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On radial and non-radial positive steady-states for Lotka-Volterra competition model on two dimensional annulus

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1 Introduction

Consider the following Lotka-Volterra competition model:

\[
\begin{aligned}
  u_t &= D \Delta u + u(a - u - bv) \quad \text{in } \Omega \times [0, \infty), \\
  v_t &= D \Delta v + v(d - v - cu) \quad \text{in } \Omega \times [0, \infty), \\
  \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times [0, \infty), \\
  u(x, 0) &= u_0 \geq 0, \\
  v(x, 0) &= v_0 \geq 0,
\end{aligned}
\]  

(1)

where \( \Omega = \{ x \in \mathbb{R}^2; R \leq |x| \leq R + 1 \} \) and \( D, a, b, c, d \) are positive constants. We will discuss the bistable case, i.e., \( bd - a > 0 \) and \( ac - d > 0 \). In this case, there exist four nonnegative constant solutions; \((0, 0), (a, 0), (0, d)\) and \((u^*, v^*) = \left( \frac{bd - a}{bc - 1}, \frac{ac - d}{bc - 1} \right)\).

They are represented in \((u,v)\)-plane in Fig. 1, where \((a, 0), (0, d)\) are stable steady-states and \((0, 0), (u^*, v^*)\) are unstable.
We are interested in the steady-state problem associated with (1);

\[
\begin{cases}
    D\Delta u + u(a - u - bv) = 0 \quad &\text{in } \Omega, \\
    D\Delta v + v(d - v - cu) = 0 \quad &\text{in } \Omega, \\
    \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad &\text{on } \partial \Omega, \\
    u \geq 0, v \geq 0, \quad &\text{in } \Omega,
\end{cases}
\]  

(2)

especially, the multiplicity of nonconstant solutions for (2).

We shall study radial solutions in section 1 and non-radial solutions in section 2.

2 Radial solutions

In this section we will study radial solutions for (2). Let \((u, v)\) be a radial solution for (2). We take polar coordinates such as \(|x| = R + s\), then \((u, v)\) satisfies

\[
\begin{cases}
    Du_{ss} + \frac{D}{R + s}u_s + u(a - u - bv) = 0 \quad &\text{in } (0, 1), \\
    Dv_{ss} + \frac{d}{R + s}v_s + v(d - v - cu) = 0 \quad &\text{in } (0, 1), \\
    u_s(0) = u_s(1) = v_s(0) = v_s(1) = 0, \quad &\text{in } (0, 1), \\
    u \geq 0, v \geq 0 \quad &\text{in } (0, 1).
\end{cases}
\]  

(3)

From now on we study (3). First we will make some definitions.

**Definition 1** Let \((u, v) = (u(s), v(s))\) be a solution for (3). Then \((u, v)\) is called an \(n\)-mode radial solution if and only if

\[
\#\{s \in (0, 1); u_s(s) = 0\} = \#\{s \in (0, 1); v_s(s) = 0\} = n - 1.
\]

Here \#A denote the number of elements of A.

![Fig. 2](image-url)
Before stating results we will prepare some notation. Let $\alpha^*$ be the positive solution of 
\[ \mu^2 + (u^* + v^*) \mu - (bc - 1)u^*v^* = 0. \]

Denote the eigenvalues of 
\[ -\frac{\partial^2}{\partial s^2} - \frac{1}{R + s} \frac{\partial}{\partial s} \]
(= $-\Delta$ in the space of radial functions) with Neumann zero boundary condition by $\lambda_n$, ($0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$). In discussing radial solutions we regard $D^{-1}$ as a parameter and consider branches of $n$–mode solutions for (3). Set 
\[ S := \{(u, v, \tau) \in C^1[0,1] \times C^1[0,1] \times \mathbb{R}^+; (u, v) \text{ is a nonconstant positive radial solution for (3) with } D^{-1} = \tau\}. \]
\[ S_n := \{(u, v, \tau) \in C^1[0,1] \times C^1[0,1] \times \mathbb{R}^+; (u, v) \text{ is an } n\text{-mode positive radial solution for (3) with } D^{-1} = \tau\}. \]

**Theorem 1**

(i) $S = \bigcup_{n=1}^{\infty} S_n$.

(ii) $S_n$ contains a connected component $B$ such that $(u^*, v^*, \frac{\lambda_n}{\alpha^*}) \in \overline{B}$ and $B$ is unbounded in $\{(u, v, \tau) \in C^1[0,1] \times C^1[0,1] \times \mathbb{R}^+; 0 \leq u \leq a, 0 \leq v \leq d\}$.

For the proof see Nakashima[11]

**Remark 1** Every positive solution for (2) has a priori estimate such that $0 \leq u \leq a$ and $0 \leq v \leq d$. Moreover, Theorem 1 implies that every solution for (3) becomes an $n$–mode solution for some $n$.

**Remark 2** When $D$ is large (or a parameter $D^{-1}$ is small), there exists no nonconstant solution for (3). (Conway-Hoff-Smoller[1].)

From the above remarks $B$ keeps the property of $n$–mode and continues up to $D^{-1} = \infty$. 

Fig. 3
3 non-radial solutions

In this section we will study the existence and multiplicity of non-radial positive solutions for (2). Here we restrict ourselves to the case $N=2$ and we show that the following result holds for (2).

For any $k \in \mathbb{N}$ there exists $R_0 = R_0(k) > 0$ such that if $R > R_0(k)$ then (2) has at least $k$ non-radial positive solutions, which are not equivalent with respect to rotation.

Such an existence result is well-studied for a single equation like

$$\Delta u + u^p = 0 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \quad u \geq 0,$$

when $1 < p < \infty$. We refer to ([2], [4], [6], [8], [9], [10]).

Taking polar coordinates $(s, \theta)$ such as

$$x = (R + s) \cos \theta, \quad y = (R + s) \sin \theta,$$

the steady-state problem for (2) is written as

$$\begin{align*}
Du_{ss} + \frac{D}{R+s}u_s + \frac{D}{(R+s)^2}u_{\theta\theta} + u(a - u - bv) &= 0 \quad \text{in } [0,1] \times [0,2\pi), \\
Dv_{ss} + \frac{D}{R+s}v_s + \frac{D}{(R+s)^2}v_{\theta\theta} + v(d - v - cu) &= 0 \quad \text{in } [0,1] \times [0,2\pi), \\
u_s(0, \theta) = u_s(1, \theta) = v_s(0, \theta) = v_s(1, \theta) &= 0, \\
u \geq 0, v \geq 0.
\end{align*}$$

(5)

From now on we will study nonconstant solutions for (5). Define a cone such as

$$C_k := \{(u, v) \in C^1([0,1] \times [0,2\pi)) \times C^1([0,1] \times [0,2\pi)); u \geq 0, v \geq 0, u_\theta \geq 0, v_\theta \leq 0 \text{ on } [0, \frac{\pi}{k}]; \text{ } u, v \text{ is symmetric with respect to } \theta = 0, \frac{\pi}{k}, \frac{2\pi}{k}, \ldots, \frac{(k-1)\pi}{k}\}.$$

\[\begin{array}{cc}
\text{C}_1 & \text{f}_g, 4 \\
0 & \pi & 2\pi
\end{array}\]
Here and henceforce we assume the following assumption (N).

(N) Every positive solution \((\phi_1, \phi_2)\) for

\[
\begin{align*}
D \frac{d^2}{dx^2} u + u(a - u - bv) &= 0 \quad \text{in } [0,1], \\
D \frac{d^2}{dx^2} v + v(d - v - cu) &= 0 \quad \text{in } [0,1], \\
u_x(0) = u_x(1) = v_x(0) = v_x(1) &= 0,
\end{align*}
\]

is nondegenerate, i.e. zero is not an eigenvalue for the linearized problem for (6) at \((\phi_1, \phi_2)\).

Remark 3 From the results of Kan-on, we know (N) holds if \(a = d, b = c\). Using the fixed point index theory on \(C_k\), we can get the following result.

**Theorem 2** Assume (N). For each \(k \in \mathbb{N}\) there exists \(R_k(D,a,b,c,d) > 0\), such that (2) has a non-radial solution in \(C_k\) for every \(R \geq R_k\).

Remark 4 Observe that \(C_k \cap C_l\) is identical with the set of radial solutions \(\Phi\) if \(k \neq l, k, l \in \mathbb{N}\). Theorem 2 implies that there exist two solutions \((u_k, v_k) \in C_k - \Phi\) and \((u_l, v_l) \in C_l - \Phi\) if we set \(R \geq \max\{R_k, R_l\}\). Since \((u_k, v_k)\) and \((u_l, v_l)\) are different with respect to rotation, we can get the multiplicity.

In this way we can find any number of nonradial positive solutions for (5) if \(R\) is sufficiently large.

## 4 Proof

In this section we show the outline of the proof of Theorem 2. We will find a non-radial solution in each \(C_k\). \((k = 1, 2, \cdots, m)\).

We will remark a priori estimate for the solutions for (5). Every solution \((u, v)\) has a priori estimate such that

\[0 \leq u \leq a \quad \text{and} \quad 0 \leq v \leq d.\]

To show this, let \(u\) has its maximum at \(x_0 \in \Omega\). We have \(a - u(x_0) - bv(x_0) = -\Delta u(x_0) \geq 0\). From the positivity of \(v\), \(u(x_0) \leq a - bv(x_0) \leq a\). We can show \(v \leq d\) in the same way.

Owing to Shaudar estimates for elliptic equations, there exist sufficiently large \(M_1, M_2\) such that \(\|u\| \leq M_1\) and \(\|v\| \leq M_2\) for every solution for (5), where \(\| \cdot \|\) denote the \(C^1\)-norm.

Set a bounded set

\[T_k := \{(u, v) \in C_k; \|u\| < M_1 + 1, \|v\| < M_2 + 1\}\]

Note that the solutions for (6) is not on the boundary of \(T_k\). Here we use the word "boundary" in the meaning of the relative topology with respect to \(C_k\).
We define a compact operator $A$ from $C^1([0,1] \times [0,2\pi]) \times C^1([0,1] \times [0,2\pi])$ into itself by

$$A \begin{pmatrix} u \\ v \end{pmatrix} = \left( -\frac{d^2}{dx^2} + p \right)^{-1} \begin{pmatrix} pu + D^{-1}(a - u - bv)u \\ pv + D^{-1}(d - v - cu)v \end{pmatrix}.$$  \hspace{1cm} (7)

where $p = \max\{D^{-1}(2M_1 + bM_2), D^{-1}(2M_2 + cM_1)\}$

Note that there is one to one correspondence between a fixed point of $A$ and a solution for (5). So $A$ has no fixed point on the boundary of $T_k$. Moreover the standard regularity theory of elliptic equations tells us that $A$ is completely continuous.

From the above fact and the following Lemma 1, we can define degree of $I - A$ on $C_k$, which is denoted by $\deg_{C_k}(I - A, \cdot)$. For the definition of $\deg_{C_k}(I - A, \cdot)$, see Dancer[3].

**Lemma 1** $A$ maps $T_k$ into $C_k$.

**Proof.** Let $(u, v) \in T_k$. First we will show the positivity of $A(u, v)$. Note that $u \geq 0$ and $v \geq 0$ and that $p$ is sufficiently large. Using the maximum principle in (7), we see that each element of $A(u, v)$ is positive.

Next we will show the monotonicity of $A(u, v)$. Defferentiating with respect to $\theta$,

$$\frac{d}{d\theta} \begin{pmatrix} A \begin{pmatrix} u \\ v \end{pmatrix} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{d^2}{dx^2} + p \end{pmatrix}^{-1} \begin{pmatrix} p + D^{-1}(a - 2u - bv) & -D^{-1}bu \\ -D^{-1}cv & p + D^{-1}(d - 2v - cu) \end{pmatrix} \begin{pmatrix} u_\theta \\ v_\theta \end{pmatrix}. $$

Since $u_\theta \geq 0$ and $v_\theta \leq 0$, it follows from the maximum principle that the first element of the above equality is nonnegative and the second is nonpositive. \hspace{1cm} $\square$

Now we can define $\deg_{C_k}(I - A, T_k)$.

**Lemma 2** $\deg_{C_k}(I - A, T_k) = 1$.

**Proof.** We use the homotopy invariance and excision property of the fixed point index theory. When we regard $D$ as a parameter, we sometimes write $A_D$ to emphasize $D$ dependence of $A$.

From the result of Conway-Hoff-Smoller[1], it is well known that there is a sufficiently large $D_0 = D_0(a, b, c, d, \Omega) > 0$ such that (5) has no nonconstant solution for $D > D_0(a, b, c, d, \Omega)$; so that the excision property gives

$$\deg_{C_k}(I - A_D, T_k) = \text{index}_{C_k}(A_D, (a, 0)) + \text{index}_{C_k}(A_D, (0, d)) + \text{index}_{C_k}(A_D, (u^*, v^*)) \quad \text{for} \quad D > D_0. $$  \hspace{1cm} (8)

To calculate the righthand side of (8), we will give the value of fixed point indices of constant solutions in the following lemma, whose proof is omitted.
\textbf{Lemma 3} (i) $\text{index}_{C_k}(A,(a,0)) = \text{index}_{C_k}(A,(0,d)) = 1$.

(ii) $\text{index}_{C_k}(A,(0,0)) = 0$.

(iii) $\text{index}_{C_k}(A,(u^*, v^*)) = -1$ if $D\lambda_1 > \alpha^*$.

We will continue the proof of Lemma 2. Since we can make $D_0$ sufficiently large, $D_0\lambda_1 > \alpha^*$ holds. So the righthand side of (8) is 1 from Lemma 3. Remember that $A_D$ has no fixed point on the boundary of $T_k$. Using the homotopy invariance property, it follows that

$$\deg_{C_k}(I - A_D, T_k) = 1 \text{ for every } D > 0.$$ 

\square

In the rest of the proof we restrict ourselves to the case when the parameter $R$ is sufficiently large.

Our strategy is as follows. In Lemma 4 we get all the positive radial solutions, and in Lemma 5 we study the fixed point indices of these radial solutions. Finally combining Lemmas 2, 4 and 5, we conclude the existence of a non-radial fixed point of $A$ in $T_k$ by contradiction.

Set $\epsilon := \frac{1}{R}$, then (5) is equivalent to

$$\begin{cases}
Du_{ss} + \frac{D\epsilon}{1 + \epsilon s}u_s + \frac{D^2 \epsilon^2}{(1 + \epsilon s)^2} u_{\theta\theta} + u(a - u - bv) = 0 & \text{in } [0,1] \times [0,2\pi), \\
Dv_{ss} + \frac{D\epsilon}{1 + \epsilon s}v_s + \frac{D^2 \epsilon^2}{(1 + \epsilon s)^2} v_{\theta\theta} + v(d - v - cu) = 0 & \text{in } [0,1] \times [0,2\pi), \\
u_s(0, \theta) = u_s(1, \theta) = v_s(0, \theta) = v_s(1, \theta) = 0, \\
u \geq 0, v \geq 0. & \text{in } [0,1] \times [0,2\pi).
\end{cases}$$

Note that (9) with $\epsilon = 0$ is equivalent to the one dimensional system (6). Denote by $\{(u_0^i, v_0^i)\}_{i=1, 2, \ldots, m}$ all the nonconstant solutions for (6). Observe that the number of such solutions is finite because of the nondigeneracy assumption.

\textbf{Lemma 4} Assume (N). For small $\epsilon > 0$, (9) has a nonconstant radial solution $(u_\epsilon^i, v_\epsilon^i)$ near $(u_0^i, v_0^i)$ for $i = 1, 2, \ldots, m$.

Moreover, if $(u, v)$ is a nonconstant radial solution, then $(u, v) = (u_\epsilon^i, v_\epsilon^i)$ for some $i \in \{1, 2, \ldots, m\}$.

The proof can be accomplished with use of the implicit function theorem.

We can calculate the fixed point index of each radial solution using Dancer's index formula.
Lemma 5 Let \((u, v)\) be a positive radial solution for (7) including \((u^*, v^*)\). If the eigenvalue problem
\[
\begin{pmatrix}
\Delta \bar{u} \\
\Delta \bar{v}
\end{pmatrix} + \frac{1}{D} \begin{pmatrix}
a - 2u - bv, \\
-cv
d - 2v - cu
\end{pmatrix} \begin{pmatrix}
\bar{u} \\
\bar{v}
\end{pmatrix} = \lambda \begin{pmatrix}
\bar{u} \\
\bar{v}
\end{pmatrix},
\]
where \(\Delta = \frac{\partial^2}{\partial s^2} + \frac{\epsilon}{1 + \epsilon} \frac{\partial}{\partial s} + \frac{\epsilon^2}{1 + \epsilon^2} \frac{\partial^2}{\partial \theta^2},\)
has a positive real eigenvalue, then
\[
\text{index}_{C_k}(A, (u, v)) = 0.
\]

Lemma 5 is useful to get the fixed point index of \((u^i_\epsilon, v^i_\epsilon)\); we study the eigenvalue problem (10) with \((u, v) = (u^i_\epsilon, v^i_\epsilon)\).

First, we consider the case \(\epsilon = 0;\)
\[
\begin{pmatrix}
\bar{u}_{xx} \\
\bar{v}_{xx}
\end{pmatrix} + \frac{1}{d} \begin{pmatrix}
a - 2u^0_0 - bv^0_0, \\
-cv^0_i 
d - 2v^0_0 - cu^0_i
\end{pmatrix} \begin{pmatrix}
\bar{u} \\
\bar{v}
\end{pmatrix} = \lambda \begin{pmatrix}
\bar{u} \\
\bar{v}
\end{pmatrix},
\]
where \(\bar{u}_x(0) = \bar{u}_x(1) = \bar{v}_x(0) = \bar{v}_x(1) = 0.\)

For \((u, v) = (u^*, v^*)\), (11) has a positive real eigenvalue. (Recall that we are discussing the bistable case.) For nonconstant solutions, we will introduce the result of Kishimoto-Weinberger[7].

Theorem 3 (Kishimoto-Weinberger)
For any nonconstant solution \((u^0_0, v^0_0)\), (11) has a positive real simple eigenvalue \(\lambda_0\).

Taking account of these results, the implicit function theorem shows the following lemma.

Lemma 6 For sufficiently small \(\epsilon > 0\), (10) with \((u, v) = (u^i_\epsilon, v^i_\epsilon)\) has a simple real eigenvalue \(\lambda_\epsilon\) near \(\lambda_0\). Therefore, \(\lambda_\epsilon\) is also positive since \(\lambda_0\) is positive.

We are ready to complete the proof of Theorem 2. It follows from Lemmas 5 and 6 that
\[
\text{index}_{C_k}(A, (u^*, v^*)) = 0,
\]
and
\[
\text{index}_{C_k}(A, (u^i_\epsilon, v^i_\epsilon)) = 0 \quad \text{for every} \quad i = 1, 2, \ldots, m.
\]

Assume that there exists no non-radial solution. From the excision property,
\[
\deg_{C_k}(I - A, T_k) = \text{index}_{C_k}(A, (a, 0)) + \text{index}_{C_k}(A, (0, d)) + \\
\text{index}_{C_k}(A, (0, 0)) + \text{index}_{C_k}(A, (u^*, v^*)) + \sum_{i=1}^{m} \text{index}_{C_k}(A, u^i_\epsilon, v^i_\epsilon).
\]

We see from (12),(13) and (i), (ii) in Lemma 3 that the righthand side is equal to 2. This contradicts to Lemma 2. Thus we can obtain a non-radial fixed point in \(C_k\).
References


