Some results on behavior of solutions to one-phase
Stefan problems for a semilinear parabolic equation

Toyohiko AIKI (愛木豊彦)
Department of Mathematics, Faculty of Education, Gifu University
Gifu 501-11, Japan

Hitoshi IMAI (今井仁司)
Faculty of Engineering, Tokushima University
Tokushima 770, Japan

1 Introduction

We consider the following one-phase Stefan problem $SP := SP(u_0, \ell_0)$ for a semilinear
parabolic equations in one-dimensional space: Find a curve (a free boundary) $x = \ell(t) > 0$
on $[0, T], 0 < T < \infty$, and a function $u = u(t, x)$ on $Q(T) := (0, T) \times (0, \infty)$ satisfying that

$$u_t = u_{xx} + u^{1+\alpha} \quad \text{in} \quad Q_\ell(T) := \{(t, x); 0 < t < T, 0 < x < \ell(t)\}, \quad (1.1)$$

$$u(0, x) = u_0(x) \quad \text{for} \quad 0 \leq x \leq \ell_0, \quad (1.2)$$

$$u_x(t, 0) = 0 \quad \text{for} \quad 0 < t < T; \quad (1.3)$$

$$u(t, x) = 0 \quad \text{for} \quad 0 < t < T \quad \text{and} \quad x \geq \ell(t), \quad (1.4)$$

$$\ell'(t) = -u_x(t, \ell(t)) \quad \text{for} \quad 0 < t < T, \quad (1.5)$$

$$\ell(0) = \ell_0, \quad (1.6)$$

where $\alpha$ and $\ell_0$ are given positive constants and $u_0$ is a given initial function on $[0, \ell_0]$.

The local existence and the uniqueness for solutions to the above problem $SP$ were already
investigated by Fasano-Primicerio [7] and Aiki-Kenmochi [1, 5, 8]. Since there are blow-up
solutions of the usual initial boundary value problem for the semilinear equation (1.1) in a
bounded domain, by using comparison principle it is clear that $SP$ has a blow-up solutions
for a large initial data. In author's previous works [2, 3, 6] we showed some theorems and
numerical experiments concerned with the behavior of free boundaries of blow-up solutions to
one-phase Stefan problems with homogenous Neumann and Dirichlet boundary condition.
On global existence (see Theorem 2.2) in [4] we obtain a solution to the problem $SP$ on $[0, \infty)$, exponential decay of $|u|_{L^\infty(0, \ell(t))}$ and boundedness of the free boundary $\ell$ for a small initial function $u_0$ in case $\alpha > 1$.

The purpose of the present paper is to establish the stability of a global solution to the problem $SP$ in the following sense: Let $\alpha > 1$ and $\{u, \ell\}$ be a solution to $SP$ on $[0, \infty)$ satisfying that there are positive constants $L, M$ and $\mu$ such that

$$\ell(t) \leq L \quad \text{for} \quad t \geq 0 \quad \text{and} \quad |u(t, x)| \leq M \exp(-\mu t) \quad \text{for} \quad t \geq 0 \text{ and } x \geq 0.$$ 

Then, there exists a positive constant $\delta$ such that if $|u_0 - \hat{u}|_{L^p(0, \hat{\ell}_0)} < \delta$, where $p > 1$ is some suitable constant, the problem $SP(\hat{u}_0, \hat{\ell}_0)$ has a solution $\{\hat{u}, \hat{\ell}\}$ on $[0, \infty)$ satisfying that the free boundary $\{\hat{\ell}(t)\}$ is bounded and $|\hat{u}(t)|_{L^\infty(0, \hat{\ell}(t))}$ decays in exponential order. We note that the global existence and stability concerned with the problem $SP$ are not proved, theoretically, for $0 < \alpha \leq 1$.

2 A main result

We give a precise definition of a solution to $SP$.

**Definition 2.1.** We say that a pair $\{u, \ell\}$ is a solution of $SP(u_0, \ell_0)$ on $[0, T]$, $0 < T < \infty$, if the following properties are fulfilled:

(S1) $u \in W^{1,2}(0, T; L^2(0, \ell(t))) \cap L^\infty(0, T; W^{1,2}(0, \ell(t)))$, and $\ell \in W^{1,2}(0, T)$ with $0 < \ell$ on $[0, T]$.

(S2) (1.1) holds in the sense of $\mathcal{D}'(Q_\ell(T))$ and (1.2) $\sim$ (1.6) are satisfied.

Also, we call that a couple $\{u, \ell\}$ is a solution of $SP$ on an interval $[0, T')$, $0 < T' \leq \infty$, if it is a solution of $SP$ on $[0, T]$ in the above sense for any $0 < T < T'$.

We introduce the following space in order to describe the class of initial functions which satisfy the compatibility condition.

$$V = \{(z, s); s > 0 \text{ and } z \in W^{1,2}(0, \infty) \text{ with } z \geq 0 \text{ on } [0, s] \text{ and } z(y) = 0 \text{ for } y \geq s\}.$$ 

First, we recall the theorem concerned with local existence, uniqueness, comparison, continuation and regularity of solutions to $SP$.

**Theorem 2.1.** (cf. [1; Theorems 1.1 and 5.1] and [7; Theorem 1]) Let $\alpha > 0$ and
(i) Then, there is a positive number $T_0$ such that the problem $SP$ has one and only one solution \( \{u, \ell\} \) on \([0, T_0]\).

(ii) We assume that \((\hat{u}_0, \hat{\ell}_0) \in V, \ell_0 \leq \hat{\ell}_0, u_0 \leq \hat{u}_0 \) on \([0, \infty)\) and \(u_0 \neq \hat{u}_0\). Let \( \{u, \ell\} \) (resp. \( \{\hat{u}, \hat{\ell}\} \)) be a solution to $SP(u_0, \ell_0)$ (resp. $SP(\hat{u}_0, \hat{\ell}_0)$) on \([0, T]\), \(0 < T < \infty\). Then, we have
\[
\ell \leq \hat{\ell} \text{ on } [0, T] \text{ and } u < \hat{u} \text{ on } Q(T).
\]

(iii) If \(u_0 \in C^1([0, \ell_0])\) and \(u_{02}(0) = 0\), then the solution \( \{u, \ell\} \) to $SP(u_0, \ell_0)$ on \([0, T]\) satisfying that \(u_x\) is continuous on \(Q_T(T)\), \(u_t\) and \(u_{xx}\) are continuous on \(Q_T(T)\) and \(\ell \in C^1([0, T])\).

(iv) Let \( \{u, \ell\} \) be a solution to $SP(u_0, \ell_0)$ on \([0, T']\), \(0 < T' < \infty\), and \(M\) be any positive number. If \(|u(t, x)| \leq M\) for \((t, x) \in Q(T')\), then the solution is extended in time beyond \(T'\).

**Remark 2.1.** By Definition 2.1 and Theorem 2.1 (iii) for a solution \( \{u, \ell\} \) to $SP$ on \([0, T]\), \(u_x\) is continuous on the set \(\{(t, x); 0 \leq x \leq \ell(t), 0 < t \leq T\}\), \(u_t\) and \(u_{xx}\) are continuous on \(Q_T(T)\) and \(\ell \in C^1([0, T])\). Hence, applying the strong maximum principle to $SP$ we get the assertion (ii) in Theorem 2.1.

Throughout this paper for the problem $SP$, we say that \([0, T]\), \(0 < T \leq +\infty\), is the maximal interval of existence of the solution, if the problem has a solution on time-interval \([0, T']\), for every \(T'\) with \(0 < T' < T\) and the solution can not be extended in time beyond \(T\). Also, for simplicity we put
\[
E(z, s) = \int_0^s z(x)dx + s \quad \text{for } (z, s) \in V,
\]
and
\[
V(M, L) = \{(z, s) \in V; s \leq L \text{ and } z(x) \leq M \text{ for } 0 < x < s\},
\]
where \(M\) and \(L\) are positive numbers.

Here, we give a theorem concerned with the global existence of solutions to $SP$.

**Theorem 2.2.** (cf. [4; Theorem 1.2]) Let \(\alpha > 1\), \((u_0, \ell_0) \in V\). Then, for any positive number \(M\) there exist positive numbers \(\delta_0 = \delta(M, \alpha) \in (0, 1]\) such that if \(\ell_0 \leq M\), \(\int_0^{\ell_0} u_{0x}^2dx \leq M\) and \(\int_0^{\ell_0} u_0^2dx \leq \delta_0\), then the problem $SP(u_0, \ell_0)$ has a solution \(\{u, \ell\}\) on \([0, \infty)\) satisfying that
\[
E(u(t), \ell(t)) \leq \left\{ C + E(\frac{1}{2})^2 \right\}^{\frac{3}{2}} \quad \text{for } t \geq \frac{1}{2},
\]
\[
\frac{d}{dt} |u(t)|_{L^2(0,\ell(t))}^2 \leq 0 \quad \text{for a.e. } t > 0,
\]
\[
|u(t)|_{L^\infty(0,\ell(t))} \leq \sqrt{2} \exp(-\mu t) \quad \text{for } t > 0,
\]
where \( C = C(\alpha), \beta = \beta(\alpha), \text{ and } \mu = \mu(\alpha, \ell_0, |u_0|_{L^\infty(0,\ell_0)}) \) are some positive constants.

For brevity we introduce the following set \( G := G(u_0, \ell_0; M, L, \mu) \) for \((u_0, \ell_0) \in V\) and positive numbers \( M, L \) and \( \mu \):
\[
G(u_0, \ell_0; M, L, \mu) = \left\{ \{u, \ell\}; \begin{array}{l}
\{u, \ell\} \text{ is a solution to } SP(u_0, \ell_0) \text{ on } [0, \infty) \text{ satisfying that } \\
|u(t)|_{L^2(0,\ell(t))} \leq M, |u(t)|_{L^\infty(0,\ell(t))} \leq M \exp(-\mu t) \text{ and } \\
\ell(t) \leq L \text{ for } t \geq 0.
\end{array} \right\}.
\]

The theorem is our main result on the stability of global solutions to \( SP \).

**Theorem 2.3.** Let \( \alpha > 1, (u_0, \ell_0) \in V, M, L \) and \( \mu \) be positive numbers and \( \{u, \ell\} \in G(u_0, \ell_0; M, L, \mu) \). Then, there is a positive number \( p_1 > 0 \) depending only on \( \alpha \) satisfying the following property:

For any positive number \( \tilde{M} \) there exists a positive constant \( \delta \) such that for any \((\tilde{u}_0, \tilde{\ell}_0) \in V(\tilde{M}) \) with \( |u_0 - \tilde{u}_0|_{L^p(0,\infty)} < \delta \) and \( |\ell_0 - \tilde{\ell}_0| < \delta \) the problem \( SP(\tilde{u}_0, \tilde{\ell}_0) \) has a solution \( \{\tilde{u}, \tilde{\ell}\} \) on \([0, \infty)\) satisfying that
\[
\tilde{\ell}(t) \leq \tilde{L} \text{ and } |\tilde{u}(t)|_{L^\infty(0,\tilde{\ell}(t))} \leq \tilde{M} \exp(-\tilde{\mu} t) \text{ for } t \geq 0,
\]
where \( \tilde{M}, \tilde{L} \) and \( \tilde{\mu} \) are positive constants depending on \( \alpha, M, L, \mu, \tilde{M} \) and \( \delta \).

We shall prove Theorem 2.3 in the following way. First, we give some useful inequalities in Sobolev spaces and an ordinary differential inequality in section 2. Secondly, some properties of a global solution belonging to the set \( G(u_0, \ell_0; M, L, \mu) \) are shown (see section 3). Next, we obtain the following decay for \( v := \tilde{u} - u \) under the condition \( \ell_0 \leq \tilde{\ell}_0 \) and \( u_0 \leq \tilde{u}_0 \):
\[
|v(t)|_{L^p(0,\infty)} \leq c(1 + t)^{-\beta} \quad \text{for } t \geq 0,
\]
where \( c \) and \( \beta \) are positive constants. Finally, we give a complete proof of Theorem 2.3 by applying Theorem 2.2.

At the final of this section we define some notations. In order to avoid surplus confusion for notations we write the set of positive constants, \( \alpha, M, L, \mu, \tilde{M} \) and \( \tilde{L} \) as \( (D) \). Since \( \alpha > 1 \)
we can take numbers satisfying that

\[
\begin{cases}
  p_1 > \max\{2 + \alpha, \frac{1 + \alpha}{\alpha - 1}\} \text{ and } p_1 \leq \frac{1}{1 + \alpha} + \frac{1}{2} < \frac{1}{r_0} < \frac{p_1}{2}, \\
  \left(\frac{1}{r_0} - \frac{1}{2}\right)\frac{1 + \alpha}{p_1} = 1 + \beta_0, \\
  p_0 = \frac{p_1 r_0}{2}.
\end{cases}
\]

(2.1)

Clearly, we obtain that $1 < p_0 < p_1$ and $0 < r_0 < 2$. These numbers play an important role in our proof.

### 3 Auxiliary lemmas

At the begin of this section we list some useful inequalities in Sobolev spaces(cf. O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva [10; Chap.2, Theorem 2.2]): Let $d$ be any positive number.

\[
\int_0^d u^{p + \alpha} dx \leq \left(\frac{q + 2}{2}\right)^{\frac{2(q - r)}{r + 2}} \left|\left(u^\frac{p}{r}\right)_x\right|_{L^2(0,d)}^{\frac{2(q - r)}{r + 2}} \left(\int_0^d u^{\frac{r}{2}} dx\right)^{\frac{r + 2}{2}} \quad \text{for } u \in W^{1,2}(0,d) \text{ with } u(d) = 0,
\]

(3.1)

where $p \geq 2$, $\alpha \geq 0$, $q = 2(p + \alpha)/p$ and $r \in (0, q)$;

\[
|u|_{L^2(0,d)} \leq 2|u|_{L^2(0,d)} \quad \text{for } u \in W^{1,2}(0,d) \text{ with } u(d) = 0;
\]

(3.2)

\[
|u|_{L^\infty(0,d)} \leq \left(\frac{q + 2}{2}\right)^{\frac{3}{2}} \left|u\right|_{L^2(0,d)}^{\frac{3}{2}} \left|u\right|_{L^2(0,d)}^{\frac{3}{2}} \quad \text{for } u \in W^{1,2}(0,d) \text{ with } u(d) = 0,
\]

(3.3)

where $q \geq 1$.

The first lemma is concerned with an ordinary differential inequality.

**Lemma 3.1.** Let $a$, $b$ and $\mu$ be positive numbers, $0 < r < 2$ and $z$ be a non-negative absolutely continuous function on $[0, T]$, $0 < T < \infty$, satisfying

\[
\frac{d}{dt} z(t) + a z^{\frac{2 + r}{2}} (t) \leq b \exp(-\mu t) \quad \text{for a.e. } t \in [0, T].
\]

Then, there is a positive constant $N_0 = N_0(a, b, r, \mu)$ such that

\[
z(t) \leq N_0(1 + z(0))(1 + a \beta t)^{-\frac{1}{2}} \quad \text{for any } t \in [0, T],
\]

(3.4)

where $\beta = \frac{2r}{2 - r}$.
Proof. Let $N_1$ be any positive number and

$$
\psi(t) = N_1(1 + a \beta t)^{-1/\beta} \quad \text{for } t \in [0, T].
$$

By elementary calculation we obtain that

$$
\frac{d}{dt}(z(t) - \psi(t)) + a \frac{2 - r}{2 + r} \psi^{2r/(2 - r)}(t)(z(t) - \psi(t)) 
\leq b \exp(-\mu t) - a(N_1^{2r/(2 - r)} - N_1)(1 + \alpha \beta t)^{-\frac{r+2}{2r}} \quad \text{for a.e. } t \in [0, T].
$$

Hence, we take a positive number $N_0 \geq 1$ such that

$$
\left(\frac{b}{a}\right)^{2r/(2 - r)} \left(1 + \frac{a \beta (2 + r)}{2r \mu e} \right) \leq (N_0^{2r/(2 - r)} - N_0)^{2r/(2 - r)},
$$

and put $N_1 = N_0(1 + z(0))$.

Therefore, we have

$$
\frac{d}{dt}(z(t) - \psi(t)) + a \frac{2 - r}{2 + r} \psi^{2r/(2 - r)}(t)(z(t) - \psi(t)) \leq 0 \quad \text{for a.e. } t \in [0, T].
$$

By using Gronwall's argument we see that

$$
z(t) - \psi(t) \leq (z(0) - \psi(0)) \exp\{-a \int_0^t \frac{2 - r}{2 + r} \psi^{2r/(2 - r)}(\tau)d\tau\}
\leq z(0) - N_1(1 + z(0)) \leq 0 \quad \text{for any } t \in [0, T].
$$

Thus, we get (3.4). \qed

Lemma 3.2. Let $p > 1$ and $d > 0$. We suppose that $u \in W^{2,2}(0, d)$ with $u_x(0) = 0$, $u(d) = 0$ and $u > 0$ on $(0, d)$. Then, $(u^{p/2})_x \in L^2(0, d)$.

Proof. It is sufficient to show that there is a function $f \in L^2(0, d)$ such that

$$
-\int_0^d u^{p/2} \eta_x dx = \int_0^d f \eta dx \quad \text{for any } \eta \in C_0^\infty([0, d]). \quad (3.5)
$$

Let $\eta \in C_0^\infty([0, d])$. Then, there is a positive number $\varepsilon$ such that $\text{supp}(\eta) \subset [\varepsilon, d - \varepsilon]$ so that $u \geq \delta > 0$ on $[\varepsilon, d - \varepsilon]$ for some positive number $\delta$. Clearly, we have

$$
-\int_0^d u^{p/2} \eta_x dx = \int_{\varepsilon}^{d-\varepsilon} (u^{p/2})_x \eta dx = \frac{p}{2} \int_{\varepsilon}^{d-\varepsilon} u_x u^{p/2 - 1} \eta dx.
$$

Hence,

$$
\left| \int_0^d u^{p/2} \eta_x dx \right| \leq \frac{p}{2} \left( \int_{\varepsilon}^{d-\varepsilon} |u_x|^2 |u|^{p-2} dx \right)^{1/2} \left( \int_0^d \eta^2 dx \right)^{1/2}.
$$
Here, we note that
\[
\int_{\varepsilon}^{d-\varepsilon} |u_x|^2 |u|^{p-2} dx \\
= \int_{\varepsilon}^{d-\varepsilon} u_x \left( \frac{1}{p-1} u^{p-1} \right) dx \\
= -\frac{1}{p-1} \int_{\varepsilon}^{d-\varepsilon} u_{xx} u^{p-1} dx + \frac{1}{p-1} \left\{ u_x (d-\varepsilon) u^{p-1} (d-\varepsilon) - u_x (\varepsilon) u^{p-1} (\varepsilon) \right\}. \tag{3.6}
\]
Letting \( \varepsilon \downarrow 0 \) in (3.6), because of continuity of \( u_x \) on \([0, d]\) we obtain that
\[
\lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{d-\varepsilon} |u_x|^2 |u|^{p-2} dx = -\frac{1}{p-1} \int_0^d u_{xx} u^{p-1} dx,
\]
that is,
\[
| - \int_0^d u^{p/2} \eta dx| \leq C |\eta|_{L^2(0,d)} \text{ for any } \eta \in C_0^\infty([0,d]),
\]
where \( C \) is some positive constant.

Immediately, we conclude that there is a function \( f \in L^2(0, d) \) satisfying (3.5). \( \square \)

\section{Properties of a global solution}

In this section we show some estimates for a global solution to \( SP \). First, we recall some useful equations for a solution to \( SP \).

**Lemma 4.1.** (cf. [9; Lemma 5.1] and [4; Lemma 2.1]) Let \((u_0, \ell_0) \in V \) and \( \{u, \ell\} \) be a solution to \( SP(u_0, \ell_0) \) on \([0, T], 0 < T < \infty \).

1. We have
\[
\frac{d}{dt} E(u(t), \ell(t)) = \int_0^t u^{1+\alpha}(t, x) dx \text{ for a.e. } t \in [0, T]. \tag{4.1}
\]

2. For a.e. \( t \in [0, T] \) we have
\[
|u_t(t)|_{L^2(0,\ell(t))}^2 + \frac{1}{2} |\ell'(t)|^2 + \frac{1}{2} \frac{d}{dt} |u_x(t)|_{L^2(0,\ell(t))}^2 = \frac{1}{2 + \alpha} \frac{d}{dt} |u(t)|_{L^2+\alpha(0,\ell(t))}^{2+\alpha}. \tag{4.2}
\]

Next, the following lemma guarantees a decay for \( u_x \).

**Lemma 4.2.** Let \( M, L \) and \( \mu \) be positive numbers, \( (u_0, \ell_0) \in V \) and \( \{u, \ell\} \in G(u_0, \ell_0; M, L, \mu) \). Then, there are positive constants \( L_1 \) and \( \mu_1 \) such that
\[
|u_x(t)|_{L^2(0,\ell(t))} \leq L_1 \exp(-\mu_1 t) \text{ for } t > 0.
\]
Proof. By the argument in the proof of [9; Lemma 5.2] we have

\[
\int_0^{\tau(t)} u_t(t) u_{xx}(t) \, dx = -\frac{1}{2} \frac{d}{dt} \int_0^{\tau(t)} |u_x(t)|^2 \, dx - |\ell'(t)|^3 \quad \text{for} \ t \geq 0. \tag{4.3}
\]

Also, from (1.1) we see that

\[
\int_0^{\tau(t)} u_t(t) u_{xx}(t) \, dx
\]

\[
= \int_0^{\tau(t)} (u_{xx}(t) + u^{1+\alpha}(t)) u_{xx}(t) \, dx
\]

\[
= \int_0^{\tau(t)} (u_{xx})^2(t) \, dx - (1+\alpha) \int_0^{\tau(t)} u^\alpha(t)(u_x)^2(t) \, dx \quad \text{for} \ t > 0. \tag{4.4}
\]

It follows from (4.3), (4.4) and (3.2) that

\[
\frac{1}{2} \frac{d}{dt} \int_0^{\tau(t)} |u_x(t)|^2 \, dx + |\ell'(t)|^3 + \frac{1}{4L^2} \int_0^{\tau(t)} |u_x(t)|^2 \, dx \leq \frac{1}{2} \frac{d}{dt} \int_0^{\tau(t)} |u_x(t)|^2 \, dx + |\ell'(t)|^3 + \int_0^{\tau(t)} |u_{xx}(t)|^2 \, dx
\]

\[
\leq (1+\alpha)M^\alpha \exp(-\alpha \mu) \int_0^{\tau(t)} |u_x(t)|^2 \, dx \quad \text{for} \ t > 0. \tag{4.5}
\]

Here, we can take a positive number \(t_0\) such that \((1+\alpha)M^\alpha \exp(-\alpha \mu) \leq \frac{1}{8L^2}\) for \(t \geq t_0\). Accordingly, for \(t \geq t_0\)

\[
\frac{d}{dt} \int_0^{\tau(t)} |u_x(t)|^2 \, dx + \frac{1}{4L^2} \int_0^{\tau(t)} |u_x(t)|^2 \, dx \leq 0,
\]

and hence

\[
\int_0^{\tau(t)} |u_x(t)|^2 \, dx \leq \exp\{-\frac{1}{4L^2}(t-t_0)\} \int_0^{\tau(t)} |u_x(t_0)|^2 \, dx.
\]

On the other hands, (4.5) implies

\[
\frac{d}{dt} \int_0^{\tau(t)} |u_x(t)|^2 \, dx \leq 2(1+\alpha)M^\alpha \int_0^{\tau(t)} |u_x(t)|^2 \, dx.
\]

By Gronwall's inequality, we have

\[
\int_0^{\tau(t)} |u_x(t)|^2 \, dx \leq \exp\{2(1+\alpha)L^\alpha t_0 + \frac{t_0}{4L^2}\} \exp\{-\frac{t}{4L^2}\}|u_{0x}|^2_{L^2(0,t_0)} \quad \text{for} \ t \in [0,t_0].
\]

Therefore, putting

\[
L_1 = \exp\{2(1+\alpha)L^\alpha t_0 + \frac{t_0}{4L^2}\}|u_{0x}|^2_{L^2(0,t_0)} \text{ and } \mu_1 = \frac{1}{4L^2},
\]
we get the assertion of the lemma.

The following lemma shows the decay of \( \ell' \), which is a key of the proof of Theorem 2.3.

**Lemma 4.3.** We suppose that the same assumptions as in Lemma 4.2 hold and \( 1 < q \leq 4 \). Then, for some positive number \( \mu_0 = \mu_0(\mu, q) \)

\[
\int_0^\infty |\ell'(t)|^q e^{\mu_0 t} dt < \infty.
\]

Clearly, the above fact implies that

\[
\int_0^\infty |\ell'(t)|^q dt < \infty.
\]

**Proof.** Let \( M_1 \) and \( \mu_1 \) be positive constants defined in Lemma 4.2. According to (3.3) and Lemma 4.2, we see that for any \( t > 0 \)

\[
|\ell'(t)|^q = |u_x(t, \ell(t))-|^q \\
\leq \sqrt{2M_1} \exp\left(-\frac{q}{2} \mu_1 t\right) |u_{xx}(t)|^2_{L^2(0, \ell(t))},
\]

and hence

\[
|\ell'(t)|^q \exp\left(\frac{\mu_1 q}{4} t\right) \leq C |u_{xx}(t)|^2_{L^2(0, \ell(t))} + C \exp\left(-\frac{\mu_1 q}{4} t\right),
\]

where \( C \) is some suitable positive constant.

By using (4.5) and Lemma 4.2, again, we have

\[
\frac{1}{2} \frac{d}{dt} \int_0^{\ell(t)} |u_x(\tau)|^2 d\tau + \int_0^{\ell(t)} |u_{xx}(\tau)|^2 d\tau
\leq (1 + \alpha) M_0 \exp\left(-\alpha \mu t\right) \int_0^{\ell(t)} |u_x(\tau)|^2 d\tau
\leq (1 + \alpha) M_0 M_1^2 \exp\left\{-(\alpha \mu + 2 \mu_1) t\right\} \quad \text{for } t > 0.
\]

Integrating this inequality over \([0, t], 0 < t < \infty\), we obtain that

\[
\int_0^{\ell(t)} |u_x(t)|^2 dt + \int_0^t \int_0^{\ell(\tau)} |u_{xx}(\tau)|^2 dx d\tau
\leq (1 + \alpha) M_0 M_1^2 \int_0^t \exp\left\{-(\alpha \mu + 2 \mu_1) \tau\right\} d\tau + \int_0^{\ell(0)} |u_{xx}|^2 dx \quad \text{for } t \geq 0.
\]

Adding to (4.6), we conclude that \( \int_0^\infty \ell'(t)e^{\mu_0 t} dt < \infty \) where \( \mu_0 = \frac{\mu_1}{q} \). \( \Box \)
5. Energy inequalities

The purpose of this section is to establish the following lemmas concerned with global estimates for the difference, \( \hat{u} - u \), of solutions to \( SP \).

**Lemma 5.1.** Let \((u_0, \ell_0), (\hat{u}_0, \hat{\ell}_0) \in V, M, L \) and \( \mu \) be positive numbers, \( \{u, \ell\} \in G(u_0, \ell_0; M, L, \mu) \), and \( \{\hat{u}, \hat{\ell}\} \) be a solution to \( SP(\hat{u}_0, \hat{\ell}_0) \) on \([0, T] \), \( 0 < T < \infty \). Moreover, we suppose that \( \ell_0 \leq \hat{\ell}, u_0 \leq \hat{u}_0 \) on \([0, \infty) \) and \( \hat{u}_0 \neq u_0 \). Then, putting \( v = \hat{u} - u \) we obtain that for \( t \in (0, T) \) and \( p \in [p_0, p_1] \) (see (2.1)),

\[
\frac{d}{dt} \int_0^{\hat{\ell}(t)} v^p(t) dx \\
\leq \left\{ -C_1 + C_2 \hat{\ell}(t)^{2-\frac{1+\alpha}{p_1}} \left( \int_0^{\hat{\ell}(t)} v^{p_1}(t) dx \right)^{\alpha} \right\} |(v^\frac{p}{2})_x(t)|_{L^2(0, \hat{\ell}(t))}^2 \\
+ C_2 \exp(-\alpha \mu t) \int_0^{\hat{\ell}(t)} v^p(t) dx + C_2 \hat{\ell}'(t)^{2-\frac{1+\alpha}{p_1}} \left( \int_0^{\hat{\ell}(t)} v^p(t) dx \right)^{\frac{p-1}{p}},
\]

where \( C_1 \) and \( C_2 \) are positive constants depending on \( \alpha, p_0, p_1 \) and \( M \).

**Proof.** For simplicity, we put \( H(t) = L^2(0, \hat{\ell}(t)) \).

First, by Theorem 2.1 (ii) we have \( v = \hat{u} - u > 0 \) on \( Q_T(t) \). Multiplying (1.1) by \( v^{p-1} \) for \( p \in [p_0, p_1] \) we apply Lemma 3.2 to the following calculation so that

\[
\frac{d}{dt} \int_0^{\hat{\ell}(t)} v^p(t) dx \\
= p \int_0^{\hat{\ell}(t)} v(t) v^{p-1}(t) dx \\
= \int_0^{\hat{\ell}(t)} (\hat{u}_{xx}(t) + \hat{u}^{1+\alpha}(t)) v^{p-1}(t) dx - p \int_0^{\hat{\ell}(t)} (u_{xx}(t) + u^{1+\alpha}(t)) v^{p-1}(t) dx \\
= -p(p-1) \int_0^{\hat{\ell}(t)} (v_x)^2(t) v^{p-2}(t) dx + p \hat{\ell}'(t)v^{p-1}(t, \hat{\ell}(t)) \\
+ p \int_0^{\hat{\ell}(t)} (\hat{u}^{1+\alpha}(t) - u^{1+\alpha}(t)) v^{p-1}(t) dx \\
\leq -\frac{4(p-1)}{p} \int_0^{\hat{\ell}(t)} ((v^{p/2})_x(t))^2 dx + p \hat{\ell}'(t)v^{p-1}(t, \hat{\ell}(t)) \\
+ p2^\alpha(1+\alpha) \int_0^{\hat{\ell}(t)} (u^{p+\alpha}(t) + u^\alpha(t)) v^{p}(t) dx \quad \text{for } t \in (0, T].
\]

Here, we note that

\[
\hat{u}^{1+\alpha} - u^{1+\alpha} \leq 2^\alpha(1+\alpha)(v^{1+\alpha} + uu^\alpha).
\]
From (3.1) and Hölder’s inequality it follows that
\[
\int_0^{\hat{t}(t)} v^{p-1}(t) dt \\
\leq \left( \frac{\alpha}{p} + 2 \right)^2 |(v^{p/2})_x(t)|_{H(t)}^2 (\int_0^{\hat{t}(t)} v^{\alpha/2}(t) dt)^2 \\
\leq \left( \frac{\alpha}{p} + 2 \right)^2 |(v^{p/2})_x(t)|_{H(t)}^2 \hat{\ell}(t)^2 \ell(t)^{-\frac{1+\alpha}{\nu_1}} (\int_0^{\hat{t}(t)} v^{p_1}(t) dt)^2 \frac{\alpha}{\nu_1} \text{ for } t \in (0, T].
\]

(5.4)

Also, according to (3.3) we have
\[
p \ell'(t) v^{p-1}(t, \ell(t)) \\
\leq p \ell'(t) |v^{p/2}(t)|_{L^\infty(0, \ell(t))}^{\frac{2(p-1)}{p}} \\
\leq 2^{\frac{1}{1-p}} p \ell'(t) |(v^{p/2})_x(t)|_{H(t)}^{p-1} |(v^{p/2})(t)|_{H(t)}^{p-1} \\
\leq \frac{2(p-1)}{p} |(v^{p/2})_x(t)|_{H(t)}^{2} + C_p |\ell'(t)|^{\frac{2p}{p+1}} (\int_0^{\hat{t}(t)} v^{p}(t) dt)^{\frac{p-1}{p+1}} \text{ for } t \in (0, T],
\]

(5.5)

where $C_p$ is a positive constant depending only on $p$.

It follows from (5.2) \sim (5.5) that for $t \in (0, T]$
\[
\frac{d}{dt} \int_0^{\hat{t}(t)} v^p(t) dt \\
\leq \left\{- C_1 + C_2 \hat{\ell}(t)^2 \ell(t)^{-\frac{\alpha}{\nu_1}} (\int_0^{\hat{t}(t)} v^{p_1}(t) dt)^2 \frac{\alpha}{\nu_1} \right\} |(v^{p/2})_x(t)|_{H(t)}^2 \\
+ C_2 \exp(-\alpha \mu t) \int_0^{\hat{t}(t)} v^p(t) dt + C_2 |\ell'(t)|^{\frac{2p}{p+1}} (\int_0^{\hat{t}(t)} v^p(t) dt)^{\frac{p-1}{p+1}},
\]

where
\[
C_1 = \frac{2(p_0 - 1)}{p_1} \text{ and } C_2 = 2^{\alpha} p_1 (2 + \frac{\alpha}{p_0})^2 (1 + \alpha) + 2^{\alpha} p_1 (1 + \alpha) M + \max_{p_0 \leq p \leq p_1} C_p.
\]

This is the conclusion of the lemma.

\[\square\]

**Lemma 5.2.** Let $B_1$ and $B_2$ be positive numbers. We suppose that the same assumptions as in Lemma 5.1 hold. Moreover, we suppose that for $p \in [p_0, p_1]$ and $t \in (0, T], 0 < T < \infty$,
\[
\frac{d}{dt} \int_0^{\hat{t}(t)} v^p(t) dt \\
\leq -B_0 |(v^{\frac{p}{2}})_x(t)|_{L^2(0, \ell(t))}^2 + B_1 \exp(-\alpha \mu t) \int_0^{\hat{t}(t)} v^p(t) dt \\
+ B_1 |\ell'(t)|^{\frac{2p}{p+1}} (\int_0^{\hat{t}(t)} v^p(t) dt)^{\frac{p-1}{p+1}}.
\]

\[\text{(5.6)}\]
Then, there is a positive constant $C_3$ depending on $\alpha, p_0, p_1, \mu, M$ and $L,$ which satisfies that

$$\int_0^{\tilde{t}(t)} \nu^p(t)dx \leq C_3(\int_0^{\tilde{t}_0} \nu^p(0)dx + 1) \quad \text{for } p \in [p_0, p_1] \text{ and } t \in [0, T].$$  \hspace{1cm} (5.7)

Proof. For simplicity, we put

$$F_p(t) = \int_0^{\tilde{t}(t)} \nu^p(t)dx \quad \text{for } p \in [p_0, p_1] \text{ and } t \in [0, T].$$

Obviously, we obtain that

$$\ell'(t)\frac{2p}{p+1}F_p(t)^{\frac{p-1}{p+1}} \leq \frac{2}{p+1} \ell'(t)\frac{2p}{p+1} + \frac{p-1}{p+1} \ell'(t)\frac{2p}{p+1}F_p(t) \quad \text{for } t \in [0, T].$$

Hence, from (5.6) it follows that for $p \in [p_0, p_1]$

$$\frac{d}{dt} F_p(t) \leq C_4 \exp(-\alpha \mu t) F_p(t) + C_4 |\ell'(t)|\frac{2p}{p+1} + C_4 |\ell'(t)|\frac{2p}{p+1} F_p(t) \quad \text{for } t \in (0, T],$$

where $C_4 = B_1 + B_1(\frac{1}{p_0 + 1} + \frac{p_1 - 1}{p_0 + 1}).$

Since $1 < \frac{2p}{p+1} < 4,$ by applying Lemma 4.3 and Gronwall's argument to the above inequality, we get

$$F_p(t) \leq (F_p(0) + C_4 \int_0^{\infty} |\ell'(t)|\frac{2p}{p+1} \exp(\int_0^{t} J_p(t)dt)) \quad \text{for } t \in [0, T],$$

where $J_p(t) = C_4 \exp(-\alpha \mu t) + C_4 |\ell'(t)|\frac{2p}{p+1}.$

Thus, this lemma has been proved.

\[ \square \]

Lemma 5.3. Let $\tilde{M}$ and $\tilde{L}$ be positive numbers. Then, under the same conditions as in Lemma 5.2 there are positive constants $C_5$ and $C_6$ depending only on (D) such that

$$\int_0^{\tilde{t}(t)} \nu^p(t)dx \leq C_5(1 + \int_0^{\tilde{t}_0} \nu^p(0)dx)(1 + C_6 t)^{\frac{3-r_0}{2r_0}} \quad \text{for } t \in [0, T],$$

where $r_0$ is a positive constant defined by (2.1).

Proof. For brevity, we put $F(t) = \int_0^{\tilde{t}(t)} \nu^p(t)dx$ for $t \in [0, T]$ and note that $\int_0^{\tilde{t}_0} \nu^{p_0}(0)dx \leq \tilde{M}^{p_0} \tilde{L}.$ According to (3.1) and the previous lemma, we infer that

$$F(t) \leq 2^{\frac{2(2-r_0)}{r_0+2}}(\nu^{p_1/2}(t)|_{H(1)}^{2(2-r_0)}(\int_0^{\tilde{t}(t)} \nu^p(t)dx)^{\frac{2}{r_0+2}}$$

$$\leq 2^{\frac{2(2-r_0)}{r_0+2}}(C_3(\tilde{M}^{p_0} \tilde{L} + 1))^{\frac{2}{r_0+2}}(\nu^{p_1/2}(t)|_{H(1)}^{\frac{2(2-r_0)}{r_0+2}} \quad \text{for } t \in (0, T],$$
and hence
\[ \left| \langle v^{p_1/2} \rangle_z(t) \right|^2_{H(t)} \geq \frac{1}{4(C_3(\tilde{M}^p \tilde{L} + 1))^{2-\theta_0}^2} F(t)^{\frac{2+\theta_0}{2-\theta_0}} \text{ for } t \in (0, T]. \]

Here, we note that
\[
\ell'(t)\frac{2p_1}{p_1+1} F(t)^{\frac{p_1-1}{p_1+1}}
\leq \frac{p_1 - 1}{p_1 + 1} \ell'(t)\frac{2p_1}{p_1+1} F(t) \exp(\mu_0 t) + \frac{2}{p_1 + 1} \exp\left( -\frac{p_1 - 1}{2} \mu_0 t \right) \text{ for } t \in (0, T],
\]
where \( \mu_0 \) is a positive constant defined in Lemma 4.3, and \( 1 < \frac{2p_1}{p_1 - 1} < 4. \)

Therefore, by adding the above inequalities together to (5.6) we have
\[
\frac{d}{dt} F(t) \leq -K_1 F(t)^{\frac{2+\theta_0}{2-\theta_0}} + K_2 |\ell'(t)|^{\frac{2p_1}{p_1+1}} F(t) \exp(\mu_0 t)
+ K_2 \exp(-\mu_2 t) + K_2 F(t) \exp(-\mu_3 t) \text{ for } t \in (0, T],
\]
where \( K_1, K_2, \mu_2 \) and \( \mu_3 \) are suitable positive constants.

For simplicity, we put
\[
J(t) = K_2 (|\ell'(t)|^{\frac{2p_1}{p_1+1}} \exp(\mu_0 t) + \exp(-\mu_3 t)) \text{ and } \Phi(t) = F(t) \exp(-\int_0^t J(\tau) d\tau).
\]
It is clear that
\[
\frac{d}{dt} \Phi(t) + K_1 \Phi(t)^{\frac{2+\theta_0}{2-\theta_0}} \exp\left( \frac{2r_0}{2 - r_0} \int_0^t J(\tau) d\tau \right) \leq K_2 \exp(-\mu_2 t) \text{ for } t \in (0, T],
\]
and
\[
\frac{d}{dt} \Phi(t) + K_1 \Phi(t)^{\frac{2+\theta_0}{2-\theta_0}} \leq K_2 \exp(-\mu_2 t) \text{ for } t \in (0, T].
\]
By Lemma 3.1, we obtain that
\[
\Phi(t) \leq N_0(1 + \Phi(0))(1 + \beta_1 K_1 t)^{-\frac{1}{\beta_1}} \text{ for } t \in [0, T],
\]
where \( \beta_1 = \frac{2r_0}{2 - r_0} \) and \( N_0 = N_0(K_1, K_2, r_0, \mu_2) > 0. \)

Accordingly, this implies that
\[
F(t) \leq N_0(1 + F(0))(1 + \beta_1 K_1 t)^{-\frac{1}{\beta_1}} \exp\left( \int_0^t J(\tau) d\tau \right) \text{ for } t \in (0, T],
\]
since \( 1 < 2p_1/(p_1 - 1) < 4, \) the integration in the above inequality makes sense. We get the assertion of Lemma 5.3. \[\Box\]
At the end of this section, we give a global estimate for $E(\hat{u}(t), \hat{t}(t))$.

**Lemma 5.4.** We suppose that the same assumptions of Lemma 5.3 hold. Then, there exists a positive constant $C_T$ depending only on $(D)$, which satisfies that

$$
E(\hat{u}(t), \hat{t}(t)) \leq C_T \{ E(\hat{u}_0, \hat{t}_0) + (\int_0^{\hat{t}_0} v^{p_1}(0) dx)^{\frac{p_1}{p_1 + \alpha}} + 1 \} \quad \text{for } t \in [0, T],
$$

(5.9)

where $\beta_0$ is a positive constant defined by (2.1).

**Proof.** For simplicity, we use the same notations as in the proof of the previous lemmas and put

$$E(t) = E(u(t), t(t)) \text{ and } \hat{E}(t) = E(\hat{u}(t), \hat{t}(t)).$$

It follows from (4.1) with help of (5.3) that

$$
\frac{d}{dt}(\hat{E}(t) - E(t))
\leq 2^{\alpha}(1 + \alpha)\{ \int_0^{\hat{t}(t)} v^{1+\alpha}(t) dx + (\int_0^{\hat{t}(t)} v^{1+\alpha}(t) dx)^{\frac{1}{1+\alpha}} (\int_0^{\hat{t}(t)} u^{1+\alpha}(t) dx)^{\frac{\alpha}{1+\alpha}} \}
\leq 2^{\alpha}(1 + \alpha)\{ \int_0^{\hat{t}(t)} v^{1+\alpha}(t) dx + \frac{1}{1+\alpha} \int_0^{\hat{t}(t)} v^{1+\alpha}(t) dx + \frac{\alpha}{1+\alpha} \int_0^{\hat{t}(t)} u^{1+\alpha}(t) dx \}
\leq 2^{\alpha}(1 + \alpha)\{ \frac{\alpha + 2}{\alpha + 1} \int_0^{\hat{t}(t)} v^{p_1}(t) dx \frac{1+\alpha}{p_1} \hat{E}(t)^{\frac{1+\alpha}{p_1}} + \frac{\alpha}{1+\alpha} M^{1+\alpha} \exp(-(1+\alpha)\mu t) \}
\leq K_3F(t)^{\frac{1+\alpha}{p_1}} E(t)^{\frac{1+\alpha}{p_1}} + K_3 \exp(-(1+\alpha)\mu t) \quad \text{for } t \in (0, T],
$$

where $K_3 = 2^{\alpha}(1 + \alpha)(\frac{\alpha + 2}{\alpha + 1} + \frac{\alpha}{1+\alpha} M^{1+\alpha})$, and hence

$$
\frac{d}{dt}(\hat{E}(t)) \hat{E}(t)^{\frac{1+\alpha}{p_1} - 1}
\leq K_3F(t)^{\frac{1+\alpha}{p_1}} E(t)^{\frac{1+\alpha}{p_1} - 1} \exp(-(1+\alpha)\mu t) + \hat{E}(t)^{\frac{1+\alpha}{p_1} - 1} \frac{d}{dt} E(t) \quad \text{for } t \in (0, T].
$$

Moreover, since $\frac{1+\alpha}{p_1} - 1 < 0$, $(\hat{E}(t))^{\frac{1+\alpha}{p_1} - 1} \leq (E(t))^{\frac{1+\alpha}{p_1} - 1}$ and $\frac{d}{dt} E(t) \geq 0$, we see that

$$
\frac{p_1}{1+\alpha} \frac{d}{d\tau}(E(\tau)^{\frac{1+\alpha}{p_1}}) \leq K_3F(\tau)^{\frac{1+\alpha}{p_1}} + K_3 E(0)^{\frac{1+\alpha}{p_1} - 1} \exp(-(1+\alpha)\mu \tau)
$$

(5.10)

and

$$
\frac{p_1}{1+\alpha} \frac{d}{d\tau}(E(\tau)^{\frac{1+\alpha}{p_1}}) + \frac{p_1}{1+\alpha} \frac{d}{d\tau}(E^0(\tau)^{\frac{1+\alpha}{p_1}}) \quad \text{for } \tau \in (0, T].
$$

Integrating (5.10) over $[0, t]$, $0 \leq t \leq T$, we conclude that

$$
\frac{p_1}{1+\alpha} (\hat{E}(t)^{\frac{1+\alpha}{p_1}} - \hat{E}(0)^{\frac{1+\alpha}{p_1}})$$
\[
\leq \frac{p_1}{1 + \alpha} (E(t)^{\frac{1 + s}{p_1}} - E(0)^{\frac{1 + s}{p_1}}) + K_3 \int_0^t F(\tau)^{\frac{1 + s}{p_1}} d\tau
+ K_3 E(0)^{\frac{1 + s}{p_1} - 1} \int_0^t \exp(-(1 + \alpha)\mu \tau) d\tau \quad \text{for } t \in [0, T].
\]

Here, it follows from Lemma 5.3 that
\[
\int_0^t F(\tau)^{\frac{1 + s}{p_1}} d\tau \leq (C_5(1 + F(0)))^{\frac{1 + s}{p_1}} \int_0^\infty (1 + C_6 \tau)^{-\frac{(1+\alpha)(2-\alpha)}{2\rho + 1 - \alpha}} d\tau
\leq (C_5(1 + F(0)))^{\frac{1 + s}{p_1}} \int_0^\infty (1 + C_6 \tau)^{-1 - \rho_0} d\tau \quad \text{for } t \in [0, T].
\]
Therefore, it is easy to check that (5.9) holds.

\[\square\]

6 Stability of global solutions

First, we shall prove Theorem 2.3 in case the following condition (*) holds:

\[(*) \quad \ell_0 \leq \hat{\ell}_0, u_0 \leq \hat{u}_0 \text{ on } [0, \infty) \text{ and } u_0 \neq \hat{u}_0.\]

**Proof of Theorem 2.3 under the condition (*)** Let \{\hat{u}, \hat{\ell}\} be a solution of SP(\hat{u}_0, \hat{\ell}_0) on \[0, T_1], 0 < T_1 < \infty, \delta \in (0, 1] \text{ and } v := \hat{u} - u.\] We assume that \(\int_0^{\hat{\ell}(t)} v^{p_1}(0) dx \leq \delta \text{ and } \hat{\ell}(t) \leq \ell_0 + \delta.\)

Since the function \(t \rightarrow \int_0^{\hat{\ell}(t)} v^{p_1}(t) dx\) is continuous, there is a positive constant \(T_2 \leq T_1\) such that
\[
\int_0^{\hat{\ell}(t)} v^{p_1}(t) dx \leq 2\delta \text{ and } \hat{\ell}(t) \leq L + 2 =: L_2 \quad \text{for } t \in [0, T_2].
\]
Lemma 5.1 implies that
\[
\frac{d}{dt} \int_0^{\hat{\ell}(t)} v^p(t) dx \leq \{-C_1 + C_2 (L_2)^{2 - \frac{1 + s}{p_1}} (2\delta)^{\frac{s}{p_1}}\} (v^{\frac{s}{2}}_2)_{x}(t)^{2}_{L^2(0, \hat{\ell}(t))}
+ C_2 \exp(-\alpha \mu t) \int_0^{\hat{\ell}(t)} v^p(t) dx
+ C_2 \ell'(t)^{2} (\int_0^{\hat{\ell}(t)} v^p(t) dx)^{\frac{p-1}{p+1}} \quad \text{for } t \in (0, T_2] \text{ and } p \in [p_0, p_1].
\]

We choose a positive number \(\delta_1\) such that
\[
C_2 (L_2)^{2 - \frac{1 + s}{p_1}} (2\delta_1)^{\frac{s}{p_1}} \leq \frac{C_1}{2},
\]
and clearly, for \(\delta \leq \delta_1\) we have
\[
\frac{d}{dt} \int_0^{\hat{\ell}(t)} v^p(t) dt \leq -\frac{C_1}{2} (v^{p/2}_2)_{x}(t)^{2}_{L^2(0, \hat{\ell}(t))} + C_2 \exp(-\alpha \mu t) \int_0^{\hat{\ell}(t)} v^p(t) dx
+ C_2 \ell'(t) (\int_0^{\hat{\ell}(t)} v^p(t) dx)^{\frac{p-1}{p+1}} \quad \text{for } t \in (0, T_2] \text{ and } p \in [p_0, p_1].
\]

(6.1)
By virtue of Lemmas 5.3 and 5.4, for \( \delta < \delta_1 \) we have
\[
\int_0^{\hat{\ell}(t)} v^{p_1}(t) dx \leq M_1(1 + \delta)(1 + M_2t)^{-\frac{2+\alpha}{2+\alpha}} \quad \text{for } t \in [0, T_2],
\]
\[
E(\hat{u}(t), \hat{\ell}(t)) \leq M_1(1 + \delta^{\frac{\alpha}{1+\alpha}}) \quad \text{for } t \in [0, T_2],
\]
where \( M_1 \) and \( M_2 \) are positive constants depending only on (D).

It follows from (4.2) that
\[
|\hat{u}_x(t)|_{L^2(0, \hat{\ell}(t))}^2 \leq |\hat{u}_0x|_{L^2(0, \hat{\ell}_0)}^2 + \frac{2}{2 + \alpha} |\hat{u}(t)|_{L^{2+\alpha}(0, \hat{\ell}(t))}^{2+\alpha} \quad \text{for } t \in [0, T_2],
\]
and hence with aid of (6.2) there is a positive constant \( M_3 \) depending on (D) such that
\[
\begin{align*}
|\hat{u}_x(t)|_{L^2(0, \hat{\ell}(t))} & \leq M_3 \\
|v(t)|_{L^\infty(0, \hat{\ell}(t))} & \leq M_3
\end{align*}
\]
for \( t \in [0, T_2]. \)

(6.3)

Also, putting \( \phi(t) = E(\hat{u}(t), \hat{\ell}(t)) - E(u(t), \ell(t)) \), we have
\[
\frac{d}{dt} \phi(t) = \int_0^{\hat{\ell}(t)} (\hat{u}^{1+\alpha}(t) - u^{1+\alpha}(t)) dx
\]
\[
\leq 2^{0}(1 + \alpha) \int_0^{\hat{\ell}(t)} (v^{1+\alpha}(t) + v(t)v^\alpha(t)) dx
\]
\[
\leq 2^{0}(1 + \alpha)(M_3^\alpha + M^\alpha \exp(-\alpha ut)) \phi(t) \quad \text{for } t \in (0, T_2].
\]

(6.4)

Accordingly, by using Gronwall’s inequality we infer that
\[
\phi(t) \leq \phi(0) \exp(M_4t) \quad \text{for } t \in [0, T_2],
\]
where \( M_4 = \exp\{2^\alpha(1 + \alpha)(M_3^\alpha + M^\alpha)\} \).

Moreover, we observe that
\[
\int_0^{\hat{\ell}(t)} v^{p_1}(t) dx \leq M_3^{p_1-1} \phi(t)
\]
\[
\leq M_3^{p_1-1} \phi(0) \exp(M_4t)
\]
\[
= M_3^{p_1-1}(\int_0^{\hat{\ell}_0} v(0) dx + \hat{\ell}_0 - \ell_0) \exp(M_4t)
\]
\[
\leq M_3^{p_1-1}(L_2^{1-1/p_1} \delta^{1/p_1} + \delta) \exp(M_4t)
\]
\[
\leq M_5 \exp(M_4t) \delta^{1/p_1} \quad \text{for } \delta < \delta_1 \text{ and } t \in (0, T_2],
\]

(6.5)
where \( M_5 = M_3^{p_1 - 1} (L_2^{1 - 1/p_1} + 1) \).

It follows from Theorem 2.1 (iv) that we can extend the solution \( \{ \tilde{u}, \tilde{\ell} \} \) on \([0, T_3)\) for some \( T_3 > T_2 \). Here, we take positive numbers \( 0 < \delta_3 < \delta_2 < \delta_1 \) and \( T_0 \) such that

\[
C_2 L_2^{2 - \frac{1}{p_1}} (3 \delta_2) \frac{\alpha}{p_1} \leq \frac{C_1}{2},
\]

\[
2M_1 (1 + M_2 T_0)^{-2 - \frac{2 - \gamma_0}{2p_0}} \leq 2 \delta_2,
\]

\[
C_2 L_2^{2 - \frac{1}{p_1}} \{2 M_5 \exp(M_4 T_0) (2 \delta_3)^{1/p_1}\} \frac{\alpha}{p_1} \leq \frac{C_1}{2}.
\]

We suppose that \( \int_0^{t_0} v^{p_1}(0) dx < \delta_3 \). Noting that \( M_5 \exp(M_4 T_0) \delta_3^{1/p_1} > \delta_3 \), if necessary we choose \( M_5 > 1 \), again. Now, if there is a positive number \( t_0 \in (0, T_0) \) such that

\[
M_5 \exp(M_4 T_0) \delta_3^{1/p_1} \leq \int_0^{\tilde{t}(t_0)} v^{p_1}(t_0) dx < 2M_5 \exp(M_4 T_0) \delta_3^{1/p_1},
\]

then the inequality (6.1) holds for \( t \in (0, t_0) \) and \( p \in [p_0, p_1] \), and hence by virtue of (6.5) we get the inequality, \( \int_0^{\tilde{t}(t)} v^{p_1}(t_0) dx < M_5 \exp(M_4 T_0) \delta_3^{1/p_1} \). This is a contradiction.

Therefore, the following inequality holds:

\[
\int_0^{\tilde{t}(t)} v^{p_1}(t) dx \leq M_5 \exp(M_4 T_0) \delta_3^{1/p_1} \quad \text{for } t \in [0, T_0].
\]

Similarly, \( \{ \tilde{u}, \tilde{\ell} \} \) is the solution on \([0, T_0] \) and on account of (6.2) we have

\[
\int_0^{\tilde{\ell}(T_0)} v^{p_1}(T_0) dx \leq 2M_1 (1 + M_2 T_0)^{-2 - \frac{2 - \gamma_0}{2p_0}} \leq 2 \delta_2.
\]

Furthermore, if there is a positive number \( t_1 > T_0 \) such that \( 2 \delta_2 < \int_0^{\tilde{t}(t_1)} v^{p_1}(t_1) dx \leq 3 \delta_2 \), then this is a contradiction to (6.2). Hence, we conclude that

\[
\int_0^{\tilde{t}(t)} v^{p_1}(t) dx \leq 2M_1 (1 + M_2 t)^{-2 - \frac{2 - \gamma_0}{2p_0}} \quad \text{for } t \geq T_0,
\]

\[
|\tilde{u}_x(t)|_{L^2(0, \ell(t)))} \leq M_3 \quad \text{for } t \geq 0,
\]

\[
E(\tilde{u}(t), \tilde{\ell}(t)) \leq 2M_1 \quad \text{for } t \geq 0.
\]

Therefore, Theorem 2.2 implies that Theorem 2.3 is valid under the condition (*)..

Finally, we give a complete proof of the theorem.

**Proof of Theorem 2.3.** First, we put \( X = L^{p_1}(0, \infty) \),

\[
\begin{align*}
u_{01} &= \min\{u_0, \hat{u}_0\}, \quad \nu_{02} = \max\{u_0, \hat{u}_0\}, \quad \ell_{01} = \min\{\ell_0, \hat{\ell}_0\} \quad \text{and} \quad \ell_{02} = \max\{\ell_0, \hat{\ell}_0\}.
\end{align*}
\]
Let \( \{u_1, \ell_1\} \) (resp. \( \{u_2, \ell_2\} \)) be a solution to \( SP(u_{01}, \ell_{01}) \) (resp. \( SP(u_{02}, \ell_{02}) \)) on \([0, T_1]\) (resp. \([0, T_2]\)). Putting \( T_3 = \min\{T_1, T_2\} \) it is clear that \( \{u_1, \ell_2\} \in G(u_0, \ell_0, M, L, \mu) \) and \( |u_{02} - u_0|_X \leq |\hat{u}_0 - u_0|_X, u_1 \leq u, \hat{u} \leq u_2 \) on \( Q(T_3) \) and \( \ell_1 \leq \ell, \hat{\ell} \leq \ell_2 \) on \([0, T_3]\),

\[
|u(t) - \hat{u}(t)|_X \leq |u_1(t) - u_2(t)|_X \\
\leq |u_2(t) - u(t)|_X + |u(t)|_X + |u_1(t)|_X.
\]

From the above argument there is a positive number \( \delta \) such that if \( |u_0 - u_{02}|_{L^{p_1}(0, \ell_0)} < \delta \) and \( \ell_0 < \ell_0 < \ell_0 + \delta \), then \( \{u_2, \ell_2\} \) is the global solution to \( SP \) and satisfies that

\[
\ell_2(t) \leq 2M_1 \quad \text{for} \quad t \geq 0, \\
|u(t) - u_2(t)|_X \leq 2M_1(1 + M_0 t)^{\frac{2 - p_0}{p_0}} \quad \text{for} \quad t \geq T_0.
\]

Therefore, if \( |\hat{u}_0 - u_0|_{L^{p_1}(0, \hat{\ell}_0)} < \delta \) and \( |\hat{\ell}_0 - \ell_0| < \delta \), then \( \{\hat{u}, \hat{\ell}\} \) satisfies the required conditions. \( \square \)

References


