frac{4}{3}\pi, \\
\tilde{E}(1,1,t^3) & \frac{4}{3}\pi < \text{arg } t < \frac{5}{3}\pi, \\
\tilde{E}(2,1,2-t^3) & \text{arg } t = \frac{5}{3}\pi, \\
\tilde{E}(0,1,1-t^3) & \frac{5}{3}\pi < \text{arg } t < 2\pi.
\end{cases}
\]

When \( b = 1 \) and \( a > b \),

\[
f_{E(a,1,0)}(t) = \begin{cases} 
\tilde{E}(a,1,a+1-t^3) & \text{arg } t = 0, \\
\tilde{E}(a-1,1,t^3-1) & 0 < \text{arg } t < \frac{1}{3}\pi, \\
\tilde{E}(a+1,1,a+1-t^3) & \text{arg } t = \frac{1}{3}\pi, \\
\tilde{E}(a,1,-t^3) & \frac{1}{3}\pi < \text{arg } t < \frac{2}{3}\pi, \\
\tilde{E}(a+1,1,a+1+t^3) & \text{arg } t = \frac{2}{3}\pi, \\
\tilde{E}(0,1,a+t^3) & \frac{2}{3}\pi < \text{arg } t < \pi, \\
\tilde{E}(a,1,a+1+t^3) & \text{arg } t = \pi, \\
\tilde{E}(a-1,1,-t^3-1) & \pi < \text{arg } t < \frac{4}{3}\pi, \\
\tilde{E}(a,1,a+1+t^3) & \text{arg } t = \frac{4}{3}\pi, \\
\tilde{E}(a,1,t^3) & \frac{4}{3}\pi < \text{arg } t < \frac{5}{3}\pi, \\
\tilde{E}(a+1,1,a+1-t^3) & \text{arg } t = \frac{5}{3}\pi, \\
\tilde{E}(0,1,a-t^3) & \frac{5}{3}\pi < \text{arg } t < 2\pi.
\end{cases}
\]
When $b \neq 1$ and $a > b$,

\[
f_{\tilde{E}(a,b,0)}(t) = \begin{cases} 
\tilde{E}(a,b,a+b-t^3) & \text{arg } t = 0, \\
\tilde{E}(a-b,b,t^3-b) & 0 < \text{arg } t < \frac{1}{3}\pi, \\
\tilde{E}(a,a+b,a+b-t^3) & \text{arg } t = \frac{1}{3}\pi, \\
\tilde{E}(a,b,-t^3) & \frac{1}{3}\pi < \text{arg } t < \frac{2}{3}\pi, \\
\tilde{E}(a+b,b,a+b+t^3) & \text{arg } t = \frac{2}{3}\pi, \\
\tilde{E}((a/b),(b-(a/b),a+t^3) & \frac{2}{3}\pi < \text{arg } t < \pi, \\
\tilde{E}(a,b,a+b+t^3) & \text{arg } t = \pi, \\
\tilde{E}(a-b,b,-t^3-b) & \pi < \text{arg } t < \frac{4}{3}\pi, \\
\tilde{E}(a,a+b,a+b+t^3) & \text{arg } t = \frac{4}{3}\pi, \\
\tilde{E}(a,b,t^3) & \frac{4}{3}\pi < \text{arg } t < \frac{5}{3}\pi, \\
\tilde{E}(a+b,b,a+b-t^3) & \text{arg } t = \frac{5}{3}\pi, \\
\tilde{E}((a/b),(b-(a/b),a-t^3) & \frac{5}{3}\pi < \text{arg } t < 2\pi.
\end{cases}
\]

For each $\tilde{E}(a,b,0)$ of type IV, the map $f_{\tilde{E}(a,b,0)}$ gives rise to a two-fold branched covering;

\[
f_{\tilde{E}(a,b,0)}(t) = f_{\tilde{E}(a,b,0)}(-t).
\]

Then $(f_{\tilde{E}(a,b,0)}, D_\epsilon, \mathbb{Z}/2\mathbb{Z})$ gives rise to a local manifold cover of $\mathcal{M}_{\mathcal{L}_{1,1}}$ around $\tilde{E}(a,b,0)$. In this way, the set $\mathcal{M}_{\mathcal{L}_{1,1}}$ is regarded as a complex $V$-manifold.

REFERENCES


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