数理解析研究所講究録
1008巻 1997年 21-39

GEOMETRIC CONSTRUCTION OF CRYSTAL BASES(*)

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1. INTRODUCTION

1.1. G.Lusztig [L3] gave a realization of the quantized universal enveloping algebras as the Grothendieck group of a category of perverse sheaves on the quiver variety. Let (I,Ω) be a finite oriented graph (=quiver), where I is the set of vertices and Ω is the set of arrows. Let us associate a complex vector space V_i to each vertex $i \in I$. We set

$$E_{V,\Omega} = \bigoplus_{\tau\in\Omega} \mathrm{Hom}\,(V_{\mathrm{out}(\tau)}, V_{\mathrm{in}(\tau)})$$

and

$$X_V = E_{V,\Omega} \oplus E^*_{V,\Omega}.$$

They are finite-dimensional vector spaces with the action of the algebraic group $G_V = \prod_{i\in I} GL(V_i)$. We regard X_V as the cotangent bundle of $E_{V,\Omega}$. Lusztig [L3] realized a half of the quantized universal enveloping algebra $U_q^-(\mathfrak{g})$ as the Grothendieck group of $\mathcal{Q}_{V,\Omega}$. Here $\mathcal{Q}_{V,\Omega}$ is a subcategory of the derived category $D^b_c(E_{V,\Omega})$ of the bounded complex of constructible sheaves on $E_{V,\Omega}$. The irreducible perverse sheaves in $\mathcal{Q}_{V,\Omega}$ form a base of $U_q^-(\mathfrak{g})$, which is called canonical basis.

In [L5] he asked the following problem.

Problem 1. If the underlying graph is of type A, D or E, then the singular support of any canonical base is irreducible.

One of the purpose of this paper is to construct a counterexample of this problem for type A.

1.2. Let G be a connected complex semisimple algebraic group, B a Borel subgroup of G and $X = G/B$ the flag variety. Let D_X denote the sheaf of differential operators on X. We denote the half sum of positive roots by ρ and the Weyl group by W. For $w \in W$, let M_w be the Verma module with highest weight $-w(\rho)-\rho$ and L_w its simple quotient. By the Beilinson-Bernstein correspondence, M_w and L_w correspond to regular holonomic D_X-modules $\mathfrak{M}_w$ and $\mathfrak{L}_w$ on X, respectively. The characteristic varieties $\mathrm{Ch}(\mathfrak{M}_w)$ and $\mathrm{Ch}(\mathfrak{L}_w)$ are Lagrangian subvarieties of the cotangent bundle

(*) This is a joint work with Masaki Kashiwara.

T^*X. Each irreducible component of $\mathrm{Ch}(\mathfrak{M}_w)$ and $\mathrm{Ch}(\mathfrak{L}_w)$ is the closure of the conormal bundle $T^*_{X_y}X$ of a Schubert cell $X_y = ByB/B$ for some $y \in W$. Let $\mathcal{M}$ be the abelian category consisting of regular holonomic systems on X whose characteristic varieties are contained in $\coprod_{w\in W} T^*_{X_w}X$. Its Grothendieck group $K(\mathcal{M})$ has two bases, $([\mathfrak{M}_w])_{w\in W}$ and $([\mathfrak{L}_w])_{w\in W}$. For $\mathfrak{M} \in \mathcal{M}$ let $\mathrm{Ch}(\mathfrak{M}) = \sum_{w\in W} m_w(\mathfrak{M})[T^*_{X_w}X]$ be the characteristic cycle. Here $m_w(\mathfrak{M})$ is the multiplicity of $\mathfrak{M}$ along $T^*_{X_w}X$. Then Ch extends to an additive map from $K(\mathcal{M})$ to the group of algebraic cycles of T^*X. Let χ be a $\mathbb{Z}$-linear isomorphism from $K(\mathcal{M})$ onto the group ring $\mathbb{Z}[W]$ defined by $\chi([\mathfrak{M}_w]) = w$. Then there exists a unique basis $\{\mathsf{b}(w)\}_{w\in W}$ of $\mathbb{Z}[W]$ such that $\mathrm{Ch}(\chi^{-1}(\mathsf{b}(w))) = [T^*_{X_w}X]$ (See [KL1] and [KT].). This basis is related to the Springer representation of the Weyl group. Set $\mathsf{a}(w) = \chi([\mathfrak{L}_w]) = \sum_{y\in W} m_y(\mathfrak{L}_w)\mathsf{b}(y)$. The basis $\{\mathsf{a}(w)\}_{w\in W}$ is related to the left cell representation of the Weyl group. Therefore an explicit knowledge of $m_y(\mathfrak{L}_w)$ gives an explicit relation between the Springer representation and the left cell representation. If $\mathrm{Ch}(\mathfrak{L}_w)$ is an irreducible variety, that is,

$$m_y(\mathfrak{L}_w) = \begin{cases} 1 & \text{if } y = w, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2.1}$$

then the Springer representation coincides with the left cell representation. Due to Tanisaki, there is a counterexample of (1.2.1) in the case of B_2 (See [T]). In [KL2] Kazhdan and Lusztig conjectured that $\mathrm{Ch}(\mathfrak{L}_w)$ is irreducible for $G = SL_n(\mathbb{C})$. In this paper, as a corollary of Problem 1, we shall show that there is a counterexample of this conjecture in the case of $G = SL_8(\mathbb{C})$ and this conjecture is true for $G = SL_n(\mathbb{C})$ with $n \leq 7$.

1.3. On the other hand, Kashiwara [K1] constructed the crystal base and the global crystal base of $U_q^-(\mathfrak{g})$ and the highest weight integrable representations of $U_q(\mathfrak{g})$ in an algebraic way. Grojnowski and Lusztig [GL] showed that the global crystal base coincides with the canonical base of Lusztig [L3].

In this paper, we shall construct the crystal base in a geometrical way. We define the nilpotent subvariety of the cotangent bundle of the quiver varieties, following Lusztig. The nilpotent variety is a Lagrangian subvariety. We shall define a crystal structure on the set of its irreducible components, and we prove that it is isomorphic to the crystal associated with $U_q^-(\mathfrak{g})$.

1.4. Let us briefly summarize the contents of this manuscript. In section 2 and 3 we give a review of the theory of crystal base [K1,2,3,4]. After recalling quiver varieties in section 4, we define the crystal structure on the set of irreducible components of the nilpotent varieties and prove that it coincides with the crystal base of $U_q^-(\mathfrak{g})$ in section 5. In section 6, we recall the relation of the quantized universal enveloping algebras and perverse sheaves on the quiver varieties. In section 7, we give a negative answer to Problem 1. In the last section, we give a counterexample of the irreducibility of the characteristic variety of the irreducible perverse sheaf with the Schubert cell as its support in the case of SL_8.

Proofs of the results announced in this manuscript appear in [KSa].

2. PRELIMINARIES

2.1. Definition of $U_q(\mathfrak{g})$. We shall give the definition of $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. We follow the notations in [K1,2,3,4].

Definition 2.1.1. Let us consider following data:

(1) a finite-dimensional $\mathbb{Q}$-vector space $\mathfrak{t}$,
(2) an index set I (of simple roots),
(3) a linearly independent subset $\{\alpha_i\,;\, i \in I\}$ of $\mathfrak{t}^*$ and a subset $\{h_i\,;\, i \in I\}$ of $\mathfrak{t}$,
(4) an inner product $(\ ,\ )$ on $\mathfrak{t}^*$ and
(5) a lattice P (a weight lattice) of $\mathfrak{t}^*$.

These data are assumed to satisfy the following conditions:

(6) $\{\langle h_i, \alpha_j\rangle\}$ is a generalized Cartan matrix
(i.e. $\langle h_i, \alpha_i\rangle = 2$, $\langle h_i, \alpha_j\rangle \in \mathbb{Z}_{\leq 0}$ for $i \neq j$ and $\langle h_i, \alpha_j\rangle = 0 \Leftrightarrow \langle h_j, \alpha_i\rangle = 0$),
(7) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$,
(8) $\langle h_i, \lambda\rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for any $i \in I$ and $\lambda \in \mathfrak{t}^*$,
(9) $\alpha_i \in P$ and $h_i \in P^* = \{h \in \mathfrak{t}\ ;\ \langle h, P\rangle \in \mathbb{Z}\}$.

Then the $\mathbb{Q}(q)$-algebra $U_q(\mathfrak{g})$ is the algebra generated by $e_i, f_i (i \in I)$ and $q^h (h \in P^*)$ with the following defining relations:

(10) $q^h = 1$ for $h = 0$ and $q^{h+h'} = q^h q^{h'}$,
(11) $q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i$ and $q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i$,
(12) $[e_i, f_j] = \delta_{i,j}(t_i - t_i^{-1})/(q_i - q_i^{-1})$ where $q_i = q^{(\alpha_i,\alpha_i)/2}$ and $t_i = q^{(\alpha_i,\alpha_i)h_i/2}$,
(13) $\sum_{n=0}^{b}(-1)^n e_i^{(n)} e_j e_i^{(b-n)} = \sum_{n=0}^{b}(-1)^n f_i^{(n)} f_j f_i^{(b-n)} = 0$
where $i \neq j$ and $b = 1 - \langle h_i, \alpha_j\rangle$.

Here we used the notations $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{k=1}^n [k]_i$, $e_i^{(n)} = e_i^n/[n]_i!$ and $f_i^{(n)} = f_i^n/[n]_i!$. We understand $e_i^{(n)} = f_i^{(n)} = 0$ for $n < 0$. We set $Q = \sum_{i\in I}\mathbb{Z}\alpha_i$, $Q_+ = \sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$ and $Q_- = -Q_+$. Let P_+ be the set of dominant integral weights.
We denote by $U_q^-(\mathfrak{g})$ the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by f_i $(i \in I)$.

As in [K], we define the $\mathbb{Q}(q)$-algebra anti-automorphism $*$ of $U_q(\mathfrak{g})$ by

$$e_i^* = e_i,\ f_i^* = f_i \text{ and } (q^h)^* = q^{-h}.$$

Note that $*^2 = 1$.

2.2. Crystal base of $U_q^-(\mathfrak{g})$. Next we shall define a crystal base of $U_q^-(\mathfrak{g})$. See [K1,2,3,4] for ditails.

Lemma 2.2.1. *For any $P \in U_q^-(\mathfrak{g})$, there exist unique Q, $R \in U_q^-(\mathfrak{g})$ such that*

$$[e_i, P] = \frac{t_i Q - t_i^{-1} R}{q_i - q_i^{-1}}.$$

By this lemma, $e_i'(P) = R$ defines an endomorphism e_i' of $U_q^-(\mathfrak{g})$.
According to [K1] we have

$$U_q^-(\mathfrak{g}) = \bigoplus_{n \geq 0} f_i^{(n)} \mathrm{Ker}\, e_i'.$$

We define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ of $U_q^-(\mathfrak{g})$ by

$$\begin{aligned} \tilde{f}_i(f_i^{(n)} u) &= f_i^{(n+1)} u \quad \text{and} \\ \tilde{e}_i(f_i^{(n)} u) &= f_i^{(n-1)} u \end{aligned}$$

for $u \in \mathrm{Ker}\, e_i'$.

Definition 2.2.2. A pair (L, B) is called a crystal base of $U_q^-(\mathfrak{g})$ if it satisfies the following conditions:

(2.3.1) L is a free sub-A-module of $U_q^-(\mathfrak{g})$ such that $U_q^-(\mathfrak{g}) \cong \mathbb{Q}(q) \otimes_A L$.

(2.3.2) B is a base of the $\mathbb{Q}$-vector space L/qL.

(2.3.3) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for any i.
Therefore $\tilde{e}_i$ and $\tilde{f}_i$ act on L/qL.

(2.3.4) $\tilde{e}_i B \subset B \sqcup \{0\}$ and $\tilde{f}_i B \subset B$.

(2.3.5)

$$L = \bigoplus_{\nu \in Q_-} L_\nu \text{ and } B = \bigsqcup_{\nu \in Q_-} B_\nu$$

where $L_\nu = L \cap U_q^-(\mathfrak{g})_\nu$, $B_\nu = B \cap (L_\nu / qL_\nu)$ and $U_q^-(\mathfrak{g})_\nu = \{P \in U_q^-(\mathfrak{g})\,;\, q^h P q^{-h} = q^{\langle h, \nu \rangle} P$ for any $h \in P^*\}$.

(2.3.6) For $b \in B$ such that $\tilde{e}_i b \neq 0$, we have $b = \tilde{f}_i \tilde{e}_i b$.

We introduce the sub-A-module $L(\infty)$ of $U_q^-(\mathfrak{g})$ generated by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \cdot 1$ and the subset $B(\infty)$ of $L(\infty)/qL(\infty)$ consisting of the non-zero vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \cdot 1$.

Theorem 2.2.3. *$(L(\infty), B(\infty))$ is a crystal base of $U_q^-(\mathfrak{g})$.*

3. Crystals

3.1. Definition of Crystal.

Definition 3.1.1. A crystal B is a set endowed with

(3.1.1) maps $\mathrm{wt}: B \to P$, $\varepsilon_i : B \to Z \sqcup \{-\infty\}$, $\varphi_i : B \to Z \sqcup \{-\infty\}$ and

(3.1.2) $\tilde{e}_i : B \to B \sqcup \{0\}$, $\tilde{f}_i : B \to B \sqcup \{0\}$.

They are subject to the following axioms:

(C 1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$.

(C 2) If $b \in B$ and $\tilde{e}_i b \in B$ then,
$\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$.

(C 2') If $b \in B$ and $\tilde{f}_i b \in B$, then
$\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$, $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.

(C 3) For $b, b' \in B$ and $i \in I$, $b' = \tilde{e}_i b$ if and only if $b = \tilde{f}_i b'$.

(C 4) For $b \in B$, if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

For two crystals B_1 and B_2, a morphism ψ from B_1 to B_2 is a map $B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ that satisfies the following conditions:

(3.1.3) $\psi(0) = 0$,

(3.1.4) If $b \in B_1$ and $\psi(b) \in B_2$, then
$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$,

(3.1.5) If $b, b' \in B_1$ and $i \in I$ satisfy $\tilde{f}_i(b) = b'$ and $\psi(b)$, $\psi(b') \in B_2$, then we have $\tilde{f}_i(\psi(b)) = \psi(b')$.

A morphism $\psi : B_1 \to B_2$ is called *strict*, if it commutes with all $\tilde{e}_i$ and $\tilde{f}_i$.

A morphism $\psi : B_1 \to B_2$ is called an *embedding*, if ψ induces an injective map from $B_1 \sqcup \{0\}$ to $B_2 \sqcup \{0\}$.

For two crystals B_1 and B_2, we define its tensor product $B_1 \otimes B_2$ as follows:

$$\begin{aligned} B_1 \otimes B_2 &= \{b_1 \otimes b_2 \ ; \ b_1 \in B_1 \ and \ b_2 \in B_2\}, \\ \varepsilon_i(b_1 \otimes b_2) &= \max\Big(\varepsilon_i(b_1),\ \varepsilon_i(b_2) - \mathrm{wt}_i(b_1)\Big), \\ \varphi_i(b_1 \otimes b_2) &= \max\Big(\varphi_i(b_1) + \mathrm{wt}_i(b_2),\ \varphi_i(b_2)\Big), \\ \mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2). \end{aligned}$$

Here $\mathrm{wt}_i(b)$ denotes $\langle h_i, \mathrm{wt}(b)\rangle$.

The action of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Example 3.1.1. For $i \in I$, B_i is the crystal defined as follows

$$\begin{aligned} &B_i = \{b_i(n)\,;\, n \in \mathbb{Z}\}, \\ &\mathrm{wt}(b_i(n)) = n\alpha_i, \\ &\varphi_i(b_i(n)) = n,\ \varepsilon_i(b_i(n)) = -n, \\ &\varphi_j(b_i(n)) = \varepsilon_j(b_i(n)) = -\infty \quad \text{for } i \neq j. \end{aligned}$$

We define the action of $\tilde{e}_i$ and $\tilde{f}_i$ by

$$\begin{aligned} &\tilde{e}_i(b_i(n)) = b_i(n+1), \\ &\tilde{f}_i(b_i(n)) = b_i(n-1), \\ &\tilde{e}_j(b_i(n)) = \tilde{f}_j(b_i(n)) = 0 \quad \text{for } i \neq j. \end{aligned}$$

We write b_i for $b_i(0)$.

Example 3.1.2. For $\lambda \in P_+$, $B(\lambda)$ denotes the crystal associated with the crystal base of the simple highest weight module with highest weight λ. For $b \in B(\lambda)$ we set $\varepsilon_i(b) = \max\{k \geq 0\ ;\ \tilde{e}_i{}^k b \neq 0\}$, $\varphi_i(b) = \max\{k \geq 0\ ;\ \tilde{f}_i{}^k b \neq 0\}$ and $\mathrm{wt}(b)$ is the weight of b.

Example 3.1.3. $B(\infty)$ is the crystal associated with the crystal base of $U_q^-(\mathfrak{g})$. For $b \in B(\infty)$ we set $\varepsilon_i(b) = \max\{k \geq 0\ ;\ \tilde{e}_i{}^k b \neq 0\}$ and $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$. We denote u_∞ by the unique element with weight 0.

3.2. We have $L(\infty)^* = L(\infty)$ and $*$ induces an endomorphism of $L(\infty)/qL(\infty)$.

Theorem 3.2.1.

$$B(\infty)^* = B(\infty).$$

We define the operators $\tilde{e}_i^*$, $\tilde{f}_i^*$ of $U_q^-(\mathfrak{g})$ by

$$\tilde{e}_i^* = *\tilde{e}_i*, \text{ and } \tilde{f}_i^* = *\tilde{f}_i*. \tag{3.2.2}$$

Theorem 3.2.2. (1) *For any i, there exists a unique strict embedding of crystals*

$$\Psi_i : B(\infty) \hookrightarrow B(\infty) \otimes B_i$$

that sends u_∞ to $u_\infty \otimes b_i$.

(2) *If $\Psi_i(b) = b' \otimes \tilde{f}_i^n b_i$ $(n \geq 0)$, then $\varepsilon_i(b^*) = n$, $\varepsilon_i(b'^*) = 0$ and $b = \tilde{f}_i^{*n} b'$.*

(3) $\mathrm{Im}\ \Psi_i = \{b \otimes \tilde{f}_i{}^n b_i\ ;\ b \in B(\infty),\ \varepsilon_i(b^*) = 0,\ n \geq 0\}$.

In fact the above properties characterize $B(\infty)$ as seen in the following proposition.

Proposition 3.2.3. *Let B be a crystal and b_0 an element of B with weight 0. Assume the following conditions.*

(1) $\mathrm{wt}(B) \subset Q_-$.
(2) *b_0 is a unique element of B with weight 0.*
(3) *$\varepsilon_i(b_0) = 0$ for every i.*
(4) *$\varepsilon_i(b) \in \mathbb{Z}$ for any b and i.*
(5) *For every i, there exists a strict embedding $\Psi_i : B \to B \otimes B_i$.*
(6) *$\Psi_i(B) \subset B \times \{\tilde{f}_i^n b_i; n \geq 0\}$.*
(7) *For any $b \in B$ such that $b \neq b_0$, there exists i such that $\Psi_i(b) = b' \otimes \tilde{f}_i^n b_i$ with $n > 0$.*

Then B is isomorphic to $B(\infty)$.

4. Quivers and associated varieties ([L5,6] and [N1,2])

4.1. Definition of quiver.

We shall recall the formulation due to Lusztig [L5,6].

Suppose a finite graph is given. In this graph, two different vertices may be joined by several edges, but any vertex is not joined with itself by any edges. Let I be the set of vertices of our graph, and let H be the set of pairs of an edge and its orientation. The precise definition is as follows.

Definition 4.1.1. Suppose that following data (1) $\sim$ (5) are given:

(1) a finite set I,
(2) a finite set H,
(3) a map $H \to I$ denoted $\tau \mapsto \mathrm{out}(\tau)$,
(4) a map $H \to I$ denoted $\tau \mapsto \mathrm{in}(\tau)$ and
(5) an involution $\tau \mapsto \bar{\tau}$ of H.

We assume that they satisfy the following conditions;

$$\mathrm{in}(\bar{\tau}) = \mathrm{out}(\tau),\ \mathrm{out}(\bar{\tau}) = \mathrm{in}(\tau) \text{ and} \tag{4.1.1}$$

$$\mathrm{out}(\tau) \neq \mathrm{in}(\tau) \text{ for all } \tau \in H. \tag{4.1.2}$$

An orientation of the graph is a choice of a subset $\Omega \subset H$ such that

$$\Omega \cup \bar{\Omega} = H \text{ and } \Omega \cap \bar{\Omega} = \phi.$$

We call a quiver a graph with an orientation.

To a graph (I, H) we associate a root system with simple roots $\{\alpha_i\}_{i \in I}$ and simple coroots $\{h_i\}_{i \in I}$ with

$$\langle h_i, \alpha_j \rangle = (\alpha_i, \alpha_j) = \begin{cases} 2, & i = j, \\ -\sharp\{\tau \in H\ ;\ \mathrm{out}(\tau) = i,\ \mathrm{in}(\tau) = j\}, & i \neq j. \end{cases}$$

We denote by $\mathfrak{g}$ the corresponding Kac-Moody Lie algebra and $U_q(\mathfrak{g})$ the corresponding quantized universal enveloping algebra.

4.2. Let $\mathcal{V}$ be the family of I-graded complex vector spaces $V = \oplus_{i\in I} V_i$. We set $\dim V = -\sum_{i\in I}(\dim V_i)\alpha_i \in Q_-$. For $\nu \in Q_-$, let $\mathcal{V}_\nu$ be the family of I-graded complex vector spaces V with $\dim V = \nu$.

Let us define the complex vector spaces $E_{V,\Omega}$ and X_V by

$$\begin{aligned} E_{V,\Omega} &= \bigoplus_{\tau\in\Omega} \operatorname{Hom}(V_{\mathrm{out}(\tau)}, V_{\mathrm{in}(\tau)}), \\ X_V &= \bigoplus_{\tau\in H} \operatorname{Hom}(V_{\mathrm{out}(\tau)}, V_{\mathrm{in}(\tau)}). \end{aligned}$$

In the sequel, a point of $E_{V,\Omega}$ or X_V will be denoted as $B = (B_\tau)$. Here B_τ is in $\operatorname{Hom}(V_{\mathrm{out}(\tau)}, V_{\mathrm{in}(\tau)})$.

We define the symplectic form ω on X_V by

$$\omega(B, B') = \sum_{\tau\in H} \varepsilon(\tau)\mathrm{tr}(B_{\bar\tau}B'_\tau), \tag{4.2.1}$$

where $\varepsilon(\tau) = 1$ if $\tau \in \Omega$, $\varepsilon(\tau) = -1$ if $\tau \in \bar\Omega$. We sometimes identify X_V and the cotangent bundle of $E_{V,\Omega}$ via ω.

The group $G_V = \prod_{i\in I} GL(V_i)$ acts on $E_{V,\Omega}$ and X_V by

$$G_V \ni g = (g_i) \quad : \quad (B_\tau) \mapsto (g_{\mathrm{in}(\tau)} B_\tau g^{-1}_{\mathrm{out}(\tau)}),$$

where $g_i \in GL(V_i)$ for each $i \in I$.

The Lie algebra of G_V is $\mathfrak{g}_V = \bigoplus_{i\in I} \operatorname{End}(V_i)$. We denote an element of $\mathfrak{g}_V$ by $A = (A_i)_{i\in I}$ with $A_i \in \operatorname{End}(V_i)$. The infinitesimal action of $A \in \mathfrak{g}_V$ on X_V at $B \in X_V$ is given by $[A, B]$. Let $\mu : X_V \to \mathfrak{g}_V$ be the moment map associated with the G_V-action on the symplectic vector space X_V. Its i-th component $\mu_i : X_V \to \operatorname{End}(V_i)$ is given by

$$\mu_i(B) = \sum_{\substack{\tau\in H \\ i=\mathrm{out}(\tau)}} \varepsilon(\tau) B_{\bar\tau} B_\tau.$$

For a non-negative integer n, we set

$$\mathfrak{S}_n = \{\sigma = (\tau_1, \tau_2, \cdots, \tau_n)\,;\, \tau_i \in H, \mathrm{in}(\tau_1) = \mathrm{out}(\tau_2), \cdots, \mathrm{in}(\tau_{n-1}) = \mathrm{out}(\tau_n)\},$$

and set $\mathfrak{S} = \bigcup_{n\geq 0} \mathfrak{S}_n$. For $\sigma = (\tau_1, \tau_2, \cdots, \tau_n)$, we set $\mathrm{out}(\sigma) = \mathrm{out}(\tau_1), \mathrm{in}(\sigma) = \mathrm{in}(\tau_n)$. For $B \in X_V$ we set $B_\sigma = B_{\tau_n} \cdots B_{\tau_1} : V_{\mathrm{out}(\tau_1)} \to V_{\mathrm{in}(\tau_n)}$. If $n = 0$, we understand that $\mathfrak{S}_n = \{1\}$ and B_1 is the identity. An element B of X_V is called nilpotent if there exists a positive integer n such that $B_\sigma = 0$ for any $\sigma \in \mathfrak{S}_n$.

Definition 4.2.1. We set

$$X_{0V} = \{B \in X_V\,;\, \mu(B) = 0\}$$

and

$$\Lambda_V = \{B \in X_V\,;\, \mu(B) = 0 \text{ and } B \text{ is nilpotent}\}.$$

It is clear that Λ_V is a G_V-stable closed subvariety of X_V. It is known that Λ_V is a Lagrangian variety [L5].

5. Lagrangian construction of crystal base

5.1. For each $\nu \in Q_-$, let us take $V(\nu) \in \mathcal{V}_\nu$ and set $X(\nu) = X_{V(\nu)}$, $X_0(\nu) = X_{0\,V(\nu)}$ and $\Lambda(\nu) = \Lambda_{V(\nu)}$. For ν, ν' and $\bar{\nu}$ in Q_- with $\nu = \nu' + \bar{\nu}$, we consider the diagram

$$X_0(\bar{\nu}) \times X_0(\nu') \xleftarrow{q_1} X_0'(\bar{\nu}, \nu') \xrightarrow{q_2} X_0(\nu). \tag{5.1}$$

Here $X_0'(\bar{\nu}, \nu')$ is the variety of $(B, \bar{\phi}, \phi')$ where $B \in X_0(\nu)$ and $\bar{\phi} = (\bar{\phi}_i)$, $\phi = (\phi_i')$ give an exact sequence

$$0 \to V(\bar{\nu})_i \xrightarrow{\bar{\phi}_i} V(\nu)_i \xrightarrow{\phi_i'} V(\nu')_i \to 0 \tag{5.2}$$

such that $\operatorname{Im}\bar{\phi}$ is stable by B. Hence B induces $\bar{B} : V(\bar{\nu}) \to V(\bar{\nu})$ and $B' : V(\nu') \to V(\nu')$. We define $q_1(B, \bar{\phi}, \phi') = (\bar{B}, B')$ and $q_2(B, \bar{\phi}, \phi') = B$.

The following lemma is easily proved.

Lemma 5.1.1. *Under the above notations the following two conditions are equivalent.*

(a) *B is nilpotent.*
(b) *Both B' and $\bar{B}$ are nilpotent.*

By this lemma, the diagram (5.1) induces the diagram

$$\Lambda(\bar{\nu}) \times \Lambda(\nu') \xleftarrow{q_1} \Lambda'(\bar{\nu}, \nu') \xrightarrow{q_2} \Lambda(\nu). \tag{5.3}$$

Here $\Lambda'(\bar{\nu}, \nu') = q_2^{-1}\big(\Lambda(\nu)\big) = q_1^{-1}\big(\Lambda(\bar{\nu}) \times \Lambda(\nu')\big)$.

For $i \in I$ and $p \in \mathbb{Z}_{\geq 0}$ we consider

$$X_0(\nu)_{i,p} = \{B \in X_0(\nu)\,;\, \varepsilon_i(B) = p\},$$

where

$$\varepsilon_i(B) = \dim \operatorname{Coker}\Big(\bigoplus_{\tau;\mathrm{in}(\tau)=i} V(\nu)_{\mathrm{out}(\tau)} \xrightarrow{(B_\tau)} V(\nu)_i\Big).$$

It is clear that $X_0(\nu)_{i,p}$ is a locally closed subvariety of $X_0(\nu)$.

5.2. In this and the next subsections, we assume that $\nu = \bar{\nu} - c\alpha_i$ for $c \in \mathbb{Z}_{\geq 0}$. We set $V = V(\nu)$ and $\bar{V} = V(\bar{\nu})$.

Let us consider the special case of (5.1). Note that $X_0(-c\alpha_i) = \{0\}$.

$$X_0(\bar{\nu}) \cong X_0(\bar{\nu}) \times X_0(-c\alpha_i) \xleftarrow{\varpi_1} X_0'(\bar{\nu}, -c\alpha_i) \xrightarrow{\varpi_2} X_0(\nu). \tag{5.4}$$

Lemma 5.2.1. *Let $p \in \mathbb{Z}_{\geq 0}$. Then we have*

$$\varpi_1^{-1}(X_0(\bar{\nu})_{i,p}) = \varpi_2^{-1}(X_0(\nu)_{i,p+c}).$$

Definition 5.2.2. We set

$$X_0'(\bar{\nu}, -c\alpha_i)_p = \varpi_1^{-1}(X_0(\bar{\nu})_{i,p}) = \varpi_2^{-1}(X_0(\nu)_{i,p+c}).$$

Suppose $p = 0$. Then we have following diagram

$$X_0(\bar{\nu}) \supset X_0(\bar{\nu})_{i,0} \xleftarrow{\varpi_1} X_0'(\bar{\nu}, -c\alpha_i)_0 \xrightarrow{\varpi_2} X_0(\nu)_{i,c} \subset X_0(\nu). \tag{5.5}$$

Note that $X_0(\bar{\nu})_{i,0}$ is an open subvariety of $X_0(\bar{\nu})$.

Lemma 5.2.3. (1) *$\varpi_2 : X_0'(\bar{\nu}, -c\alpha_i)_0 \longrightarrow X_0(\nu)_{i,c}$ is a principal fiber bundle with $GL(\mathbb{C}^c) \times \prod_{j \in I} GL(V(\bar{\nu})_j)$ as fiber.*

(2) *$\varpi_1 : X_0'(\bar{\nu}, -c\alpha_i)_0 \longrightarrow X_0(\bar{\nu})_{i,0}$ is a smooth map whose fiber is a connected rational variety of dimension $\sum_{j\in I} \left(\dim V(\nu)_j\right)^2 - c\,(\alpha_i, \bar{\nu})$.*

Now we denote by $B(\infty; \nu)$ the set of irreducible components of $\Lambda(\nu)$. For $\Lambda \in B(\infty; \nu)$, we define $\varepsilon_i(\Lambda) = \varepsilon_i(B)$ by taking a generic point B of Λ . For $l \in \mathbb{Z}_{\geq 0}$, we set $B(\infty; \nu)_{i,l}$ the set of all elements of $B(\infty; \nu)$ such that $\varepsilon_i(\Lambda) = l$.

The preceding lemma implies the following proposition.

Proposition 5.2.4.

$$B(\infty; \bar{\nu})_{i,0} \cong B(\infty; \nu)_{i,c}.$$

Definition 5.2.5. Suppose that $\bar{\Lambda} \in B(\infty; \bar{\nu})_0$ corresponds to $\Lambda \in B(\infty; \nu)_c$ by this isomorphism. Then we define maps $\tilde{f}_i^{\,c} : B(\infty; \bar{\nu})_0 \to B(\infty; \nu)_c$ and $\tilde{e}_i^{\max} : B(\infty; \nu)_c \to B(\infty; \bar{\nu})_0$ by

$$\tilde{f}_i^{\,c}(\bar{\Lambda}) = \Lambda,$$
$$\tilde{e}_i^{\max}(\Lambda) = \bar{\Lambda}.$$

Furthermore we define the maps

$$\tilde{e}_i : \bigsqcup_{\nu} B(\infty; \nu) \to \bigsqcup_{\nu} B(\infty; \nu) \sqcup \{0\} \text{ and}$$

$$\tilde{f}_i : \bigsqcup_{\nu} B(\infty; \nu) \to \bigsqcup_{\nu} B(\infty; \nu)$$

as follows. If $c > 0$ then we define

$$\tilde{e}_i : B(\infty; \nu)_c \xrightarrow{\tilde{e}_i^{\max}} B(\infty; \bar{\nu})_0 \xrightarrow{\tilde{f}_i^{\,c-1}} B(\infty; \nu + \alpha_i)_{c-1},$$

and $\tilde{e}_i(\Lambda) = 0$ for $\Lambda \in B(\infty;\nu)_0$. We define $\tilde{f}_i$ by

$$\tilde{f}_i : B(\infty;\nu)_c \xrightarrow{\tilde{e}_i^{\max}} B(\infty;\bar{\nu})_0 \xrightarrow{\tilde{f}_i^{c+1}} B(\infty;\nu-\alpha_i)_{c+1}.$$

Then the maps $\tilde{e}_i^{\max}$ (resp. $\tilde{f}_i^{c}$) which is constructed in the definition may be considered as the c-th power of $\tilde{e}_i$ (resp. $\tilde{f}_i$). Let us define a map $\mathrm{wt} : \bigsqcup_\nu B(\infty;\nu) \to P$ by $\mathrm{wt}(\Lambda) = \nu \in P$ for $\Lambda \in B(\infty;\nu)$. We set $\varphi_i(\Lambda) = \varepsilon_i(\Lambda) + \langle h_i, \mathrm{wt}(\Lambda)\rangle$.

Theorem 5.2.6. *$\bigsqcup_\nu B(\infty;\nu)$ is a crystal in the sense of Definition* 3.1.1.

Lemma 5.2.7. *If $\Lambda \in B(\infty;\nu)$ satisfies $\varepsilon_i(\Lambda) = 0$ for every i, then $\nu = 0$.*

5.3. We shall use the diagram (5.1.1) in the opposite way to (5.4).

$$X_0(\bar{\nu}) \cong X_0(-c\alpha_i) \times X_0(\bar{\nu}) \xleftarrow{\varpi_1'} X_0'(-c\alpha_i,\bar{\nu}) \xrightarrow{\varpi_2'} X_0(\nu). \tag{5.6}$$

We define for $B \in X_0(\nu)$

$$\varepsilon_i^*(B) = \dim \mathrm{Ker}\,(V(\nu)_i \xrightarrow{(B_\tau)} \bigoplus_{\tau;\mathrm{out}(\tau)=i} V(\nu)_{\mathrm{in}(\tau)}).$$

For $\Lambda \in B(\infty;\nu)$ we define $\varepsilon_i^*(\Lambda)$ as $\varepsilon_i^*(B)$ by taking a generic point B of Λ. We set

$$X_0(\nu)_i^p = \{B \in X_0(\nu)\,;\, \varepsilon_i^*(B) = p\},$$
$$B(\infty;\nu)_i^p = \{\Lambda \in B(\infty;\nu)\,;\, \tilde{e}_i^*(\Lambda) = p\}.$$

We choose an isomorphism between $V(\nu)_i$ and its dual for every i. Then $* : B \mapsto {}^tB$ gives an automorphism of $X_0(\nu)$ and $\Lambda(\nu)$ is invariant by this automorphism. This induces an automorphism $* : B(\infty;\nu) \to B(\infty;\nu)$. Since $\Lambda(\nu)$ is $G_{V(\nu)}$-invariant, this does not depend of the choice of isomorphisms $V(\nu)^* \simeq V(\nu)$. The diagrams (5.4) and (5.6) are transformed by $*$. We have

$$\varepsilon_i^*(\Lambda) = \varepsilon_i(\Lambda^*).$$

We define

$$\begin{aligned} \tilde{e}_i^{*\max} &= * \circ \tilde{e}_i^{\max} \circ *, \\ \tilde{e}_i^{*} &= * \circ \tilde{e}_i \circ *, \\ \tilde{f}_i^{*} &= * \circ \tilde{f}_i \circ *, \\ \varphi_i^*(\Lambda) &= \varphi(\Lambda^*). \end{aligned}$$

Note that $\tilde{e}_i^*$ and $\tilde{f}_i^*$ may be defined as $\tilde{e}_i$ and $\tilde{f}_i$ using (5.6) instead of (5.4). We have

$$\tilde{e}_i^{*\max} : B(\infty;\nu)^c \xrightarrow{\sim} B(\infty;\bar{\nu})^0$$

Proposition 5.3.1. *Let Λ be an irreducible component of $\Lambda(\nu)$. We set $c = \varepsilon_i^*(\Lambda)$ and $\bar{\Lambda} = \tilde{e}_i^{*\max}\Lambda$. Then we have*

(1)
$$\varepsilon_i(\Lambda) = \max(\varepsilon_i(\bar{\Lambda}), c - (\alpha_i, \bar{\nu})).$$

(2) *for $i \neq j$,*
$$\varepsilon_i^*(\tilde{e}_j(\Lambda)) = c,$$
$$\tilde{e}_i^{*\max}(\tilde{e}_j(\Lambda)) = \tilde{e}_j(\bar{\Lambda}).$$

(3) *Assume $\varepsilon_i(\Lambda) > 0$. Then we have*
$$\varepsilon_i^*(\tilde{e}_i(\Lambda)) = \begin{cases} c & \text{if } \varepsilon_i(\bar{\Lambda}) \geq c - (\alpha_i, \bar{\nu}), \\ c - 1 & \text{if } \varepsilon_i(\bar{\Lambda}) < c - (\alpha_i, \bar{\nu}), \end{cases}$$

and
$$\tilde{e}_i^{*max}(\tilde{e}_i(\Lambda)) = \begin{cases} \tilde{e}_i(\bar{\Lambda}), & \text{if } \varepsilon_i(\bar{\Lambda}) \geq c - (\alpha_i, \bar{\nu}), \\ \bar{\Lambda}, & \text{if } \varepsilon_i(\bar{\Lambda}) < c - (\alpha_i, \bar{\nu}). \end{cases}$$

We recall that $B(\infty)$ is the crystal base of $U_q^-(\mathfrak{g})$.

Theorem 5.3.2. *We have an isomorphism of crystals*
$$\bigsqcup_{\nu \in Q_-} B(\infty; \nu) \cong B(\infty) .$$

We denote by $\Lambda_b \in \bigsqcup_{\nu \in Q_-} B(\infty; \nu)$ the corresponding element to $b \in B(\infty)$ under this isomorphism. The following proposition is proved by Lusztig.

Proposition 5.3.3. *$\Lambda(\nu)$ is a Lagrangian subvariety of $X_0(\nu)$.*

By this result, any Λ_b in $B(\infty; \nu)$ is an irreducible Lagrangian subvariety of $X_0(\nu)$.

6. Review of the theory of canonical base

6.1. Canonical base. Let us recall the results on Lusztig on canonical bases. We write $\mathcal{D}(X)$ for the bounded derived category of complexes of sheaves of $\mathbb{C}$-vector spaces on the associated complex variety with an algebraic variety X over $\mathbb{C}$. Objects of $\mathcal{D}(X)$ are referred to as complexes. We shall use the notations of [BBD]; in particular, $[d]$ denotes a shift by $[d]$ degrees, and for a morphism f of algebraic varieties, f^* denotes the inverse image functor, $f_!$ denotes direct image with compact support, etc.

We fix an orientation Ω of quiver. Let $\nu \in Q_-$ and let S_ν be the set of all pairs (i, a) where $\mathrm{i} = (i_1, i_2, \cdots, i_m)$ is a sequence of elements of I and $\mathrm{a} = (a_1, a_2, \cdots, a_m)$ is a sequence of non-negative integers such that $\sum_l a_l \alpha_{i_l} = -\nu$. Now let $V \in \mathcal{V}_\nu$ and let $(\mathrm{i}, \mathrm{a}) \in S_\nu$. A flag of type (i, a) is, by definition, a sequence $\phi = (V = V^0 \supset V^1 \supset \cdots \supset V^m = 0)$ of I-graded subspace of V such that, for any $l = 1, 2, \cdots, m$, the I-graded vector space V^{l-1}/V^l is zero in degrees $\neq i_l$ and has dimension a_l in degree i_l. We define a variety $\tilde{\mathcal{F}}_{\mathrm{i},\mathrm{a}}$ of all pairs (B, ϕ) such that $B \in E_{V,\Omega}$ and ϕ is a B-stable flag of type (i, a). The group G_V acts on $\tilde{\mathcal{F}}_{\mathrm{i},\mathrm{a}}$ in natural way. We denote by

$\pi_{\mathrm{i,a}} : \tilde{\mathcal{F}}_{\mathrm{i,a}} \to E_{V,\Omega}$ the natural projection. We note that $\pi_{\mathrm{i,a}}$ is a G_V-equivariant proper morphism. We set $L_{\mathrm{i,a};\Omega} = (\pi_{\mathrm{i,a}})_!(1) \in \mathcal{D}(E_{V,\Omega})$. Here $1 \in \mathcal{D}(\tilde{\mathcal{F}}_{\mathrm{i,a}})$ is the constant sheaf on $\tilde{\mathcal{F}}_{\mathrm{i,a}}$. By the decomposition theorem [BBD], $L_{\mathrm{i,a};\Omega}$ is a semisimple complex.

Let $\mathcal{P}_{V,\Omega}$ be the set of isomorphism class of simple perverse sheaves L on $E_{V,\Omega}$ such that $L[d]$ appears as direct summand of $L_{\mathrm{i,a};\Omega}$ for some $(\mathrm{i,a}) \in S_\nu$ and some $d \in \mathbb{Z}$. We write $\mathcal{Q}_{V,\Omega}$ for the subcategory of $\mathcal{D}(E_{V,\Omega})$ consisting of all complexes that are isomorphic to finite direct sums of complexes of the form $L[d]$ for various simple perverse sheaves $L \in \mathcal{P}_{V,\Omega}$ and various $d \in \mathbb{Z}$. Any complex in $\mathcal{Q}_{V,\Omega}$ is semisimple and G_V-equivariant.

Take $V \in \mathcal{V}_\nu, V' \in \mathcal{V}_{\nu'}, \bar{V} \in \mathcal{V}_{\bar{\nu}}$ for $\nu = \nu' + \bar{\nu}$ in Q_-). We consider the diagram

$$E_{\bar{V},\Omega} \times E_{V',\Omega} \xleftarrow{p_1} E' \xrightarrow{p_2} E'' \xrightarrow{p_3} E_{V,\Omega}\,. \tag{6.1}$$

Here E' is the variety of $(B, \bar{\phi}, \phi')$ where $B \in E_{V,\Omega}$ and $0 \to \bar{V} \xrightarrow{\bar{\phi}} V \xrightarrow{\phi'} V' \to 0$ is a B-stable exact sequence of I-graded vector spaces, and E'' is the variety of (B, C) where $B \in E_{V,\Omega}$ and C is a B-stable I-graded subspace of V with $\dim C = \bar{\nu}$. The morphisms p_1, p_2 and p_3 are defined by $p_1(B, \bar{\phi}, \phi') = (B|_{\bar{V}}, B|_{V'})$, $p_2(B, \bar{\phi}, \phi') = (B, \mathrm{Im}\,(\bar{\phi}))$ and $p_3(B, C) = B$. Note that p_1 is smooth with connected fiber, p_2 is a principal $G_{V'} \times G_{\bar{V}}$-bundle, and p_3 is proper.

Let $L' \in \mathcal{Q}_{V',\Omega}$ and $\bar{L} \in \mathcal{Q}_{\bar{V},\Omega}$. Consider the exterior tenser product $\bar{L} \boxtimes L'$. Then there is $(p_2)_\flat p_1^*(\bar{L} \boxtimes L') \in \mathcal{D}(E'')$ such that $(p_2)^*(p_2)_\flat p_1^*(\bar{L} \boxtimes L') \cong p_1^*(\bar{L} \boxtimes L')$. We define $L' * \bar{L} \in \mathcal{Q}_{V,\Omega}$ by $(p_3)_!(p_2)_\flat p_1^*(\bar{L} \boxtimes L')[d_1 - d_2]$ where d_i is the fiber dimension of p_i $(i = 1, 2)$. Let $\mathcal{K}_{V,\Omega}$ be the Grothendieck group of $\mathcal{Q}_{V,\Omega}$. We considered as a $\mathbb{Z}[q, q^{-1}]$-module by $q(L) = L[1]$, $q^{-1}(L) = L[-1]$. Then $\mathcal{K}_\Omega = \oplus_{\nu \in Q_-} \mathcal{K}_{V(\nu),\Omega}$ has a structure of an associative graded $\mathbb{Z}[q, q^{-1}]$-algebra by the operation $*$. We denote by $F_i \in \mathcal{K}_{V(-\alpha_i),\Omega}$ the element attached to $1 \in \mathcal{D}(E_{V(-\alpha_i),\Omega})$.

Theorem 6.1.1. [L3] *There is a unique* $\mathbb{Q}(q)$*-algebra isomorphism*

$$\Gamma_\Omega : U_q^-(\mathfrak{g}) \to \mathcal{K}_\Omega \underset{\mathbb{Z}[q,q^{-1}]}{\otimes} \mathbb{Q}(q)$$

such that $\Gamma_\Omega(f_i) = F_i$.

Let us identify $L \in \mathcal{P}_{V,\Omega}$ with $L \otimes 1 \in \mathcal{K}_\Omega \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$. We set $\mathbf{B} = \Gamma_\Omega^{-1}(\sqcup_{V \in \mathcal{V}} \mathcal{P}_{V,\Omega})$ and call it the canonical basis of $U_q^-(\mathfrak{g})$. By [GL], $\mathbf{B}$ and $B(\infty)$ are canonically identified. For $b \in B(\infty)$ the corresponding perverse sheaf is denoted by $L_{b,\Omega}$.

6.2. Let Y be a smooth algebraic variety. For any $L \in \mathcal{D}(Y)$, we denote by $SS(L)$ the singular support (or the characteristic variety) of L. It is known that $SS(L)$ is a closed Lagrangian subvariety of T^*Y (See [KS]).

We recall that $T^*E_{V,\Omega}$ is identified with X_V. By the Fourier transform method, we have

Theorem 6.2.1. [L5] $SS(L_{b,\Omega})$ *does not depend on the choice of* Ω.

We say $i \in I$ is sink (resp. source) of Ω if there is no arrow $i \to j$ (resp. $j \to i$) in Ω.

Theorem 6.2.2. (1) *For any $L \in \mathcal{P}_{V,\Omega}$ the singular support $SS(L)$ is a union of irreducible components of Λ_V.*

(2) *For any $b \in B(\infty)$ and $i \in I$, we have*

$$\Lambda_b \subset SS(L_{b,\Omega}) \subset \Lambda_b \cup \bigcup_{\varepsilon_i(b')>\varepsilon_i(b)} \Lambda_{b'}. \tag{6.2}$$

Note that if there is a bijection $s : B(\infty) \to B(\infty)$ such that $SS(L_{b,\Omega}) \supset \Lambda_{s(b)}$ for any $b \in B(\infty)$, then s must be the identity (*cf.* Problem in [L5]). In fact, by the decreasing induction on $\varepsilon_i(b)$, (6.2) implies $s(b) = b$.

The following problem is also asked by Lusztig [L5].

Problem 1. If the underlying graph is of type A, D, E, then the singular support of any $L \in \mathcal{P}_{V,\Omega}$ is irreducible.

Furthermore he noted that the next conjecture [KL2] follows from Problem 1 for type A (see §8.1). In fact it is easy to see that they are equivalent.

Conjecture 2. Let X be the flag manifold for $SL(n)$ and let X_w be the Schubert variety of $SL(n)$ which corresponds to the element w of the Weyl group W. Then the singular support of ${}^{\pi}\mathbb{C}_{X_w}$ is irreducible.

In the next section we construct a counterexample of Problem 1 for a graph of type A.

7. Counterexample to Problem 1

7.1. In this and the next section we assume that the underlying graph is of type A.

Let us take $\nu \in Q_-$ and $V \in \mathcal{V}_\nu$. Let $\mathcal{O}_\Omega$ be a G_V-orbit in $E_{V,\Omega}$. As the underlying graph is of type A, $E_{V,\Omega}$ has finitely many G_V-orbits. By [L3] we know that there is one-to-one correspondence between G_V-orbits $\mathcal{O}$ in $E_{V,\Omega}$ between the crystal basis $b \in B(\infty)$ of $U_q^-(\mathfrak{g})$ of weight ν by $\Lambda_b = T^*_{\mathcal{O}} E_{V,\Omega}$. For $b \in B(\infty)$, we denote by $\mathcal{O}_{b,\Omega}$ the corresponding G_V-orbit. The next theorem is due to Lusztig (See [L3].).

Theorem 7.1.1. *Let $b \in B(\infty)$. Then we have*

$$L_{b.\Omega} = {}^{\pi}\mathbb{C}_{\mathcal{O}_{b,\Omega}}$$

where $\mathbb{C}_{\mathcal{O}_{b,\Omega}}$ is the constant sheaf on $\mathcal{O}_{b,\Omega}$ and ${}^{\pi}\cdot$ is the minimal extension functor.

Note that $SS(L_{b.\Omega})$ depends only on $b \in B(\infty)$ and not on Ω (cf. Theorem 6.2.1).

7.2. In the rest of the section, we shall present a counterexample of Problem 1 when the underlying graph is of type A_5. Let us take a graph of type A_5 and its orientation Ω as follows;

$$\overset{1}{\circ} \xleftarrow{\tau_1} \overset{2}{\circ} \xleftarrow{\tau_2} \overset{3}{\circ} \xleftarrow{\tau_3} \overset{4}{\circ} \xleftarrow{\tau_4} \overset{5}{\circ}.$$

Let $\nu = -2\alpha_1 - 4\alpha_2 - 4\alpha_3 - 4\alpha_4 - 2\alpha_5$. Set $b = \tilde{f}_2\tilde{f}_1\tilde{f}_3\tilde{f}_2\tilde{f}_4^2\tilde{f}_3^2\tilde{f}_2\tilde{f}_1\tilde{f}_5^2\tilde{f}_4^2\tilde{f}_3\tilde{f}_2 u_\infty$ and $b' = \tilde{f}_2^2\tilde{f}_1^2\tilde{f}_3^2\tilde{f}_4^2\tilde{f}_3^2\tilde{f}_2^2\tilde{f}_5^2\tilde{f}_4^2 u_\infty$. Then the following points B_0 and B_0' of $E_{V,\Omega}$ are in $\mathcal{O}_{b,\Omega}$ and $\mathcal{O}_{b',\Omega}$, respectively.

$$(B_0)_{\tau_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, (B_0)_{\tau_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(B_0)_{\tau_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, (B_0)_{\tau_4} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$(B_0')_{\tau_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, (B_0')_{\tau_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(B_0')_{\tau_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, (B_0')_{\tau_4} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we can state a counterexample of Conjecture 1.

Theorem 7.2.1.

$$SS({}^{\pi}\mathbb{C}_{\mathcal{O}_{b,\Omega}}) \supset \Lambda_b \cup \Lambda_{b'}.$$

Remark 1. In fact, although we don't give a proof (relying on Lemmas 8.2.1 and 8.2.2), they coincide.

8. Relation with Schubert cells

8.1. We consider the Dynkin diagram of type A_{2n-1} and take its orientation Ω_0 as follows:

$$\Omega_0;\ \overset{1}{\circ} \xleftarrow{\tau_1} \overset{2}{\circ} \xleftarrow{\tau_2} \cdots \xleftarrow{\tau_{2n-2}} \overset{2n-1}{\circ}.$$

Let $\nu_{cl} = \sum_{i=1}^{2n-1} -\nu_{cl}(i)\alpha_i$ where $\nu_{cl}(i) = i$ (for $1 \le i \le n$), $= 2n - i$ (for $n \le i \le 2n-1$) and let $V_{cl} \in \mathcal{V}_{\nu_{cl}}$.

Let us set

$$E^{\natural}_{V_{cl}} = \{B \in E_{V_{cl},\Omega_0}\ ;\ B_{\tau_i} \text{ is injective for } 1 \le i \le n-1 \text{ and surjective } n \le i \le 2n-2\}.$$

It is clear that $E^{\natural}_{V_{cl}}$ is G_V-invariant.

Let G be $GL(n,\mathbb{C})$, B a Borel subgroup of G, W the Weyl group of G and $X = G/B$ the flag variety. We set $X_w = BwB/B$ $(w \in W)$. Then $X = \bigsqcup_{w\in W} X_w$ gives a cellular decomposition of X.

The decomposition of $X \times X$ to G-orbits is given by $X \times X = \bigsqcup_{w\in W} Y_w$ with $Y_w = G \cdot (\{eB\} \times X_w)$. Then, the following two conditions are equivalent:

(8.1.1) $\qquad SS({}^{\pi}\mathbb{C}_{X_w})$ *is an irreducible variety.*

(8.1.2) $\qquad SS({}^{\pi}\mathbb{C}_{Y_w})$ *is an irreducible variety.*

We have a G-equivariant isomorphism

$$E^{\natural}_{V_{cl}} / \prod_{j\neq n} GL(V_{clj}) \simeq X \times X.$$

Therefore there is a one-to-one correspondence between G-orbits of $X \times X$ and $G_{V_{cl}}$-orbits of $E^{\natural}_{V_{cl}}$. Let us denote by $\mathcal{O}_{w,\Omega_0}$ the $G_{V_{cl}}$-orbit of $E^{\natural}_{V_{cl}}$ corresponding to Y_w. Then we have

(8.1) The irreducibility of $SS({}^{\pi}\mathbb{C}_{X_w})$ is equivalent to that of $SS({}^{\pi}\mathbb{C}_{\mathcal{O}_{w,\Omega_0}})$.

8.2. For an orientation Ω we say that $i \in I$ is sink (resp. source) of Ω if there is no arrow $i \to j$ (resp. $j \to i$) in Ω.

Lemma 8.2.1. (1) *Let $b \in B(\infty)$. If $SS(L_{b,\Omega}) \supset \Lambda_{b'}$, then $\varepsilon_i(b) \le \varepsilon_i(b')$ for any $i \in I$.*

(2) *If $\varepsilon_i(b) = \varepsilon_i(b')$, then the condition $SS(L_{b,\Omega}) \supset \Lambda_{b'}$ is equivalent to*

$$SS(L_{\tilde{e}_i^{\max} b,\Omega}) \supset \Lambda_{\tilde{e}_i^{\max} b'}.$$

For an orientation Ω, let $s_i\Omega$ $(i \in I)$ be the orientation obtained from Ω by reversing each arrow that ends or starts at i.

We define a map $S_i : \{b \in B(\infty)\,;\, \varepsilon_i(b) = 0\} \to \{b \in B(\infty)\,;\, \varepsilon_i^*(b) = 0\}$ by $S_i(b) = \tilde{f}_i^{\varphi_i^*(b)} \tilde{e}_i^{*\max} b$. Then S_i is bijective. Note that $\mathrm{wt}(S_i(b)) = s_i(\mathrm{wt}(b))$ (see [S]). Here s_i is the simple reflection.

Lemma 8.2.2. *Assume that $b, b' \in B(\infty)$ has the same weight and that $i \in I$ satisfies $\varepsilon_i(b) = \varepsilon_i(b') = 0$. Then the following two conditions are equivalent;*

(1) $SS(L_{b,\Omega}) \supset \Lambda_{b'}$,

(2) $SS(L_{S_i(b),s_i\Omega}) \supset \Lambda_{S_i(b')}$.

8.3. Only by using Lemma 8.2.1 and 8.2.2 we can show

Proposition 8.3.1. *Conjecture 2 is true for* $1 \leq n \leq 7$.

In fact we used a computer to check this.

There is a counterexample in the $n = 8$ case derived by the counterexample in Theorem 7.2.1.

Example 8.3.1. Let

$$\begin{aligned} w &= s_1 s_3 s_2 s_4 s_3 s_5 s_4 s_3 s_2 s_1 s_6 s_7 s_6 s_5 s_4 s_3 \quad \text{and} \\ w' &= s_1 s_3 s_4 s_3 s_5 s_4 s_3 s_7. \end{aligned}$$

Here $\{s_i\}_{i \in I}$ are the standard generators of symmetric group. Then we have

$$SS({}^{\pi}\mathbb{C}_{\mathcal{O}_{w,\Omega_0}}) = \overline{T^*_{\mathcal{O}_{w,\Omega_0}} E_{V,\Omega_0}} \cup \overline{T^*_{\mathcal{O}_{w',\Omega_0}} E_{V,\Omega_0}}.$$

This singularity is also realized by a partial flag manifold as follows. Let X' be the set of flags $\{F_j\}$ of $\mathbb{C}^8$ with $0 = F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 = \mathbb{C}^8$ and $\dim F_j = 2j$ ($j = 1, 2, 3$). Set $Z = X' \times X' = \{(F, F') \in X' \times X'\}$. Let Z_1 be the $SL(8)$-orbit of Z given by the following table of $\dim \mathrm{Gr}_i^F \mathrm{Gr}_j^{F'}$:

$j\backslash i$	1	2	3	4
1	1	0	1	0
2	0	1	0	1
3	1	0	1	0
4	0	1	0	1

and Z_2 is given by

$j\backslash i$	1	2	3	4
1	2	0	0	0
2	0	0	2	0
3	0	2	0	0
4	0	0	0	2

Then $\overline{Y_w}$ (resp. $\overline{Y_{w'}}$) is the inverse image of $\overline{Z_1}$ (resp. $\overline{Z_2}$) by the canonical morphism $X \times X \to X' \times X'$. Hence the characteristic variety of the intersection cohomology sheaf of Z_1 contains the conormal bundle of Z_2. The singularity of Z_1 at Z_2 is the same as the one of the counterexample in Theorem 7.2.1.

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