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Some results on behavior of solutions to one-phase Stefan problems for a semilinear parabolic equation

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1 Introduction

We consider the following one-phase Stefan problem $SP := SP(u_0, \ell_0)$ for a semilinear parabolic equations in one-dimensional space: Find a curve (a free boundary) $x = \ell(t) > 0$ on $[0, T]$, $0 < T < \infty$, and a function $u = u(t, x)$ on $Q(T) := (0, T) \times (0, \infty)$ satisfying that

$$u_t = u_{xx} + u^{1+\alpha} \quad \text{in} \quad Q_\ell(T) := \{(t, x); 0 < t < T, 0 < x < \ell(t)\}, \tag{1.1}$$

$$u(0, x) = u_0(x) \quad \text{for} \quad 0 \le x \le \ell_0, \tag{1.2}$$

$$u_x(t, 0) = 0 \quad \text{for} \quad 0 < t < T, \tag{1.3}$$

$$u(t, x) = 0 \quad \text{for} \quad 0 < t < T \text{ and } x \ge \ell(t), \tag{1.4}$$

$$\ell'(t) = -u_x(t, \ell(t)) \quad \text{for} \quad 0 < t < T, \tag{1.5}$$

$$\ell(0) = \ell_0, \tag{1.6}$$

where α and ℓ_0 are given positive constants and u_0 is a given initial function on $[0, \ell_0]$.

The local existence and the uniqueness for solutions to the above problem SP were already investigated by Fasano-Primicerio [7] and Aiki-Kenmochi [1, 5, 8]. Since there are blow-up solutions of the usual initial boundary value problem for the semilinear equation (1.1) in a bounded domain, by using comparison principle it is clear that SP has a blow-up solutions for a large initial data. In author's previous works [2, 3, 6] we showed some theorems and numerical experiments concerned with the behavior of free boundaries of blow-up solutions to one-phase Stefan problems with homogenuous Neumann and Dirichlet boundary condition.

On global existence (see Theorem 2.2) in [4] we obtain a solution to the problem SP on $[0,\infty)$, exponential decay of $|u|_{L^\infty(0,\ell(t))}$ and boundedness of the free boundary ℓ for a small initial function u_0 in case $\alpha > 1$.

The purpose of the present paper is to establish the stability of a global solution to the problem SP in the following sense: Let $\alpha > 1$ and $\{u,\ell\}$ be a solution to SP on $[0,\infty)$ satisfying that there are positive constants L, M and μ such that

$$\ell(t) \le L \text{ for } t \ge 0 \text{ and } |u(t,x)| \le M\exp(-\mu t) \text{ for } t \ge 0 \text{ and } x \ge 0.$$

Then, there exists a positive constant δ such that if $|u_0 - \hat{u}|_{L^p(0,\hat{\ell}_0)} < \delta$, where $p > 1$ is some suitable constant, the problem $SP(\hat{u}_0,\hat{\ell}_0)$ has a solution $\{\hat{u},\hat{\ell}\}$ on $[0,\infty)$ satisfying that the free boundary $\{\hat{\ell}(t)\}$ is bounded and $|\hat{u}(t)|_{L^\infty(0,\hat{\ell}(t))}$ decays in exponential order. We note that the global existence and stability concerned with the problem SP are not proved, theoritically, for $0 < \alpha \le 1$.

2 A main result

We give a precise definition of a solution to SP.

Definition 2.1. We say that a pair $\{u,\ell\}$ is a solution of $SP(u_0,\ell_0)$ on $[0,T]$, $0 < T < \infty$, if the following properties are fulfilled:

(S1) $u \in W^{1,2}(0,T;L^2(0,\ell(t))) \cap L^\infty(0,T;W^{1,2}(0,\ell(t)))$, and $\ell \in W^{1,2}(0,T)$ with $0 < \ell$ on $[0,T]$.

(S2) (1.1) holds in the sense of $\mathcal{D}'(Q_\ell(T))$ and (1.2) $\sim$ (1.6) are satisfied.

Also, we call that a couple $\{u,\ell\}$ is a solution of SP on an interval $[0,T')$, $0 < T' \le \infty$, if it is a solution of SP on $[0,T]$ in the above sense for any $0 < T < T'$.

We introduce the following space in order to describe the class of initial functions which satisfy the compatiblity condition.

$$V = \{(z,s); s > 0 \text{ and } z \in W^{1,2}(0,\infty) \text{ with } z \ge 0 \text{ on } [0,s] \text{ and } z(y) = 0 \text{ for } y \ge s\}.$$

First, we recall the theorem concerned with local existence, uniqueness, comparison, continuation and regularity of solutions to SP.

Theorem 2.1. *(cf. [1; Theorems 1.1 and 5.1] and [7; Theorem 1]) Let* $\alpha > 0$ *and*

$(u_0, \ell_0) \in V$. *(i) Then, there is a positive number* T_0 *such that the problem* SP *has one and only one solution* $\{u, \ell\}$ *on* $[0, T_0]$.
(ii) We assume that $(\hat{u}_0, \hat{\ell}_0) \in V$, $\ell_0 \leq \hat{\ell}_0$, $u_0 \leq \hat{u}_0$ *on* $[0, \infty)$ *and* $u_0 \not\equiv \hat{u}_0$. *Let* $\{u, \ell\}$ *(resp.* $\{\hat{u}, \hat{\ell}\}$*) be a solution to* $SP(u_0, \ell_0)$ *(resp.* $SP(\hat{u}_0, \hat{\ell}_0)$*) on* $[0, T]$, $0 < T < \infty$. *Then, we have*

$$\ell \leq \hat{\ell} \text{ on } [0, T] \text{ and } u < \hat{u} \text{ on } Q(T).$$

(iii) If $u_0 \in C^1([0, \ell_0])$ *and* $u_{0x}(0) = 0$, *then the solution* $\{u, \ell\}$ *to* $SP(u_0, \ell_0)$ *on* $[0, T]$ *satisfying that* u_x *is continuous on* $\overline{Q_\ell(T)}$, u_t *and* u_{xx} *are continuous on* $Q_\ell(T)$ *and* $\ell \in C^1([0, T])$.
(iv) Let $\{u, \ell\}$ *be a solution to* $SP(u_0, \ell_0)$ *on* $[0, T')$, $0 < T' < \infty$, *and* M *be any positive number. If* $|u(t, x)| \leq M$ *for* $(t, x) \in Q(T')$, *then the solution is extended in time beyond* T'.

Remark 2.1. By Definition 2.1 and Theorem 2.1 (iii) for a solution $\{u, \ell\}$ to SP on $[0, T]$, u_x is continuous on the set $\{(t, x); 0 \leq x \leq \ell(t), 0 < t \leq T\}$, u_t and u_{xx} are continuous on $Q_\ell(T)$ and $\ell \in C^1([0, T])$. Hence, applyng the strong maximum principle to SP we get the assertion (ii) in Theorem 2.1.

Throughout this paper for the problem SP, we say that $[0, T)$, $0 < T \leq +\infty$, is the maximal interval of existence of the solution, if the problem has a solution on time-interval $[0, T']$, for every T' with $0 < T' < T$ and the solution can not be extended in time beyond T. Also, for simplicity we put

$$E(z, s) = \int_0^s z(x)dx + s \quad \text{for } (z, s) \in V,$$

and

$$V(M, L) = \{(z, s) \in V; s \leq L \text{ and } z(x) \leq M \text{ for } 0 < x < s\},$$

where M and L are positive numbers.

Here, we give a theorem concerned with the global existence of solutions to SP.
Theorem 2.2. *(cf. [4; Theorem1.2]) Let* $\alpha > 1$, $(u_0, \ell_0) \in V$. *Then, for any positive number* M *there exist positive numbers* $\delta_0 = \delta(M, \alpha) \in (0, 1]$ *such that if* $\ell_0 \leq M$, $\int_0^{\ell_0} u_{0x}^2 dx \leq M$ *and* $\int_0^{\ell_0} u_0^2 dx \leq \delta_0$, *then the problem* $SP(u_0, \ell_0)$ *has a solution* $\{u, \ell\}$ *on* $[0, \infty)$ *satisfying that*

$$E(u(t), \ell(t)) \leq \left\{C + E(\frac{1}{2})^\beta\right\}^{\frac{1}{\beta}} \quad \text{for } t \geq \frac{1}{2},$$

$$\frac{d}{dt}|u_x(t)|^2_{L^2(0,\ell(t))} \le 0 \quad \textit{for a.e. } t > 0,$$
$$|u(t)|_{L^\infty(0,\ell(t))} \le \sqrt{2}\exp(-\mu t) \quad \textit{for } t > 0,$$

where $C = C(\alpha)$, $\beta = \beta(\alpha)$, *and* $\mu = \mu(\alpha, \ell_0, |u_0|_{L^\infty(0,\ell_0)})$ *are some positive constants.*

For brevity we introduce the following set $G := G(u_0, \ell_0; M, L, \mu)$ for $(u_0, \ell_0) \in V$ and positive numbers M, L and μ:

$$G(u_0,\ell_0;M,L,\mu) = \left\{ \{u,\ell\};\ \begin{array}{l} \{u,\ell\} \text{ is a solution to } SP(u_0,\ell_0) \text{ on } [0,\infty) \text{ satisfying that} \\ |u_x(t)_{L^2(0,\ell(t))} \le M, |u(t)|_{L^\infty(0,\ell(t))} \le M\exp(-\mu t) \text{ and} \\ \ell(t) \le L \text{ for } t \ge 0. \end{array} \right\}.$$

The theorem is our main result on the stability of global solutions to SP.

Theorem 2.3. *Let* $\alpha > 1$, $(u_0, \ell_0) \in V$, M, L *and* μ *be positive numbers and* $\{u, \ell\} \in G(u_0, \ell_0; M, L, \mu)$. *Then, there is a positive number* $p_1 > 0$ *depending only on* α *satisfying the following property:*

For any positive number $\tilde{M}$ *there exists a positive constant* δ *such that for any* $(\hat{u}_0, \hat{\ell}_0) \in V(\tilde{M})$ *with* $|u_0 - \hat{u}_0|_{L^{p_1}(0,\infty)} < \delta$ *and* $|\ell_0 - \hat{\ell}_0| < \delta$ *the problem* $SP(\hat{u}_0, \hat{\ell}_0)$ *has a solution* $\{\hat{u}, \hat{\ell}\}$ *on* $[0,\infty)$ *satisfying that*

$$\hat{\ell}(t) \le \hat{L} \ \textit{and}\ |\hat{u}(t)|_{L^\infty(0,\hat{\ell}(t))} \le \hat{M}\exp(-\hat{\mu}t) \ \textit{for } t \ge 0,$$

where $\hat{M}$, $\hat{L}$ *and* $\hat{\mu}$ *are positive constants depending on* α, M, L, μ, $\tilde{M}$ *and* δ.

We shall prove Theorem 2.3 in the following way. First, we give some useful inequalities in Sobolev spaces and an ordinary differential inequality in section 2. Secondly, some properties of a global solution belonging to the set $G(u_0, \ell_0; M, L, \mu)$ are shown (see section 3). Next, we obtain the following decay for $v := \hat{u} - u$ under the condition $\ell_0 \le \hat{\ell}_0$ and $u_0 \le \hat{u}_0$:

$$|v(t)|_{L^{p_1}(0,\infty)} \le c(1+t)^{-\beta} \quad \text{for } t \ge 0,$$

where c and β are positive constants. Finally, we give a complete proof of Theorem 2.3 by applying Theorem 2.2.

At the final of this section we define some notations. In order to avoid surplus confusion for notations we write the set of positive constants, α, M, L, μ, $\tilde{M}$ and $\tilde{L}$ as (D). Since $\alpha > 1$

we can take numbers satisfying that

$$\begin{cases} p_1 > \max\{2+\alpha, \dfrac{1+\alpha}{\alpha-1}\} \text{ and } \dfrac{p_1}{1+\alpha} + \dfrac{1}{2} < \dfrac{1}{r_0} < \dfrac{p_1}{2}, \\ (\dfrac{1}{r_0} - \dfrac{1}{2})\dfrac{1+\alpha}{p_1} = 1 + \beta_0, \\ p_0 = \dfrac{p_1 r_0}{2}. \end{cases} \tag{2.1}$$

Clearly, we obtain that $1 < p_0 < p_1$ and $0 < r_0 < 2$. These numbers play an important role in our proof.

3 Auxiliary lemmas

At the begin of this section we list some useful inequalities in Sobolev spaces(cf. O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva [10; Chap.2, Theorem 2.2]): Let d be any positive number.

$$\int_0^d u^{p+\alpha} dx \le \left(\frac{q+2}{2}\right)^{\frac{2(q-r)}{r+2}} |(u^{\frac{p}{2}})_x|_{L^2(0,d)}^{\frac{2(q-r)}{r+2}} \left(\int_0^d u^{\frac{pr}{2}} dx\right)^{\frac{q+2}{r+2}} \quad \text{for } u \in W^{1,2}(0,d) \text{ with } u(d) = 0, \tag{3.1}$$

where $p \ge 2$, $\alpha \ge 0$, $q = 2(p+\alpha)/p$ and $r \in (0,q)$;

$$|u|_{L^2(0,d)} \le 2d|u_x|_{L^2(0,d)} \quad \text{for } u \in W^{1,2}(0,d) \text{ with } u(d) = 0; \tag{3.2}$$

$$|u|_{L^\infty(0,d)} \le \left(\frac{q+2}{2}\right)^{\frac{2}{q+2}} |u_x|_{L^2(0,d)}^{\frac{2}{q+2}} |u|_{L^q(0,d)}^{\frac{q}{q+2}} \quad \text{for } u \in W^{1,2}(0,d) \text{ with } u(d) = 0, \tag{3.3}$$

where $q \ge 1$.

The first lemma is concerned with an ordinary differential inequality.

Lemma 3.1. *Let a, b and μ be positive numbers, $0 < r < 2$ and z be a non-negative absolutely continuous function on $[0,T]$, $0 < T < \infty$, satisfying*

$$\frac{d}{dt} z(t) + a z^{\frac{2+r}{2-r}}(t) \le b \exp(-\mu t) \quad \textit{for a.e. } t \in [0,T].$$

Then, there is a positive constant $N_0 = N_0(a,b,r,\mu)$ such that

$$z(t) \le N_0(1 + z(0))(1 + a\beta t)^{-\frac{1}{\beta}} \quad \textit{for any } t \in [0,T], \tag{3.4}$$

where $\beta = \dfrac{2r}{2-r}$.

Proof. Let N_1 be any positive number and

$$\psi(t) = N_1(1 + a\beta t)^{-1/\beta} \quad \text{for } t \in [0,T].$$

By elementary calculation we obtain that

$$\begin{aligned} &\frac{d}{dt}(z(t) - \psi(t)) + a\frac{2-r}{2+r}\psi^{\frac{2r}{2-r}}(t)(z(t) - \psi(t)) \\ \le\ & b\exp(-\mu t) - a(N_1^{\frac{2+r}{2-r}} - N_1)(1 + a\beta t)^{-\frac{r+2}{2r}} \quad \text{for a.e. } t \in [0,T]. \end{aligned}$$

Hence, we take a positive number $N_0 \ge 1$ such that

$$(\frac{b}{a})^{\frac{2r}{2+r}}(1 + \frac{a\beta(2+r)}{2r\mu e}) \le (N_0^{\frac{2r}{2-r}} - N_0)^{\frac{2r}{2+r}},$$

and put $N_1 = N_0(1 + z(0))$.

Therefore, we have

$$\frac{d}{dt}(z(t) - \psi(t)) + a\frac{2-r}{2+r}\psi^{\frac{2r}{2-r}}(t)(z(t) - \psi(t)) \le 0 \qquad \text{for a.e. } t \in [0,T].$$

By using Gronwall's argument we see that

$$\begin{aligned} z(t) - \psi(t) &\le (z(0) - \psi(0))\exp\{-a\int_0^t \frac{2-r}{2+r}\psi^{\frac{2r}{2-r}}(\tau)d\tau\} \\ &\le z(0) - N_1(1 + z(0)) \le 0 \quad \text{for any } t \in [0,T]. \end{aligned}$$

Thus, we get (3.4). $\square$

Lemma 3.2. *Let $p > 1$ and $d > 0$. We suppose that $u \in W^{2,2}(0,d)$ with $u_x(0) = 0$, $u(d) = 0$ and $u > 0$ on $(0,d)$. Then, $(u^{p/2})_x \in L^2(0,d)$.*

Proof. It is sufficient to show that there is a function $f \in L^2(0,d)$ such that

$$-\int_0^d u^{p/2}\eta_x dx = \int_0^d f\eta dx \quad \text{for any } \eta \in C_0^\infty([0,d]). \tag{3.5}$$

Let $\eta \in C_0^\infty([0,d])$. Then, there is a positive number ε such that $supp(\eta) \subset [\varepsilon, d-\varepsilon]$ so that $u \ge \delta > 0$ on $[\varepsilon, d-\varepsilon]$ for some positive number δ. Clearly, we have

$$-\int_0^d u^{p/2}\eta_x dx = \int_\varepsilon^{d-\varepsilon} (u^{p/2})_x \eta dx = \frac{p}{2}\int_\varepsilon^{d-\varepsilon} u_x u^{\frac{p}{2}-1}\eta dx.$$

Hence,

$$\left| -\int_0^d u^{p/2}\eta_x dx \right| \le \frac{p}{2}\left(\int_\varepsilon^{d-\varepsilon} |u_x|^2|u|^{p-2}dx\right)^{1/2}\left(\int_0^d \eta^2 dx\right)^{1/2}.$$

Here, we note that

$$
\begin{aligned}
&\int_{\varepsilon}^{d-\varepsilon} |u_x|^2|u|^{p-2}dx \\
= &\int_{\varepsilon}^{d-\varepsilon} u_x(\frac{1}{p-1}u^{p-1})_x dx \\
= &-\frac{1}{p-1}\int_{\varepsilon}^{d-\varepsilon} u_{xx}u^{p-1}dx + \frac{1}{p-1}\{u_x(d-\varepsilon)u^{p-1}(d-\varepsilon) - u_x(\varepsilon)u^{p-1}(\varepsilon)\}. \qquad (3.6)
\end{aligned}
$$

Letting $\varepsilon \downarrow 0$ in (3.6), because of continuity of u_x on $[0, d]$ we obtain that

$$\lim_{\varepsilon\downarrow 0}\int_{\varepsilon}^{d-\varepsilon} |u_x|^2|u|^{p-2}dx = -\frac{1}{p-1}\int_0^d u_{xx}u^{p-1}dx,$$

that is,

$$|-\int_0^d u^{p/2}\eta_x dx| \leq C|\eta|_{L^2(0,d)} \quad \text{for any } \eta \in C_0^\infty([0,d]),$$

where C is some positive constant.

Immediately, we conclude that there is a function $f \in L^2(0, d)$ satisfying (3.5). □

4 Properties of a global solution

In this section we show some estimates for a global solution to SP. First, we recall some useful equations for a solution to SP.

Lemma 4.1. *(cf. [9; Lemma 5.1] and [4; Lemma 2.1]) Let $(u_0, \ell_0) \in V$ and $\{u, \ell\}$ be a solution to $SP(u_0, \ell_0)$ on $[0, T]$, $0 < T < \infty$.*

(1) We have

$$\frac{d}{dt}E(u(t), \ell(t)) = \int_0^{\ell(t)} u^{1+\alpha}(t, x)dx \quad \textit{for a.e. } t \in [0, T]. \qquad (4.1)$$

(2) For a.e. $t \in [0, T]$ we have

$$|u_t(t)|^2_{L^2(0,\ell(t))} + \frac{1}{2}|\ell'(t)|^3 + \frac{1}{2}\frac{d}{dt}|u_x(t)|^2_{L^2(0,\ell(t))} = \frac{1}{2+\alpha}\frac{d}{dt}|u(t)|^{2+\alpha}_{L^{2+\alpha}(0,\ell(t))}. \qquad (4.2)$$

Next, the following lemma guarantees a decay for u_x.

Lemma 4.2. *Let M, L and μ be positive numbers, $(u_0, \ell_0) \in V$ and $\{u, \ell\} \in G(u_0, \ell_0; M, L, \mu)$. Then, there are positive constants L_1 and μ_1 such that*

$$|u_x(t)|_{L^2(0,\ell(t))} \leq L_1 \exp(-\mu_1 t) \quad \textit{for } t > 0.$$

Proof. By the argument in the proof of [9; Lemma 5.2] we have

$$\int_0^{\ell(t)} u_t(t)u_{xx}(t)dx = -\frac{1}{2}\frac{d}{dt}\int_0^{\ell(t)} |u_x(t)|^2 dx - |\ell'(t)|^3 \quad \text{for } t \geq 0. \tag{4.3}$$

Also, from (1.1) we see that

$$\begin{aligned} & \int_0^{\ell(t)} u_t(t)u_{xx}(t)dx \\ = & \int_0^{\ell(t)} (u_{xx}(t) + u^{1+\alpha}(t))u_{xx}(t)dx \\ = & \int_0^{\ell(t)} (u_{xx})^2(t)dx - (1+\alpha)\int_0^{\ell(t)} u^\alpha(t)(u_x)^2(t)dx \quad \text{for } t > 0. \end{aligned} \tag{4.4}$$

It follows from (4.3), (4.4) and (3.2) that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_0^{\ell(t)} |u_x(t)|^2 dx + |\ell'(t)|^3 + \frac{1}{4L^2}\int_0^{\ell(t)} |u_x(t)|^2 dx \\ \leq & \frac{1}{2}\frac{d}{dt}\int_0^{\ell(t)} |u_x(t)|^2 dx + |\ell'(t)|^3 + \int_0^{\ell(t)} |u_{xx}(t)|^2 dx \\ \leq & (1+\alpha)M^\alpha \exp(-\alpha\mu t)\int_0^{\ell(t)} |u_x(t)|^2 dx \quad \text{for } t > 0. \end{aligned} \tag{4.5}$$

Here, we can take a positive number t_0 such that $(1+\alpha)M^\alpha \exp(-\alpha\mu t) \leq \frac{1}{8L^2}$ for $t \geq t_0$. Accordingly, for $t \geq t_0$

$$\frac{d}{dt}\int_0^{\ell(t)} |u_x(t)|^2 dx + \frac{1}{4L^2}\int_0^{\ell(t)} |u_x(t)|^2 dx \leq 0,$$

and hence

$$\int_0^{\ell(t)} |u_x(t)|^2 dx \leq \exp\{-\frac{1}{4L^2}(t-t_0)\}\int_0^{\ell(t)} |u_x(t_0)|^2 dx.$$

On the other hands, (4.5) implies

$$\frac{d}{dt}\int_0^{\ell(t)} |u_x(t)|^2 dx \leq 2(1+\alpha)M^\alpha \int_0^{\ell(t)} |u_x(t)|^2 dx.$$

By Gronwall's inequality, we have

$$\int_0^{\ell(t)} |u_x(t)|^2 dx \leq \exp\{2(1+\alpha)L^\alpha t_0 + \frac{t_0}{4L^2}\}\exp\{-\frac{t}{4L^2}\}|u_{0x}|^2_{L^2(0,\ell_0)} \quad \text{for } t \in [0, t_0].$$

Therefore, putting

$$L_1 = \exp\{2(1+\alpha)L^\alpha t_0 + \frac{t_0}{4L^2}\}|u_{0x}|^2_{L^2(0,\ell_0)} \text{ and } \mu_1 = \frac{1}{4L^2},$$

we get the assertion of the lemma. □

The following lemma shows the decay of ℓ', which is a key of the proof of Theorem 2.3.

Lemma 4.3. *We suppose that the same assumptions as in Lemma 4.2 hold and* $1 < q \leq 4$. *Then, for some positive number* $\mu_0 = \mu_0(\mu, q)$

$$\int_0^\infty |\ell'(t)|^q e^{\mu_0 t} dt < \infty.$$

Clearly, the above fact implies that

$$\int_0^\infty |\ell'(t)|^q dt < \infty.$$

Proof. Let M_1 and μ_1 be positive constants defined in Lemma 4.2. According to (3.3) and Lemma 4.2, we see that for any $t > 0$

$$\begin{aligned} |\ell'(t)|^q &= |u_x(t, \ell(t)-)|^q \\ &\leq \sqrt{2M_1} \exp(-\frac{q}{2}\mu_1 t) |u_{xx}(t)|^{q/2}_{L^2(0,\ell(t))}, \end{aligned}$$

and hence

$$|\ell'(t)|^q \exp(\frac{\mu_1 q}{4} t) \leq C |u_{xx}(t)|^2_{L^2(0,\ell(t))} + C \exp(-\frac{\mu_1 q}{4-q} t), \tag{4.6}$$

where C is some suitable positive constant.

By using (4.5) and Lemma 4.2, again, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\tau} \int_0^{\ell(\tau)} |u_x(\tau)|^2 dx + \int_0^{\ell(\tau)} |u_{xx}(\tau)|^2 dx \\ \leq\ & (1+\alpha) M^\alpha \exp(-\alpha\mu\tau) \int_0^{\ell(\tau)} |u_x(\tau)|^2 dx \\ \leq\ & (1+\alpha) M^\alpha M_1^2 \exp\{-(\alpha\mu + 2\mu_1)\tau\} \quad \text{for } \tau > 0. \end{aligned}$$

Integrating this inequality over $[0, t]$, $0 < t < \infty$, we obtain that

$$\begin{aligned} &\int_0^{\ell(t)} |u_x(t)|^2 dx + \int_0^t \int_0^{\ell(\tau)} |u_{xx}(\tau)|^2 dx d\tau \\ \leq\ & (1+\alpha) M^\alpha M_1^2 \int_0^t \exp\{-(\alpha\mu + 2\mu_1)\tau\} d\tau + \int_0^{\ell_0} |u_{0x}|^2 dx \quad \text{for } t \geq 0. \end{aligned}$$

Adding to (4.6), we conclude that $\int_0^\infty \ell'(t) e^{\mu_0 t} dt < \infty$ where $\mu_0 = \frac{\mu_1}{q}$. □

5 Energy inequalities

The purpose of this section is to establish the following lemmas concerned with global estimates for the difference, $\hat{u}-u$, of solutions to SP.

Lemma 5.1. *Let $(u_0,\ell_0),(\hat{u}_0,\hat{\ell}_0)\in V$, M, L and μ be positive numbers, $\{u,\ell\}\in G(u_0,\ell_0;M,L,\mu)$, and $\{\hat{u},\hat{\ell}\}$ be a solution to $SP(\hat{u}_0,\hat{\ell}_0)$ on $[0,T]$, $0<T<\infty$. Moreover, we suppose that $\ell_0\le\hat{\ell}$, $u_0\le\hat{u}_0$ on $[0,\infty)$ and $\hat{u}_0\not\equiv u_0$. Then, putting $v=\hat{u}-u$ we obtain that for $t\in(0,T]$ and $p\in[p_0,p_1]$ (see (2.1)).*

$$
\begin{aligned}
&\frac{d}{dt}\int_0^{\hat{\ell}(t)} v^p(t)dx\\
\le\ & \left\{-C_1+C_2\hat{\ell}(t)^{2-\frac{1+\alpha}{p_1}}\left(\int_0^{\hat{\ell}(t)} v^{p_1}(t)dx\right)^{\frac{\alpha}{p_1}}\right\}|(v^{\frac{p}{2}})_x(t)|^2_{L^2(0,\hat{\ell}(t))} \qquad (5.1)\\
&+C_2\exp(-\alpha\mu t)\int_0^{\hat{\ell}(t)} v^p(t)dx+C_2\ell'(t)^{\frac{2p}{p+1}}\left(\int_0^{\hat{\ell}(t)} v^p(t)dx\right)^{\frac{p-1}{p+1}},
\end{aligned}
$$

where C_1 and C_2 are positive constants depending on α, p_0, p_1 and M.

Proof. For simplicity, we put $H(t)=L^2(0,\hat{\ell}(t))$.

First, by Theorem 2.1 (ii) we have $v=\hat{u}-u>0$ on $Q_{\hat{\ell}}(T)$. Multiplying (1.1) by v^{p-1} for $p\in[p_0,p_1]$ we apply Lemma 3.2 to the following caluculation so that

$$
\begin{aligned}
&\frac{d}{dt}\int_0^{\hat{\ell}(t)} v^p(t)dx\\
=\ & p\int_0^{\hat{\ell}(t)} v_t(t)v^{p-1}(t)dx\\
=\ & p\int_0^{\hat{\ell}(t)}(\hat{u}_{xx}(t)+\hat{u}^{1+\alpha}(t))v^{p-1}(t)dx-p\int_0^{\ell(t)}(u_{xx}(t)+u^{1+\alpha}(t))v^{p-1}(t)dx\\
=\ & -p(p-1)\int_0^{\hat{\ell}(t)}(v_x)^2(t)v^{p-2}(t)dx+p\ell'(t)v^{p-1}(t,\ell(t))\\
&+p\int_0^{\hat{\ell}(t)}(\hat{u}^{1+\alpha}(t)-u^{1+\alpha}(t))v^{p-1}(t)dx\\
\le\ & -\frac{4(p-1)}{p}\int_0^{\hat{\ell}(t)}((v^{p/2})_x(t))^2dx+p\ell'(t)v^{p-1}(t,\ell(t)) \qquad (5.2)\\
&+p2^\alpha(1+\alpha)\int_0^{\hat{\ell}(t)}(v^{p+\alpha}(t)+u^\alpha(t)v^p(t))dx \quad \text{for } t\in(0,T].
\end{aligned}
$$

Here, we note that

$$\hat{u}^{1+\alpha}-u^{1+\alpha}\le 2^\alpha(1+\alpha)(v^{1+\alpha}+vu^\alpha). \qquad (5.3)$$

From (3.1) and Hölder's inequality it follows that

$$\begin{aligned}
&\int_0^{\hat\ell(t)} v^{p+\alpha}(t)dx \\
\le\ & (\frac{\alpha}{p}+2)^2|(v^{p/2})_x(t)|^2_{H(t)}(\int_0^{\hat\ell(t)} v^{\alpha/2}(t)dx)^2 \\
\le\ & (\frac{\alpha}{p}+2)^2|(v^{p/2})_x(t)|^2_{H(t)}\hat\ell(t)^{2-\frac{1+\alpha}{p_1}}(\int_0^{\hat\ell(t)} v^{p_1}(t)dx)^{\frac{\alpha}{p_1}} \quad \text{for } t\in(0,T]. \qquad (5.4)
\end{aligned}$$

Also, according to (3.3) we have

$$\begin{aligned}
&p\ell'(t)v^{p-1}(t,\ell(t)) \\
\le\ & p\ell'(t)|v^{p/2}(t)|^{\frac{2(p-1)}{p}}_{L^\infty(0,\hat\ell(t))} \\
\le\ & 2^{\frac{p-1}{p}}p\ell'(t)|(v^{p/2})_x(t)|^{\frac{p-1}{p}}_{H(t)}|(v^{p/2})(t)|^{\frac{p-1}{p}}_{H(t)} \\
\le\ & \frac{2(p-1)}{p}|(v^{p/2})_x(t)|^2_{H(t)}+C_p|\ell'(t)|^{\frac{2p}{p+1}}(\int_0^{\hat\ell(t)} v^p(t)dx)^{\frac{p-1}{p+1}} \quad \text{for } t\in(0,T], \qquad (5.5)
\end{aligned}$$

where C_p is a positive constant depending only on p.

It follows from (5.2) $\sim$ (5.5) that for $t \in (0,T]$

$$\begin{aligned}
\frac{d}{dt}\int_0^{\hat\ell(t)} v^p(t)dx \le\ & \left\{-C_1+C_2\hat\ell(t)^{2-\frac{\alpha}{p_1}}(\int_0^{\hat\ell(t)} v^{p_1}(t)dx)^{\frac{\alpha}{p_1}}\right\}|(v^{p/2})_x(t)|^2_{H(t)} \\
&+C_2\exp(-\alpha\mu t)\int_0^{\hat\ell(t)} v^p(t)dx+C_2|\ell'(t)|^{\frac{2p}{p+1}}(\int_0^{\hat\ell(t)} v^p(t)dx)^{\frac{p-1}{p+1}},
\end{aligned}$$

where

$$C_1=\frac{2(p_0-1)}{p_1} \text{ and } C_2=2^\alpha p_1(2+\frac{\alpha}{p_0})^2(1+\alpha)+2^\alpha p_1(1+\alpha)M+\max_{p_0\le p\le p_1} C_p.$$

This is the conclusion of the lemma. □

Lemma 5.2. *Let B_1 and B_2 be positive numbers. We suppose that the same assumptions as in Lemma 5.1 hold. Moreover, we suppose that for $p\in[p_0,p_1]$ and $t\in(0,T]$, $0<T<\infty$,*

$$\begin{aligned}
\frac{d}{dt}\int_0^{\hat\ell(t)} v^p(t)dx \le\ & -B_0|(v^{\frac{p}{2}})_x(t)|^2_{L^2(0,\hat\ell(t))}+B_1\exp(-\alpha\mu t)\int_0^{\hat\ell(t)} v^p(t)dx \qquad (5.6) \\
&+B_1\ell'(t)^{\frac{2p}{p+1}}(\int_0^{\hat\ell(t)} v^p(t)dx)^{\frac{p-1}{p+1}}.
\end{aligned}$$

Then, there is a positive constants C_3 depending on α, p_0, p_1, μ, M and L, which satisfies that

$$\int_0^{\hat{\ell}(t)} v^p(t)dx \le C_3(\int_0^{\hat{\ell}_0} v^p(0)dx + 1) \quad \textit{for } p \in [p_0, p_1] \textit{ and } t \in [0,T]. \tag{5.7}$$

Proof. For simplicity, we put

$$F_p(t) = \int_0^{\hat{\ell}(t)} v^p(t)dx \quad \text{for } p \in [p_0, p_1] \text{ and } t \in [0,T].$$

Obviously, we obtain that

$$\ell'(t)^{\frac{2p}{p+1}} F_p(t)^{\frac{p-1}{p+1}} \le \frac{2}{p+1}\ell'(t)^{\frac{2p}{p+1}} + \frac{p-1}{p+1}\ell'(t)^{\frac{2p}{p+1}} F_p(t) \quad \text{for } t \in [0,T].$$

Hence, from (5.6) it follows that for $p \in [p_0, p_1]$

$$\frac{d}{dt}F_p(t) \le C_4 \exp(-\alpha\mu t)F_p(t) + C_4|\ell'(t)|^{\frac{2p}{p+1}} + C_4|\ell'(t)|^{\frac{2p}{p+1}} F_p(t) \quad \text{for } t \in (0,T],$$

where $C_4 = B_1 + B_1(\frac{1}{p_0+1} + \frac{p_1-1}{p_0+1})$.

Since $1 < \dfrac{2p}{p+1} < 4$, by applying Lemma 4.3 and Gronwall's argument to the above inequality, we get

$$F_p(t) \le (F_p(0) + C_4\int_0^\infty |\ell'|^{\frac{2p}{p+1}}) \exp(\int_0^\infty J_p(t)dt) \quad \text{for } t \in [0,T],$$

where $J_p(t) = C_4 \exp(-\alpha\mu t) + C_4|\ell'(t)|^{\frac{2p}{p-1}}$.

Thus, this lemma has been proved. □

Lemma 5.3. *Let $\tilde{M}$ and $\tilde{L}$ be positive numbers. Then, under the same conditions as in Lemma 5.2 there are positive constants C_5 and C_6 depending only on (D) such that*

$$\int_0^{\hat{\ell}(t)} v^{p_1}(t)dx \le C_5(1 + \int_0^{\hat{\ell}_0} v^{p_1}(0)dx)(1 + C_6 t)^{-\frac{2-r_0}{2r_0}} \quad \textit{for } t \in [0,T], \tag{5.8}$$

where r_0 is a positive constant defined by (2.1).

Proof. For brevity, we put $F(t) = \int_0^{\hat{\ell}(t)} v^{p_1}(t)dx$ for $t \in [0,T]$ and note that $\int_0^{\hat{\ell}_0} v^{p_0}(0)dx \le \tilde{M}^{p_0}\tilde{L}$. According to (3.1) and the previous lemma, we infer that

$$\begin{aligned} F(t) &\le 2^{\frac{2(2-r_0)}{r_0+2}} |(v^{p_1/2})_x(t)|_{H(t)}^{\frac{2(2-r_0)}{r_0+2}} (\int_0^{\hat{\ell}(t)} v^{p_0}(t)dx)^{\frac{2}{r_0+2}} \\ &\le 2^{\frac{2(2-r_0)}{r_0+2}} (C_3(\tilde{M}^{p_0}\tilde{L} + 1))^{\frac{2}{r_0+2}} |(v^{p_1/2})_x(t)|_{H(t)}^{\frac{2(2-r_0)}{r_0+2}} \quad \text{for } t \in (0,T], \end{aligned}$$

and hence

$$|(v^{p_1/2})_x(t)|^2_{H(t)} \geq \frac{1}{4(C_3(\tilde{M}^{p_0}\tilde{L}+1))^{\frac{2}{2-r_0}}} F(t)^{\frac{2+r_0}{2-r_0}} \quad \text{for } t \in (0,T].$$

Here, we note that

$$\begin{aligned} & \ell'(t)^{\frac{2p}{p_1+1}} F(t)^{\frac{p_1-1}{p_1+1}} \\ \leq\ & \frac{p_1-1}{p_1+1}\ell'(t)^{\frac{2p_1}{p_1-1}} F(t)\exp(\mu_0 t) + \frac{2}{p_1+1}\exp(-\frac{p_1-1}{2}\mu_0 t) \quad \text{for } t \in (0,T], \end{aligned}$$

where μ_0 is a positive constant defined in Lemma 4.3, and $1 < \dfrac{2p_1}{p_1-1} < 4$.

Therefore, by adding the above inequalities together to (5.6) we have

$$\begin{aligned} \frac{d}{dt}F(t) \ &\leq\ -K_1F(t)^{\frac{2+r_0}{2-r_0}} + K_2|\ell'(t)|^{\frac{2p_1}{p_1-1}} F(t)\exp(\mu_0 t) \\ &\quad +K_2\exp(-\mu_2 t) + K_2F(t)\exp(-\mu_3 t) \quad \text{for } t \in (0,T], \end{aligned}$$

where K_1, K_2, μ_2 and μ_3 are suitable positive constants.

For simplicity, we put

$$J(t) = K_2(|\ell'(t)|^{\frac{2p_1}{p_1-1}}\exp(\mu_0 t) + \exp(-\mu_3 t)) \text{ and } \Phi(t) = F(t)\exp(-\int_0^t J(\tau)d\tau).$$

It is clear that

$$\frac{d}{dt}\Phi(t) + K_1\Phi(t)^{\frac{2+r_0}{2-r_0}}\exp(\frac{2r_0}{2-r_0}\int_0^t J(\tau)d\tau) \leq K_2\exp(-\mu_2 t) \quad \text{for } t \in (0,T],$$

and

$$\frac{d}{dt}\Phi(t) + K_1\Phi(t)^{\frac{2+r_0}{2-r_0}} \leq K_2\exp(-\mu_2 t) \quad \text{for } t \in (0,T].$$

By Lemma 3.1, we obtain that

$$\Phi(t) \leq N_0(1+\Phi(0))(1+\beta_1K_1t)^{-\frac{1}{\beta_1}} \quad \text{for } t \in [0,T],$$

where $\beta_1 = \dfrac{2r_0}{2-r_0}$ and $N_0 = N_0(K_1,K_2,r_0,\mu_2) > 0$.

Accordingly, this implies that

$$F(t) \leq N_0(1+F(0))(1+\beta_1K_1t)^{-\frac{1}{\beta_1}}\exp(\int_0^\infty J(\tau)d\tau) \quad \text{for } t \in (0,T],$$

since $1 < 2p_1/(p_1-1) < 4$, the integration inthe above inequality makes sense. We get the assertion of Lemma 5.3. $\square$

At the end of this section, we give a global estimate for $E(\hat{u}(t), \hat{\ell}(t))$.

Lemma 5.4. *We suppose that the same assumptions of Lemma 5.3 hold. Then, there exists a positive constant C_7 depending only on (D), which satisfies that*

$$E(\hat{u}(t), \hat{\ell}(t)) \leq C_7\{E(\hat{u}_0, \hat{\ell}_0) + (\int_0^{\hat{\ell}_0} v^{p_1}(0)dx)^{\frac{p_1\beta_0}{1+\alpha}} + 1\} \quad \textit{for } t \in [0, T], \tag{5.9}$$

where β_0 is a positive constant defined by (2.1).

Proof. For simplicity, we use the same notations as in the proof of the previous lemmas and put

$$E(t) = E(u(t), \ell(t)) \text{ and } \hat{E}(t) = E(\hat{u}(t), \hat{\ell}(t)).$$

It follows from (4.1) with help of (5.3) that

$$\begin{aligned}
&\frac{d}{dt}(\hat{E}(t) - E(t)) \\
\leq\ & 2^\alpha(1+\alpha)\{\int_0^{\hat{\ell}(t)} v^{1+\alpha}(t)dx + (\int_0^{\hat{\ell}(t)} v^{1+\alpha}(t)dx)^{\frac{1}{1+\alpha}}(\int_0^{\ell(t)} u^{1+\alpha}(t)dx)^{\frac{\alpha}{1+\alpha}}\} \\
\leq\ & 2^\alpha(1+\alpha)\{\int_0^{\hat{\ell}(t)} v^{1+\alpha}(t)dx + \frac{1}{1+\alpha}\int_0^{\hat{\ell}(t)} v^{1+\alpha}(t)dx + \frac{\alpha}{1+\alpha}\int_0^{\ell(t)} u^{1+\alpha}(t)dx\} \\
\leq\ & 2^\alpha(1+\alpha)\{\frac{\alpha+2}{\alpha+1}(\int_0^{\hat{\ell}(t)} v^{p_1}(t)dx)^{\frac{1+\alpha}{p_1}}\hat{\ell}(t)^{1-\frac{1+\alpha}{p_1}} + \frac{\alpha}{1+\alpha}M^{1+\alpha}\exp(-(1+\alpha)\mu t)\} \\
\leq\ & K_3 F(t)^{\frac{1+\alpha}{p_1}}\hat{E}(t)^{1-\frac{1+\alpha}{p_1}} + K_3\exp(-(1+\alpha)\mu t) \quad \text{for } t \in (0, T],
\end{aligned}$$

where $K_3 = 2^\alpha(1+\alpha)(\frac{\alpha+2}{\alpha+1} + \frac{\alpha}{1+\alpha}M^{1+\alpha})$, and hence

$$\begin{aligned}
&(\frac{d}{dt}\hat{E}(t))\hat{E}(t)^{\frac{1+\alpha}{p_1}-1} \\
\leq\ & K_3F(t)^{\frac{1+\alpha}{p_1}} + K_3\hat{E}(t)^{\frac{1+\alpha}{p_1}-1}\exp(-(1+\alpha)\mu t) + \hat{E}(t)^{\frac{1+\alpha}{p_1}-1}\frac{d}{dt}E(t) \quad \text{for } t \in (0, T].
\end{aligned}$$

Moreover, since $\frac{1+\alpha}{p_1} - 1 < 0$, $(\hat{E}(t))^{\frac{1+\alpha}{p_1}-1} \leq (E(t))^{\frac{1+\alpha}{p_1}-1}$ and $\frac{d}{dt}E(t) \geq 0$, we see that

$$\begin{aligned}
\frac{p_1}{1+\alpha}\frac{d}{d\tau}(\hat{E}(\tau)^{\frac{1+\alpha}{p_1}}) \leq\ & K_3F(\tau)^{\frac{1+\alpha}{p_1}} + K_3E(0)^{\frac{1+\alpha}{p_1}-1}\exp(-(1+\alpha)\mu\tau) \\
& + \frac{p_1}{1+\alpha}\frac{d}{d\tau}(E(\tau)^{\frac{1+\alpha}{p_1}}) \quad \text{for } \tau \in (0, T].
\end{aligned} \tag{5.10}$$

Integrating (5.10) over $[0, t]$, $0 \leq t \leq T$, we conclude that

$$\frac{p_1}{1+\alpha}(\hat{E}(t)^{\frac{1+\alpha}{p_1}} - \hat{E}(0)^{\frac{1+\alpha}{p_1}})$$

$$\leq \frac{p_1}{1+\alpha}(E(t)^{\frac{1+\alpha}{p_1}} - E(0)^{\frac{1+\alpha}{p_1}}) + K_3\int_0^t F(\tau)^{\frac{1+\alpha}{p_1}}d\tau$$
$$+K_3E(0)^{\frac{1+\alpha}{p_1}-1}\int_0^t \exp(-(1+\alpha)\mu\tau)d\tau \quad \text{for } t\in[0,T].$$

Here, it follows from Lemma 5.3 that

$$\int_0^t F(\tau)^{\frac{1+\alpha}{p_1}}d\tau \leq \{C_5(1+F(0))\}^{\frac{1+\alpha}{p_1}}\int_0^\infty (1+C_6\tau)^{-\frac{(1+\alpha)(2-r_0)}{2p_1r_0}}d\tau$$
$$\leq \{C_5(1+F(0))\}^{\frac{1+\alpha}{p_1}}\int_0^\infty (1+C_6\tau)^{-1-\beta_0}d\tau \quad \text{for } t\in[0,T].$$

Therefore, it is easy to check that (5.9) holds. □

6 Stability of global solutions

First, we shall prove Theorem 2.3 in case the following condition (*) holds:

(*) $$\ell_0 \leq \hat{\ell}_0, u_0 \leq \hat{u}_0 \text{ on } [0, infty) \text{ and } u_0 \not\equiv \hat{u}_0.$$

Proof of Theorem 2.3 under the condition (*) Let $\{\hat{u},\hat{\ell}\}$ be a solution of $SP(\hat{u}_0,\hat{\ell}_0)$ on $[0,T_1]$, $0<T_1<\infty$, $\delta\in(0,1]$ and $v:=\hat{u}-u$. We assume that $\int_0^{\hat{\ell}_0} v^{p_1}(0)dx \leq \delta$ and $\hat{\ell}_0 \leq \ell_0+\delta$. Since the function $t\to \int_0^{\hat{\ell}(t)} v^{p_1}(t)dx$ is continuous, there is a positive constant $T_2\leq T_1$ such that

$$\int_0^{\hat{\ell}(t)} v^{p_1}(t)dx \leq 2\delta \text{ and } \hat{\ell}(t)\leq L+2 =: L_2 \quad \text{for } t\in[0,T_2].$$

Lemma 5.1 implies that

$$\frac{d}{dt}\int_0^{\hat{\ell}(t)} v^p(t)dx \leq \{-C_1 + C_2(L_2)^{2-\frac{1+\alpha}{p_1}}(2\delta)^{\frac{\alpha}{p_1}}\}|(v^{\frac{p}{2}})_x(t)|^2_{L^2(0,\hat{\ell}(t))}$$
$$+C_2\exp(-\alpha\mu t)\int_0^{\hat{\ell}(t)} v^p(t)dx$$
$$+C_2\ell'(t)^{\frac{2p}{p+1}}(\int_0^{\hat{\ell}(t)} v^p(t)dx)^{\frac{p-1}{p+1}} \quad \text{for } t\in(0,T_2] \text{ and } p\in[p_0,p_1].$$

We choose a positive number δ_1 such that

$$C_2(L_2)^{2-\frac{1+\alpha}{p_1}}(2\delta_1)^{\frac{\alpha}{p_1}} \leq \frac{C_1}{2},$$

and clearly, for $\delta\leq\delta_1$ we have

$$\frac{d}{dt}\int_0^{\hat{\ell}(t)} v^p(t)dt \leq -\frac{C_1}{2}|(v^{p/2})_x(t)|^2_{L^2(0,\hat{\ell}(t))} + C_2\exp(-\alpha\mu t)\int_0^{\hat{\ell}(t)} v^p(t)dx \tag{6.1}$$
$$+C_2\ell'(t)(\int_0^{\hat{\ell}(t)} v^p(t)dx)^{\frac{p-1}{p+1}} \quad \text{for } t\in(0,T_2] \text{ and } p\in[p_0,p_1].$$

By virtue of Lemmas 5.3 and 5.4, for $\delta < \delta_1$ we have

$$\int_0^{\hat{\ell}(t)} v^{p_1}(t)dx \le M_1(1+\delta)(1+M_2t)^{-\frac{2-r_0}{2r_0}} \quad \text{for } t \in [0,T_2], \tag{6.2}$$

$$E(\hat{u}(t),\hat{\ell}(t)) \le M_1(1+\delta^{\frac{p_1\beta_0}{1+\alpha}}) \quad \text{for } t \in [0,T_2],$$

where M_1 and M_2 are positive constants depending only on (D).

It follows from (4.2) that

$$|\hat{u}_x(t)|^2_{L^2(0,\hat{\ell}(t))} \le |\hat{u}_{0x}|^2_{L^2(0,\hat{\ell}_0)} + \frac{2}{2+\alpha}|\hat{u}(t)|^{2+\alpha}_{L^{2+\alpha}(0,\hat{\ell}(t))} \quad \text{for } t \in [0,T_2],$$

and hence with aid of (6.2) there is a positive constant M_3 depending on (D) such that

$$\left.\begin{array}{l} |\hat{u}_x(t)|_{L^2(0,\hat{\ell}(t))} \le M_3 \\ |v(t)|_{L^\infty(0,\hat{\ell}(t))} \le M_3 \end{array}\right\} \quad \text{for } t \in [0,T_2]. \tag{6.3}$$

Also, putting $\phi(t) = E(\hat{u}(t),\hat{\ell}(t)) - E(u(t),\ell(t))$, we have

$$\begin{aligned} \frac{d}{dt}\phi(t) &= \int_0^{\hat{\ell}(t)} (\hat{u}^{1+\alpha}(t) - u^{1+\alpha}(t))dx \\ &\le 2^\alpha(1+\alpha)\int_0^{\hat{\ell}(t)} (v^{1+\alpha}(t) + v(t)u^\alpha(t))dx \\ &\le 2^\alpha(1+\alpha)(M_3^\alpha + M^\alpha \exp(-\alpha\mu t))\phi(t) \quad \text{for } t \in (0,T_2]. \end{aligned} \tag{6.4}$$

Accordingly, by using Gronwall's inequality we infer that

$$\phi(t) \le \phi(0)\exp(M_4t) \quad \text{for } t \in [0,T_2],$$

where $M_4 = \exp\{2^\alpha(1+\alpha)(M_3^\alpha + M^\alpha)\}$.

Moreover, we observe that

$$\begin{aligned} \int_0^{\ell(t)} v^{p_1}(t)dx &\le M_3^{p_1-1}\phi(t) \\ &\le M_3^{p_1-1}\phi(0)\exp(M_4t) \\ &= M_3^{p_1-1}\Big(\int_0^{\hat{\ell}_0} v(0)dx + \hat{\ell}_0 - \ell_0\Big)\exp(M_4t) \\ &\le M_3^{p_1-1}(L_2^{1-1/p_1}\delta^{1/p_1} + \delta)\exp(M_4t) \\ &\le M_5\exp(M_4t)\delta^{1/p_1} \quad \text{for } \delta < \delta_1 \text{ and } t \in (0,T_2], \end{aligned} \tag{6.5}$$

where $M_5 = M_3^{p_1-1}(L_2^{1-1/p_1}+1)$.

It follows from Theorem 2.1 (iv) that we can extend the solution $\{\hat{u},\hat{\ell}\}$ on $[0,T_3)$ for some $T_3 > T_2$. Here, we take positive numbers $0<\delta_3<\delta_2<\delta_1$ and T_0 such that

$$C_2 L_2^{2-\frac{1+\alpha}{p_1}}(3\delta_2)^{\frac{\alpha}{p_1}} \le \frac{C_1}{2},$$

$$2M_1(1+M_2T_0)^{-\frac{2-r_0}{2r_0}} \le 2\delta_2,$$

$$C_2 L_2^{2-\frac{1+\alpha}{p_1}}\{2M_5\exp(M_4T_0)(2\delta_3)^{1/p_1}\}^{\frac{\alpha}{p_1}} \le \frac{C_1}{2}.$$

We suppose that $\int_0^{\hat{\ell}_0} v^{p_1}(0)dx < \delta_3$. Noting that $M_5\exp(M_4T_0)\delta_3^{1/p_1} > \delta_3$, if necessary we choose $M_5>1$, again. Now, if there is a positive number $t_0\in(0,T_0)$ such that

$$M_5\exp(M_4T_0)\delta_3^{\frac{1}{p_1}} \le \int_0^{\hat{\ell}(t_0)} v^{p_1}(t_0)dx < 2M_5\exp(M_4T_0)\delta_3^{\frac{1}{p_1}},$$

then the inequality (6.1) holds for $t\in(0,t_0]$ and $p\in[p_0,p_1]$, and hence by virtue of (6.5) we get the inequality, $\int_0^{\hat{\ell}(t_0)} v^{p_1}(t_0)dx < M_5\exp(M_4T_0)\delta_3^{\frac{1}{p_1}}$. This is a contradiction.

Therefore, the following inequality holds:

$$\int_0^{\hat{\ell}(t)} v^{p_1}(t)dx \le M_5\exp(M_4T_0)\delta_3^{\frac{1}{p_1}} \quad \text{for } t\in[0,T_0].$$

Similarly, $\{\hat{u},\hat{\ell}\}$ is the solution on $[0,T_0]$ and on account of (6.2) we have

$$\int_0^{\hat{\ell}(T_0)} v^{p_1}(T_0)dx \le 2M_1(1+M_2T_0)^{-\frac{2-r_0}{2r_0}} \le 2\delta_2.$$

Furthermore, if there is a positive number $t_1>T_0$ such that $2\delta_2 < \int_0^{\hat{\ell}(t_1)} v^{p_1}(t_1)dx \le 3\delta_2$, then this is a contradiction to (6.2). Hence, we conclude that

$$\int_0^{\hat{\ell}(t)} v^{p_1}(t)dx \le 2M_1(1+M_2t)^{-\frac{2-r_0}{2r_0}} \quad \text{for } t\ge T_0,$$

$$|\hat{u}_x(t)|_{L^2(0,\hat{\ell}(t))} \le M_3 \quad \text{for } t\ge 0,$$

$$E(\hat{u}(t),\hat{\ell}(t)) \le 2M_1 \quad \text{for } t\ge 0.$$

Therefore, Theorem 2.2 implies that Theorem 2.3 is valid under the condition (*).

Finally, we give a complete proof of the theorem.

Proof of Theorem 2.3. First, we put $X = L^{p_1}(0,\infty)$,

$$u_{01}=\min\{u_0,\hat{u}_0\},\, u_{02}=\max\{u_0,\hat{u}_0\},\, \ell_{01}=\min\{\ell_0,\hat{\ell}_0\} \text{ and } \ell_{02}=\max\{\ell_0,\hat{\ell}_0\}.$$

Let $\{u_1, \ell_1\}$ (resp. $\{u_2, \ell_2\}$) be a solution to $SP(u_{01}, \ell_{01})$ (resp. $SP(u_{02}, \ell_{02})$) on $[0, T_1]$ (resp. $[0, T_2]$). Putting $T_3 = \min\{T_1, T_2\}$ it is clear that $\{u_1, \ell_2\} \in G(u_0, \ell_0; M, L, \mu)$ and $|u_{02} - u_0|_X \leq |\hat{u}_0 - u_0|_X$, $u_1 \leq u$, $\hat{u} \leq u_2$ on $Q(T_3)$ and $\ell_1 \leq \ell$, $\hat{\ell} \leq \ell_2$ on $[0, T_3]$,

$$\begin{aligned} |u(t) - \hat{u}(t)|_X &\leq |u_1(t) - u_2(t)|_X \\ &\leq |u_2(t) - u(t)|_X + |u(t)|_X + |u_1(t)|_X. \end{aligned}$$

From the above argument there is a positive number δ such that if $|u_0 - u_{02}|_{L^{p_1}(0,\ell_{02})} < \delta$ and $\ell_0 < \ell_{02} < \ell_0 + \delta$, then $\{u_2, \ell_2\}$ is the global solution to SP and satisfies that

$$\begin{aligned} &\ell_2(t) \leq 2M_1 \quad \text{for } t \geq 0, \\ &|u(t) - u_2(t)|_X \leq 2M_1(1 + M_6 t)^{-\frac{2-r_0}{2r_0}} \quad \text{for } t \geq T_0. \end{aligned}$$

Therefore, if $|\hat{u}_0 - u_0|_{L^{p_1}(0,\hat{\ell}_0)} < \delta$ and $|\hat{\ell}_0 - \ell_0| < \delta$, then $\{\hat{u}, \hat{\ell}\}$ satisfies the required conditions. $\square$

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