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The Linearized Boltzmann Equation with an External Force

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§1 INTRODUCTION

The nonlinear Boltzmann equation with an external force describes the evolution of the density of rarefied gas acted upon by the external force. We admit that the force depends on the space variable and the velocity variable. We assume that the gas particles are confined in a bounded domain $\Omega\subset\mathbb{R}^3$ by being reflected perfectly from the boundary $\partial\Omega$. The equation has the form,

$$\partial f/\partial t+\Lambda f=Q(f,f), \qquad \text{(NBE)}$$

where we denote by $f=f(t,x,\xi)$ the unknown function which represents the density of gas particles that have a velocity $\xi\in\mathbb{R}^3$ at time $t\geqq 0$ and at a point $x\in\Omega$. $Q=Q(\cdot,\cdot)$ denotes the nonlinear collision operator[1-2], and Λ is a differential operator defined as follows:

$$\Lambda\equiv\xi\cdot\nabla_x+\mathbb{F}\cdot\nabla_\xi, \qquad (1.1)$$

where $\mathbb{F}=\mathbb{F}(x,\xi)$ denotes the external force. We assume that $\mathbb{F}=\mathbb{F}(x,\xi)$ has the form,

$$\mathbb{F}(x,\xi)\equiv-\nabla\phi(x)+\mathbf{b}(x,\xi), \qquad (1.2)$$

where $\phi=\phi(x)$ is a sufficiently smooth, real-valued function of $x\in\Omega$, and $\mathbf{b}=\mathbf{b}(x,\xi)$ is a sufficiently smooth, 3-dimensional-real-vector-valued function of $(x,\xi)\in\Omega\times\mathbb{R}^3$ such that

$$\xi\cdot \mathbf{b}(x,\xi) = 0 \text{ for each } (x,\xi)\in\Omega\times\mathbb{R}^3. \tag{1.3}$$

Under the assumption of cut-off hard potentials in the sense of Grad[1-2], we linearize (NBE) around the equilibrium state $M\equiv e^{-\phi(x)-|\xi|^2/2}$. Substituting $f = M+M^{1/2}u$ in (NBE), applying (1.2-3), and dropping the nonlinear term, we obtain the *linearized Boltzmann equation with an external force,*

$$\partial u/\partial t = Bu, \tag{LBE}$$

$$B \equiv -\Lambda + e^{-\phi}(-\nu+K), \tag{1.4}$$

where $\nu = \nu(\xi)$ is a multiplication operator, and K is an integral operator. ν and K act only on the velocity variable ξ, and have the following properties[1-2]:

Lemma 1.1. (i) There exist positive constants ν_j, j=0,1, such that $\nu_0\leqq \nu(\xi)\leqq \nu_1(1+|\xi|)$ for each $\xi\in\mathbb{R}^3$.

(ii) K is a self-adjoint compact operator in $L^2(\mathbb{R}^3_\xi)$.

(iii) $(-\nu+K)$ is a nonpositive operator in $L^2(\mathbb{R}^3_\xi)$ whose null space is spanned by $\xi_j e^{-|\xi|^2/4}$, j = 1,2,3, $e^{-|\xi|^2/4}$, and $|\xi|^2 e^{-|\xi|^2/4}$, where we denote the j-th component of ξ by ξ_j, j = 1,2,3, i.e., $\xi = (\xi_1, \xi_2, \xi_3)$.

The purpose of the present paper is to study decay of solutions to the mixed problem for (LBE) with the perfectly reflective boundary condition. In order to study this subject, we need to investigate the structure of the spectrum of the operator B. We can prolong solutions of the following system of ordinary differential equations in time by the law of perfect reflection:

$$dx/dt = \xi,\quad d\xi/dt = \mathbb{F}(x,\xi), \tag{SODE}$$

and the continued characteristic curves of Λ is described by these prolonged solutions (cf. p. 391[3]). Noting that spectral properties of Λ are deeply connected to the continued characteristic curves of Λ and that the operator B is strongly influenced by Λ, we can reasonably conclude that spectral properties of B must be closely related to the behavior of the prolonged solutions of (SODE). Therefore we need to investigate the behavior of the prolonged solutions of (SODE) (cf. § 1[4], pp. 742-746[5], pp. 754-756[5], and p. 1273[6]).

However it must be noted that the behavior of the prolonged solutions of (SODE) is very complicated; it is difficult to observe the solutions globally in time. Hence it is advisable to simplify the behavior of the prolonged solutions by imposing some assumptions. We note that the complexity of the behavior of the prolonged solutions depends largely on geometry of the boundary surface $\partial\Omega$. For example, as the geometry of $\partial\Omega$ becomes more complex, the solutions of (SODE) are prolonged by being reflected from $\partial\Omega$ in a more complicated manner. Hence the behavior of the prolonged solutions is also more complex. Conversely, as the geometry of $\partial\Omega$ becomes simpler, the behavior of the prolonged solutions is also simplified. For these reasons we will impose the following assumption, in addition to the assumption that Ω is bounded:

Assumption Ω. $\partial\Omega$ is a 2-dimensional piecewise linear manifold.

We note that the solutions of (SODE) considerably changes at the time when the particle $x = x(t)$ collides with $\partial\Omega$ (see § 7 for the details). In order to prevent the characteristic curves of Λ from being very largely influenced by the external force when the characteristic curves collide with $\partial\Omega$, we will impose the following assumption:

Assumption $\mathbb{F}$. For almost all $(X,\Xi)\in\mathbf{F}_+$,

$$\{(\gamma_+\mathbb{F}(\cdot,\cdot))(X,\Xi)+(\gamma_-\mathbb{F}(\cdot,\cdot))(X,\Xi-2(\mathbf{n}(X)\cdot\Xi)\mathbf{n}(X))\}\perp\mathbf{n}(X),$$

$$\{(\gamma_+\mathbb{F}(\cdot,\cdot))(X,\Xi) - (\gamma_-\mathbb{F}(\cdot,\cdot))(X,\Xi - 2(\mathbf{n}(X)\cdot\Xi)\mathbf{n}(X))\} // \mathbf{n}(X).$$

Here we denote by $\gamma_\pm$ the trace operators along the characteristic curves of Λ onto

$$\mathsf{F}_\pm \equiv \{(X,\Xi) \in F(\partial\Omega)\times\mathbb{R}^3;\ \pm\mathbf{n}(X)\cdot\Xi > 0\}. \tag{1.5}$$

We denote by $\mathbf{n} = \mathbf{n}(x)$ the outer unit normal of $\partial\Omega$ at $x \in F(\partial\Omega) \equiv \partial\Omega \setminus E(\partial\Omega)$, where $E(\partial\Omega) \equiv \{x \in \partial\Omega;\ x$ is contained in an edge of $\partial\Omega\}$. By virtue of Assumption $\mathbb{F}$ and Assumption Ω, we can simplify the behavior of the prolonged solutions (in Remark 7.5 we will fully discuss the roles of these assumptions). The main result of this paper is Main Theorem in § 3, which is as follows: *the semigroup generated by B decays exponentially in time.*

Under the spatial periodicity condition, we have already investigated decay of solutions of (LBE) in the case where $\mathbf{b}(x,\xi) \equiv 0$[4]. Hence, it seems to be promising to attempt to apply the method in [4] also to the problem of this paper. However, if we try to do so, then we immediately encounter the difficulty which is caused not only by the fact that the external force depends on the velocity variable but also by the fact that there is a possibility that some continued characteristic curves of Λ tend to follow $\partial\Omega$ (see p. 391[3]). The possibility presents a great difficulty in attacking our problem (we will discuss this difficulty in § 7 fully).

§ 2 PRELIMINARIES

(1) Assumptions. We impose the following two assumptions on the external force in addition to Assumption $\mathbb{F}$ (see (1.2)):

Assumption ϕ. $\sup_{x\in\Omega} |\partial^2 \phi(x)/\partial x_i \partial x_j| < +\infty$, $i,j = 1,2,3$, where we denote by x_i the i-th component of x, $i = 1,2,3$, i.e., $x = (x_1,x_2,x_3)$.

Assumption $\mathbf{b}$. (i) $\sup_{x\in\Omega,|\xi|\le r} |\partial \mathbf{b}(x,\xi)/\partial x_i|$, $\sup_{x\in\Omega,|\xi|\le r} |\partial \mathbf{b}(x,\xi)/\partial \xi_i| < +\infty$, $i = 1,2,3$, for each $r \geqq 1$, where we denote by ξ_i the i-th component of ξ, $i =$

1,2,3.

(ii) $\nabla_\xi \cdot \mathbf{b}(x, \xi) = 0$ for each $(x, \xi) \in \Omega \times \mathbb{R}^3$.

(2) **Function spaces.** By $\mathbb{B}(X,Y)$ ($\mathbb{C}(X,Y)$, respectively) we denote the set of all bounded (compact, respectively) linear operators from a Banach space X to a Banach space Y. For simplicity, we write $\mathbb{B}(X)$ and $\mathbb{C}(X)$ as $\mathbb{B}(X,X)$ and $\mathbb{C}(X,X)$ respectively. By E_α, $\alpha \geq 0$, we denote a Hilbert space of complex-valued functions of $(x, \xi) \in \Omega \times \mathbb{R}^3$ with the following inner product (recall Remark 2.1, (ii)):

$$(u,v)_\alpha \equiv \int_{\Omega \times \mathbb{R}^3} u(x, \xi)\overline{v(x, \xi)}(1+E(x, \xi))^\alpha dxd\xi, \qquad (2.1)$$

where $E(x, \xi) \equiv \phi(x) + |\xi|^2/2$. Define $\|u\|_\alpha \equiv ((u,u)_\alpha)^{1/2}$. Write $|||\cdot|||_\alpha$ as the norm of operators of $\mathbb{B}(E_\alpha)$. Write $\|\cdot\|$, $|||\cdot|||$ as $\|\cdot\|_0$, $|||\cdot|||_0$, respectively for simplicity.

(3) **The domains of operators.** We denote the domain of an operator L by D(L). Let us define the domain of Λ (see (1.1)) as follows: $D(\Lambda) \equiv \{u = u(x, \xi) \in E_\alpha;\ \Lambda u \in E_\alpha$, and $u = u(x, \xi)$ satisfies the perfectly reflective boundary condition,

$$(\gamma_+ u(\cdot,\cdot))(X,\Xi) = (\gamma_- u(\cdot,\cdot))(X,\Xi - 2(\mathbf{n}(X)\cdot\Xi)\mathbf{n}(X)), \qquad \text{(PRBC)}$$

for a.e. $(X, \Xi) \in \mathbf{F}_+\}$. See (1.5) for $\mathbf{n}(X)$ and $\mathbf{F}_\pm$. We define the domain of the operator,

$$A \equiv -\Lambda + e^{-\phi}(-\nu), \qquad (2.2)$$

as follows: $D(A) \equiv \{u = u(x, \xi) \in E_\alpha;\ Au \in E_\alpha$, and $u = u(x, \xi)$ satisfies (PRBC) for a.e. $(X,\Xi) \in \mathbf{F}_+\}$. Applying Remark 2.1, (ii), we deduce that

$$\mathbf{K} \equiv e^{-\phi}K \in \mathbb{B}(E_\alpha, E_{\alpha+1}), \text{ for each } \alpha \geq 0, \qquad (2.3)$$

in the same way as Lemma 2.1, (iv)[4]. Hence, we can define $D(\mathrm{B}) \equiv D(\mathrm{A})$ (see (1.4)).

(4) **The purely imaginary point spectrum** of B. Noting that Ω is bounded, and applying Assumption Ω, we see that Ω has no axis of symmetry. Making use of this fact, and performing calculations similar to those in Theorem 4.1[6], we can obtain the following lemma (see (2.1) for $\mathrm{E}(x, \xi)$):

Lemma 2.2. The intersection of $\{\mu \in \mathbb{C};\ \mathrm{Re}\,\mu \geqq 0\}$ and the point spectrum of B is equal to $\{0\}$. The null space of B is spanned by $e^{-\mathrm{E}(x,\xi)/2}$ and $\mathrm{E}(x, \xi)e^{-\mathrm{E}(x,\xi)/2}$.

We denote by $\mathbf{P}$ the projection operator (in E_0) upon the null space of B. It follows from the lemma above and Remark 2.1, (ii) that

$$\mathbf{P} \in \mathbb{C}(\mathrm{E}_\alpha, \mathrm{E}_\beta) \text{ for each } \alpha, \beta \geqq 0. \tag{2.4}$$

§ 3 MAIN THEOREM

We denote by $\mathrm{E}_{\alpha,\perp}$ the set of all functions of E_α which are perpendicular (in E_0) to the null space of B. In what follows throughout the paper, we denote some positive constants by c, and we use the letter c as a *generic* constant replacing any other constants by c. The following theorem is the main result of this paper:

Main Theorem. For each $\alpha \geqq 0$, the operator B generates a strongly continuous semigroup in E_α, which satisfies that, for each $u_0 \in \mathrm{E}_{\alpha,\perp}$,

$$e^{tB}u_0 \in \mathrm{E}_{\alpha,\perp},\quad \| e^{tB}u_0 \|_\alpha \leqq c\| u_0 \|_\alpha \exp(-c_{3.0}t), \text{ for each } t \geqq 0,$$

where $c_{3.0}$ is a positive constant independent of t and u_0.

In order to prove the Main Theorem, we will apply the following lemma, which will be proved in § 5 (see (2.2) for A):

Lemma 3.1. (i) For any $\alpha \geqq 0$, the operator A generates a strongly continuous semigroup in E_α, which satisfies (see Remark 2.1, (ii), for $c_{2.1}$),

$$|||e^{tA}|||_\alpha \leqq \exp(-c_{2.1}t), \text{ for each } t \geqq 0.$$

(ii) Let α, $c_{3.1}$ and C be constants such that $\alpha \geqq 0$, $c_{2.1} > c_{3.1} > 0$, and $C > 0$. If $f = f(t)$ is a continuous function from $[0, +\infty)_t$ to E_α such that (see (2.3) for $\mathbf{K}$)

$$\| f(t) \|_\alpha \leqq C\exp(-c_{3.1}t), \text{ for each } t \geqq 0,$$

then

$$\left\| \int_0^t e^{(t-s)A}\mathbf{K} f(s)ds \right\|_{\alpha+1} \leqq cC\exp(-c_{3.1}t), \text{ for each } t \geqq 0.$$

(iii) Write $\mu = \gamma + i\delta$, $\gamma, \delta \in \mathbb{R}$. If $\beta \equiv \gamma + c_{2.1} > 0$ and $f \in E_0$, then,

$$\int_{-\infty}^{+\infty} \| (\mu - A)^{-1} f \|^2 d\delta, \int_{-\infty}^{+\infty} \| (\mu - A^*)^{-1} f \|^2 d\delta \leqq c\| f \|^2/\beta.$$

In addition to the lemma above, we need the following key lemma:

Lemma 3.2. $\mathbb{L} = \mathbb{L}(\mu) \equiv (\mathbf{K} - \mathbf{P})(\mu - A)^{-1}$ is an analytic operator-valued function of $\mu \in \mathbb{D} \equiv \{\mu \in \mathbb{C};\ \mathrm{Re}\,\mu \geqq -c_{2.1}/2\}$, and satisfies that $\mathbb{L}^4(\mu) \in \mathbb{C}(E_0)$ for each $\mu \in \mathbb{D}$ and that $|||\mathbb{L}^4(\mu)||| \to 0$ as $|\mu| \to +\infty$, $\mu \in \mathbb{D}$.

Proof of the Main Theorem. It follows from Lemma 3.1, (i), that $(\mu - A)^{-1}$ is an analytic operator-valued function of $\mu \in \mathbb{D}$. Hence we can set the resolvent equation,

$$(\mu - \mathbb{B})^{-1} = (\mu - A)^{-1} + (\mu - A)^{-1}(1 - \mathbb{L}(\mu))^{-1}\mathbb{L}(\mu), \qquad (3.1)$$

where $\mu \in \mathbb{D}$ and $\mathbb{B} \equiv B - \mathbf{P}$. We consider the operator $\mathbb{B}$ in place of B, in order to remove the null space of B (cf. p. 1833[4]). By (2.3-4) and Lemma 3.1, (i), we easily see that $\mathbb{B}$ generates a strongly continuous semigroup in E_α, which is represented in terms of the inverse Laplace transformation of (3.1). Applying Lemmas 3.1-2 and Lemma 2.2 in the same way as pp. 1833-1834[4], we obtain the theorem.

Remark 3.3. (i) In pp. 1833-1834[4] we do not need to directly apply the spatial periodicity condition. Hence, without the aid of the condition, we can apply Lemmas 3.1-2 in the proof above. For the same reason as § 6[4], we consider the 4-th power $\mathbb{L}^4(\mu)$ in Lemma 3.2. Cf. pp. 433-434[3].

(ii) We can decompose $\mathbb{L}(\mu)$ as follows: $\mathbb{L}(\mu) = L_{\mathbf{K}}(\mu) + L_{\mathbf{P}}(\mu)$, where $L_{\mathbf{K}}(\mu) \equiv \mathbf{K}(\mu - A)^{-1}$ and $L_{\mathbf{P}}(\mu) \equiv -\mathbf{P}(\mu - A)^{-1}$. We can easily derive Lemma 3.2 from the following (3.2-5):

$$L_{\mathbf{K}}^4(\mu) \in \mathbb{C}(E_0) \text{ for each } \mu \in \mathbb{D}, \tag{3.2}$$

$$|||L_{\mathbf{K}}^4(\mu)||| \rightarrow 0 \text{ as } |\mu| \rightarrow +\infty, \ \mu \in \mathbb{D}, \tag{3.3}$$

$$L_{\mathbf{P}}(\mu) \in \mathbb{C}(E_0) \text{ for each } \mu \in \mathbb{D}, \tag{3.4}$$

$$|||L_{\mathbf{P}}(\mu)||| \rightarrow 0 \text{ as } |\mu| \rightarrow +\infty, \ \mu \in \mathbb{D}. \tag{3.5}$$

Making use of (2.4) and Lemma 2.2, and performing calculations similar to, but much easier than, those in proving (3.2-3), we can obtain (3.4-5). Hence we will prove (3.2-3) only. (3.2-3) will be proved in § 6-9.

§ 4. THE SOLUTIONS TO (SODE)

By (CP) we denote the Cauchy problem for (SODE) with the initial data,

$$(x, \xi)(0) = (X, \Xi) \in (\Omega \times \mathbb{R}^3) \cup \mathbf{F}_-. \tag{4.1}$$

See (1.5) for $\mathbf{F}_-$. Recall Remark 2.1, (iv). In the same way as pp. 1284-1285[6], we can prolong the solution of (CP) by the law of perfect reflection (cf. (PRBC)), i.e., by

$$(x(s+0), \xi(s+0)) \\ = (x(s-0), \xi(s-0)-2(\mathbf{n}(x(s-0))\cdot\xi(s-0))\mathbf{n}(x(s-0))), \quad (4.2)$$

where $x(s\pm 0)\in F(\partial\Omega)$. See § 1 for $F(\partial\Omega)$. We denote the prolonged solution of (CP) by

$$(x, \xi) = (x(t,X,\Xi), \xi(t,X,\Xi)). \quad (4.3)$$

We can decompose $\partial\Omega\times\mathbb{R}^3$ into four disjoint subsets as follows: $\partial\Omega\times\mathbb{R}^3 = \mathbf{E}\cup\mathbf{F}_0\cup\mathbf{F}_+\cup\mathbf{F}_-$, where $\mathbf{E}\equiv E(\partial\Omega)\times\mathbb{R}^3$ and $\mathbf{F}_0 \equiv \{(x,\xi)\in F(\partial\Omega)\times\mathbb{R}^3;\ \mathbf{n}(x)\cdot\xi = 0\}$. For $E(\partial\Omega)$, see § 1. If $(x,\xi) = (x(t,X,\Xi), \xi(t,X,\Xi))$ does not go into $\mathbf{E}\cup\mathbf{F}_0$, and if $x = x(t,X,\Xi)$ does not collide with $F(\partial\Omega)$ an infinite number of times in a finite time interval, then we can prolong the solution of (CP) globally in time $t\in\mathbb{R}$ in the same way as pp. 1284-1285[6].

§ 5. THE OPERATOR A

Proof of Lemma 3.1. Applying the conservation law of energy (see (2.1)),

$$E(X,\Xi) = E(x(-t,X,\Xi), \xi(-t,X,\Xi)), \quad (5.1)$$

and Lemma 4.2, (ii), we easily see that A (see (2.2)) generates a strongly continuous semigroup in E_α for each $\alpha\geqq 0$. The semigroup e^{tA} has the form,

$$(e^{tA}f(\cdot,\cdot))(X,\Xi) = f(x(-t,X,\Xi), \xi(-t,X,\Xi))e(t,X,\Xi), \quad (5.2)$$

where $e(t,X,\Xi) \equiv \exp(-\int_0^t e^{-\phi(x(-s,X,\Xi))}\nu(\xi(-s,X,\Xi))ds)$. Remark 2.1, (ii), implies that

$$e(t,X,\Xi) \leqq \exp(-c_{2.1}t), \quad \text{for each } t \geqq 0. \tag{5.3}$$

Applying (5.3), (2.3), Lemma 4.2, (ii), and (5.1), we can obtain Lemma 3.1 in the same way as Lemma 3.1[4] and Lemma 3.3[4].

Restricting the domain of integration of the Laplace transformation of (5.2) within a Lebesgue measurable set $M \subseteq [0,+\infty)$, we define the following operator:

$$(\mathbb{R}(\mu,M)f(\cdot,\cdot))(X,\Xi) \equiv \int_{t\in M} R(\mu,t,X,\Xi)f(x(-t,X,\Xi),\xi(-t,X,\Xi))dt, \tag{5.4}$$

where $R(\mu,t,X,\Xi) \equiv e(t,X,\Xi)\exp(-\mu t)$, $\mu \in \mathbb{D}$. See Lemma 3.2 for $\mathbb{D}$. We can obtain the following lemma in the same way as Lemma 3.2, (i)[4]:

Lemma 5.1. If $\beta \equiv \mathrm{Re}\,\mu + c_{2.1} > 0$ and $M \subseteq [0,+\infty)$, then

$$|||\mathbb{R}(\mu,M)||| \leqq \int_{t\in M} e^{-\beta t}dt.$$

§6. DISCUSSION ON (3.2-3)

We will seek estimates which imply (3.2-3). Consider operators of the following form:

$$\mathbb{G}(\mu,T) \equiv \prod_{j=1}^{4}\{e^{-\phi(x)}k_j\mathbb{R}(\mu,[0,T])\}, \tag{6.1}$$

where $\mu \in \mathbb{D}$, $1 \leqq T < +\infty$, and $k_j \in Q(\mathbb{R}^3_\xi)$, $j=1,\dots,4$. See Lemma 3.2 for $\mathbb{D}$. Here we denote by $Q(\mathbb{R}^3_\xi)$ the set of all the one-rank operators of the form, $(ku(\cdot,\cdot))(x,\xi) = (u(x,\cdot),f(\cdot))g(\xi)$, where the brackets $(\cdot,\cdot)$ denote the inner product in $L^2(\mathbb{R}^3_\xi)$. $f=f(\xi)$ and $g=g(\xi)$ are infinitely differentiable functions of $\xi \in \mathbb{R}^3$ which have compact supports. By $\prod_{j=1}^{m} A_j$ we denote the product $A_m A_{m-1} \cdots A_2 A_1$ for operators A_j, $j = 1,\dots,m$. Making use of Lemma 1.1, (ii), and

Theorem VI.13[16], we deduce that the operator K can be approximated in $\mathbb{B}(E_0)$ with a finite sum of operators of $Q(\mathbb{R}^3_\xi)$. Applying this result and Lemma 5.1, we can derive (3.2-3) from the following (G.1-2):

$$\mathbb{G}(\mu,T) \in \mathbb{C}(E_0) \text{ for each } \mu \in \mathbb{D} \text{ and } T \in [1,+\infty), \qquad \text{(G.1)}$$

$$|||\mathbb{G}(\mu,T)||| \to 0 \text{ as } |\mu| \to +\infty,\ \mu \in \mathbb{D}, \text{ for each } T \in [1,+\infty). \qquad \text{(G.2)}$$

Hence, we have only to show (G.1-2), which will be proved in §9.

We write $(x_4, \xi_4) \in \Omega \times \mathbb{R}^3$ as the variable of $\mathbb{G}(\mu,T)u$, i.e., we write $\mathbb{G}(\mu,T)u = (\mathbb{G}(\mu,T)u(\cdot,\cdot))(x_4, \xi_4)$. In the same way as p. 1837[4], we can extract the integration kernel of $\mathbb{G}(\mu,T)$ as follows:

$$(\mathbb{G}(\mu,T)u(\cdot,\cdot))(x_4,\xi_4) = \int_{0\leq t_j\leq T, |\eta_j|\leq r,\ j=1,\dots,4} Gu(x_0,\xi_0)dtd\eta, \qquad (6.2)$$

$$G=G(\mu,x_4,\xi_4;\eta_4,t_4,\eta_3,t_3,\eta_2,t_2,\eta_1,t_1)$$
$$\equiv \prod_{j=1}^{4}\{e^{-\phi(x_j)}k_j(\xi_j,\eta_j)R(\mu,t_j,x_j,\eta_j)\}, \qquad (6.3)$$

$$x_j \equiv x(-t_{j+1},x_{j+1},\eta_{j+1}),\ \xi_j \equiv \xi(-t_{j+1},x_{j+1},\eta_{j+1}),\ j=0,\dots,3, \qquad (6.4)$$

where $dt \equiv dt_1\cdots dt_4$, and $d\eta \equiv d\eta_1\cdots d\eta_4$. Recall (4.3). By $k_j(\xi_j, \eta_j)$ we denote the integration kernel of $k_j \in Q(\mathbb{R}^3_\xi)$, $j=1,\dots,4$, i.e., $(k_j f(\cdot,\cdot))(x_j,\xi_j) = \int k_j(\xi_j,\eta_j)f(x_j,\eta_j)d\eta_j$, $j=1,\dots,4$. $r>0$ is a constant so large that supp $k_j(\cdot,\cdot)$, $j = 1,\dots,4$, can be contained in $\{\xi; |\xi|\leq r\}\times\{\eta; |\eta|\leq r\}$.

§7. ESTIMATES FOR J = J(t,X,Ξ)

The purpose of this section is to obtain estimates for $J=J(t,X,\Xi)$. See (4.4). By x_j, ξ_j, X_j, Ξ_j we denote the j-th components of x, ξ, X, Ξ, respectively, $j=1,2,3$. We denote the (i,j) component of $J = J(t,X,\Xi)$ by $m_{ij} = m_{ij}(t,X,\Xi)$, i, j = 1,...,6, i.e., if $1\leq i, j \leq 3$, then $m_{ij}(t,X,\Xi) \equiv \partial x_i(-t,X,\Xi)/\partial X_j$. If $1 \leq i \leq 3$ and $4 \leq j \leq 6$, then $m_{ij}(t,X,\Xi) \equiv \partial x_i(-t,X,\Xi)/\partial \Xi_{j-3}$. If $4 \leq i \leq 6$ and 1

$\leqq j \leqq 3$, then $m_{ij}(t,X,\Xi) \equiv \partial \xi_{i-3}(-t,X,\Xi)/\partial X_j$. If $4 \leqq i, j \leqq 6$, then $m_{ij}(t,X,\Xi) \equiv \partial \xi_{i-3}(-t,X,\Xi)/\partial \Xi_{j-3}$. We denote the i-th row vectors of $J = J(t,X,\Xi)$ by $J_i = J_i(t,X,\Xi)$, i.e., we define $J_i = J_i(t,X,\Xi) \equiv (m_{i1},\ldots,m_{i6})$, $i = 1,\ldots,6$.

Let b_j, $j = 1,\ldots,N$, be linearly independent vectors in $\mathbb{R}^n$, $n,N \in \mathbb{N}$. We orthogonalize these vectors, i.e., we define $b_{j,\perp}$, $j = 1,\ldots,N$, as follows (we do not normalize them):

$$b_{1,\perp} \equiv b_1, \quad b_{j+1,\perp} \equiv b_{j+1} - \sum_{k=1}^{j} (b_{j+1} \cdot b_{k,\perp}) b_{k,\perp} / |b_{k,\perp}|^2, \quad j = 1,\ldots,N-1.$$

Lemma 7.1. Let $1 \leqq r,T < +\infty$. If $t \in [0,T]$ and $(X,\Xi) \in D(r)$ satisfy (4.6-7), then, for $i = 1,\ldots,6$,

$$6^{-5/2}\exp(-5c_{4.2}t) \leqq |J_{i,\perp}(t,X,\Xi)| \leqq |J_i(t,X,\Xi)| \leqq 6^{1/2}\exp(c_{4.2}t).$$

Remark 7.2. (i) It does not follow from Lemma 4.1 that the integer m of (II) is essentially bounded in $\Omega \times \mathbb{R}^3$. In fact, there is a possibility that this number goes to infinity as (X,Ξ) moves, i.e., that the trajectory of $x = x(-t,X,\Xi)$ tends to follow the boundary surface $\partial\Omega$. See p. 391[3]. Hence we need to prove that the inequalities of Lemma 7.1 hold uniformly for m. Therefore we have to carefully inspect the change of $J = J(t,X,\Xi)$ before and after $x = x(-t,X,\Xi)$ collides with $F(\partial\Omega)$, i.e., the difference between $J(t^n(X,\Xi)+0,X,\Xi)$ and $J(t^n(X,\Xi)-0,X,\Xi)$, $n = 1,\ldots,m$.

(ii) Considering the methods employed in prolonging solutions of (CP) (see § 4 and pp. 1284-1285[6]), we easily see that if t = t and t = s satisfy (4.7), then

$$x(-t,X,\Xi) = x(-\tau, x(-s,X,\Xi), \xi(-s,X,\Xi)), \qquad (7.1.1)$$

$$\xi(-t,X,\Xi) = \xi(-\tau, x(-s,X,\Xi), \xi(-s,X,\Xi)), \qquad (7.1.2)$$

where $\tau \equiv t-s$. Therefore we can regard $(x(-t,X,\Xi), \xi(-t,X,\Xi))$ as function of τ and $(x(-s,X,\Xi), \xi(-s,X,\Xi))$. Hence we can define the following

Jacobian matrix:

$$J(t,s,X,\Xi) \equiv \partial(x(-t,X,\Xi), \xi(-t,X,\Xi))/\partial(x(-s,X,\Xi), \xi(-s,X,\Xi)). \quad (7.2)$$

From (4.4) and (7.2), we easily obtain $J(t,X,\Xi) = J(t,s,X,\Xi)J(s,X,\Xi)$. Let $t \downarrow t^n(X,\Xi)$ and $s \uparrow t^n(X,\Xi)$ in this equality. Then we have

$$J(t^n(X,\Xi)+0,X,\Xi) = U^n(X,\Xi)J(t^n(X,\Xi)-0,X,\Xi),\ n = 1,\dots,m, \quad (7.3)$$

where $U^n(X,\Xi) \equiv \lim_{t\downarrow t^n(X,\Xi),s\uparrow t^n(X,\Xi)} J(t,s,X,\Xi)$, $n = 1,\dots,m$. We will prove that $U^n(X,\Xi)$, $n = 1,\dots,m$, are orthogonal matrices.

First we will prove that $U^1(X,\Xi)$ is an orthogonal matrix. In order to obtain the concrete form of $U^1(X,\Xi)$, we let $t \downarrow t^1(X,\Xi)$ and $s \uparrow t^1(X,\Xi)$ in (7.2). Hence we assume that t and s satisfy

$$0 \leqq s < t^1(X,\Xi) < t < t^2(X,\Xi). \quad (7.4)$$

For simplicity, we write $(\underline{X},\underline{\Xi})$ as $(x(-s,X,\Xi), \xi(-s,X,\Xi))$, and we write t^j and $\underline{t}^j$ as $t^j(X,\Xi)$ and $t^j(\underline{X},\underline{\Xi})$ respectively, $j = 1,2$. We easily see that $t^j - s = \underline{t}^j$, $j = 1,2$. Hence, it follows from (7.4) that

$$0 < \underline{t}^1 < \tau < \underline{t}^2, \quad (7.5)$$

where $\tau \equiv t-s$. Moreover we deduce that if $t \downarrow t^1$ and $s \uparrow t^1$, then

$$\tau \downarrow 0, \text{ and } (\underline{X},\underline{\Xi}) \to (X^1,\Xi^{1-0}) \text{ with } (\underline{X},\underline{\Xi}) \in \Gamma^1(X,\Xi), \quad (7.6)$$

where $\Gamma^1(X,\Xi) \equiv \{(x,\xi) = (x(-s,X,\Xi), \xi(-s,X,\Xi));\ s \text{ satisfies } (7.4)\}$, that is, $\Gamma^1(X,\Xi)$ denotes a characteristic curve of Λ which connects (X,Ξ) and

$$(X^1,\Xi^{1-0}) \equiv (x(-t^1,X,\Xi), \xi(-(t^1-0),X,\Xi)). \quad (7.7)$$

Substituting $(\underline{X},\underline{\Xi})$ and (7.1.1-2) in (7.2), and considering (4.4), we obtain $J(t,s,X,\Xi) = J(\tau,\underline{X},\underline{\Xi})$. From (7.5-6) we see that $\lim_{t\downarrow t^1, s\uparrow t^1}$ is equivalent to

$$\lim_{\tau\downarrow 0, t^1<\tau<t^2, (\underline{X},\underline{\Xi})\to(X^1,\Xi^{1-0}), (\underline{X},\underline{\Xi})\in\Gamma^1(X,\Xi)}.$$ Hence,

$$U^1(X,\Xi) = \lim_{\tau\downarrow 0, t^1<\tau<t^2, (\underline{X},\underline{\Xi})\to(X^1,\Xi^{1-0}), (\underline{X},\underline{\Xi})\in\Gamma^1(X,\Xi)} J(\tau,\underline{X},\underline{\Xi}). \qquad (7.8)$$

Let us calculate the right hand side of (7.8). In order to simplify the calculations, we will introduce a 3-dimensional rectangular coordinate system (x_1,x_2,x_3) in such a way that the origin coincides with X^1, that the x_2x_3 plane $(x_1 = 0)$ includes the face of $\partial\Omega$ which contains X^1, and that $\mathbf{n}(X^1)= (1,0,0)$. See § 1 for $\mathbf{n}=\mathbf{n}(x)$. By x_j, ξ_j, Ξ_j^{1-0}, $\underline{X}_j$, and $\underline{\Xi}_j$ we denote the j-th components of x, ξ, Ξ^{1-0}, $\underline{X}$, and $\underline{\Xi}$, respectively, j=1,2,3. We denote by $F_i(x,\xi)$ the i-th component of $\mathbb{F}(x,\xi)$, i = 1,2,3. We define ρ_i, i=1,2,3, as follows: $\rho_1\equiv -1$, $\rho_2\equiv 1$, and $\rho_3\equiv 1$. Define $\Xi^{1+0}\equiv \xi(-(t^1+0),X,\Xi)$.

Lemma 7.3. If $\tau\downarrow 0$ with (7.5), and if $(\underline{X},\underline{\Xi})\to(X^1,\Xi^{1-0})$ along $\Gamma^1(X,\Xi)$, then $J(\tau,\underline{X},\underline{\Xi})$ converges to a 6×6 matrix as follows:

$$\partial x_i(-\tau,\underline{X},\underline{\Xi})/\partial\underline{X}_j,\ \partial\xi_i(-\tau,\underline{X},\underline{\Xi})/\partial\underline{\Xi}_j \to \rho_i\delta_{ij},\quad i,j = 1,2,3,$$

$$\partial x_i(-\tau,\underline{X},\underline{\Xi})/\partial\underline{\Xi}_j \to 0,\quad i,j = 1,2,3,$$

$$\partial\xi_i(-\tau,\underline{X},\underline{\Xi})/\partial\underline{X}_1 \to (F_i(X^1,\Xi^{1+0})-\rho_i F_i(X^1,\Xi^{1-0}))/\Xi_1^{1-0},\quad i = 1,2,3, \qquad (7.9)$$

$$\partial\xi_i(-\tau,\underline{X},\underline{\Xi})/\partial\underline{X}_j \to 0,\quad i = 1,2,3,\ j = 2,3.$$

Applying Assumption $\mathbb{F}$ to (7.9) of this lemma, we have

$$\partial\xi_i(-\tau,\underline{X},\underline{\Xi})/\partial\underline{X}_1 \to 0,\quad i = 1,2,3. \qquad (7.10)$$

It follows from Lemma 7.3 and (7.10) that $U^1(X,\Xi)$ is an orthogonal matrix.

Performing the same calculations as above, we see that $U^n(X,\Xi)$, $n = 2,\ldots,m$, also are orthogonal matrices. Applying these results to (7.3), we have $\|J(t^n(X,\Xi)+0,X,\Xi)\| = \|J(t^n(X,\Xi)-0,X,\Xi)\|$, $n = 1,\ldots,m$. Combining these equalities and Lemma 4.2, (i), we see that

$$\|J(t,X,\Xi)\| \leqq \|J(0,X,\Xi)\| e^{c_{4.2}t}, \quad \text{for each } t \geqq 0.$$

Making use of this inequality, Lemma 4.2, (ii), and the following inequalities and equalities, we can obtain Lemma 7.1:

$$|J_{i,\perp}(t,X,\Xi)| \leqq |J_i(t,X,\Xi)| \leqq \|J(t,X,\Xi)\|, \; i = 1,\ldots,6,$$

$$|J_{1,\perp}(t,X,\Xi)|\cdots|J_{6,\perp}(t,X,\Xi)| = |\det(J(t,X,\Xi))|, \quad \|J(0,X,\Xi)\| = 6^{1/2}.$$

§ 8. ESTIMATES FOR $\mathbb{J} = \mathbb{J}(\omega_1)$

The purpose of this section is to calculate the rank of $\mathbb{J} = \mathbb{J}(\omega_1)$ (see (6.6)). In this section we will prove Lemmas 8.1-4. By making use of Lemma 8.j, we prove Lemma 8.j+1, j=1,2,3, respectively. The main result of this section is Lemma 8.4, which will be employed in § 9 to prove (G.1-2).

Let u_j, $j=1,\ldots,N$, be vectors of $\mathbb{R}^m$, $m,N \in \mathbb{N}$. Let $\varepsilon > 0$. By $U(u_1,\ldots,u_N;\varepsilon)$ we denote the set of all vectors of $\mathbb{R}^m$ whose distance from the subspace spanned by u_j, $j = 1,\ldots,N$, is greater than or equal to ε. Let T be a positive constant. Write $\Omega_4(\mathbf{r},T)$ as the set of all vectors of the following form: $\omega_4 = (x_4, \eta_4, t_4) \in \Omega \times \mathbb{R}^3 \times [0,T]$, where $|\eta_4| \leqq \mathbf{r}$; $\mathbf{r}$ is the positive constant defined in (6.2). If

$$\omega_4 \in \Omega_4(\varepsilon,\mathbf{r},T) \equiv \{\omega_4 = (x_4, \eta_4, t_4) \in \Omega_4(\mathbf{r},T);\; \eta_4 \in U(0;\varepsilon)\},$$

then we can obtain estimates for the rank of the Jacobian matrix $\partial(x_3)/\partial(\eta_4, t_4)$ as follows (we can regard x_3 in (6.4) as function of ω_4):

Lemma 8.1. Let $\varepsilon, T > 0$ be constants. We can choose one column vector from the Jacobian matrix $\partial(x_3)/\partial(\eta_4, t_4)$ for $\omega_4 \in \mathfrak{Q}_4(\varepsilon, \mathbf{r}, T)$ almost everywhere so that the essential infimum (in $\mathfrak{Q}_4(\varepsilon, \mathbf{r}, T)$) of the norm of the chosen column vector is positive, i.e., there exists a Lebesgue measurable, integer-valued function $G = G(\omega_4)$ which satisfies the following (i-ii):

(i) $G(\omega_4) \in \{1,\ldots,4\}$ for almost all $\omega_4 \in \mathfrak{Q}_4(\varepsilon, \mathbf{r}, T)$,

(ii) $\text{ess inf}\,\{\,|a(G(\omega_4), \omega_4)|\,;\ \omega_4 \in \mathfrak{Q}_4(\varepsilon, \mathbf{r}, T)\} > 0$.

We denote by $a(k, \omega_4)$ the k-th column vector of $\partial(x_3)/\partial(\eta_4, t_4)$, $k = 1,\ldots,4$, i.e.,

$$a(k, \omega_4) \equiv (\partial/\partial\eta_{4,k})^t(x_{3,1}, x_{3,2}, x_{3,3}),\ k = 1,2,3,\quad a(4, \omega_4) \equiv (\partial/\partial t_4)^t(x_{3,1}, x_{3,2}, x_{3,3}),$$

where by $x_{i,j}$, $\eta_{i,j}$ we denote the j-th components of x_i, $\eta_i \in \mathbb{R}^3$, $j = 1,2,3$, respectively, i.e., $x_i = (x_{i,1}, x_{i,2}, x_{i,3})$, $\eta_i = (\eta_{i,1}, \eta_{i,2}, \eta_{i,3})$.

Let ε, $T > 0$. Write $\Omega_3(\mathbf{r}, T)$ as the set of all vectors of the following form: $\omega_3 = (x_4, \eta_4, t_4, \eta_3, t_3)$, where $x_4 \in \Omega$, $|\eta_j| \leqq \mathbf{r}$, $\eta_j \in \mathbb{R}^3$, $0 \leqq t_j \leqq T$, $j = 3,4$. We define

$$\mathfrak{Q}_3(\varepsilon, \mathbf{r}, T) \equiv \{\omega_3 = (x_4, \eta_4, t_4, \eta_3, t_3) \in \Omega_3(\mathbf{r}, T);$$
$$\omega_4 = (x_4, \eta_4, t_4) \in \mathfrak{Q}_4(\varepsilon, \mathbf{r}, T),\ \eta_3 \in U(a(G(\omega_4), \omega_4); \varepsilon)\},$$

where $a(\cdot, \omega_4)$ and $G = G(\omega_4)$ are those in Lemma 8.1. If $\omega_3 \in \mathfrak{Q}_3(\varepsilon, \mathbf{r}, T)$, then we can obtain estimates for the rank of $\partial(x_2)/\partial(\eta_4, t_4, \eta_3, t_3)$ as follows (we can regard x_2 in (6.4) as function of ω_3):

Lemma 8.2. Let $\varepsilon, T > 0$. We can choose two column vectors from the Jacobian matrix $\partial(x_2)/\partial(\eta_4, t_4, \eta_3, t_3)$ for $\omega_3 \in \mathfrak{Q}_3(\varepsilon, \mathbf{r}, T)$ almost everywhere so that the chosen column vectors are uniformly linearly independent for almost all $\omega_3 \in \mathfrak{Q}_3(\varepsilon, \mathbf{r}, T)$, i.e., there exist Lebesgue measurable, integer-valued functions $H_j = H_j(\omega_3)$, $j = 1,2$, such that, for almost all $\omega_3 \in \mathfrak{Q}_3(\varepsilon,$

$\mathbf{r}$,T),

(i) $H_j(\omega_3) \in \{1,\dots,8\}$, $j = 1,2$, and $H_1(\omega_3) \neq H_2(\omega_3)$,

(ii) $b(H_j(\omega_3), \omega_3)$, $j = 1,2$, are uniformly linearly independent.

We denote by $b(k, \omega_3)$ the k-th column vector of $\partial(x_2)/\partial(\eta_4, t_4, \eta_3, t_3)$, $k = 1,\dots,8$, i.e.,

$$b(k, \omega_3) \equiv (\partial / \partial \eta_{4,k})^t(x_{2,1}, x_{2,2}, x_{2,3}),\ k = 1,2,3,$$

$$b(4, \omega_3) \equiv (\partial / \partial t_4)^t(x_{2,1}, x_{2,2}, x_{2,3}),$$

$$b(k, \omega_3) \equiv (\partial / \partial \eta_{3,k-4})^t(x_{2,1}, x_{2,2}, x_{2,3}),\ k = 5,6,7,$$

$$b(8, \omega_3) \equiv (\partial / \partial t_3)^t(x_{2,1}, x_{2,2}, x_{2,3}).$$

Let ε, $T > 0$. Write $\Omega_2(\mathbf{r},T)$ as the set of all vectors of the following form: $\omega_2 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2)$, where $x_4 \in \Omega$, $|\eta_j| \leqq \mathbf{r}$, $\eta_j \in \mathbb{R}^3$, $0 \leqq t_j \leqq T$, $j = 2,\dots,4$. Define

$$\boldsymbol{\Omega}_2(\varepsilon, \mathbf{r}, T) \equiv \{\omega_2 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2) \in \Omega_2(\mathbf{r},T);$$
$$\omega_3 = (x_4, \eta_4, t_4, \eta_3, t_3) \in \boldsymbol{\Omega}_3(\varepsilon, \mathbf{r}, T),\ \eta_2 \in U(b(H_1(\omega_3), \omega_3), b(H_2(\omega_3), \omega_3); \varepsilon)\},$$

where $b(\cdot, \omega_3)$ and $H_j = H_j(\omega_3)$, $j=1,2$, are those in Lemma 8.2. If $\omega_2 \in \boldsymbol{\Omega}_2(\varepsilon, \mathbf{r}, T)$, then we can obtain estimates for the rank of $\partial(x_1)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2)$ as follows (we can regard x_1 in (6.4) as function of ω_2):

Lemma 8.3. Let ε, $T > 0$. We can choose three column vectors from the Jacobian matrix $\partial(x_1)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2)$ for $\omega_2 \in \boldsymbol{\Omega}_2(\varepsilon, \mathbf{r}, T)$ almost everywhere so that the chosen column vectors are uniformly linearly independent for almost all $\omega_2 \in \boldsymbol{\Omega}_2(\varepsilon, \mathbf{r}, T)$, i.e., there exist Lebesgue measurable, integer-valued functions $I_j = I_j(\omega_2)$, $j = 1,2,3$, such that, for almost all $\omega_2 \in \boldsymbol{\Omega}_2(\varepsilon, \mathbf{r}, T)$,

(i) $I_j(\omega_2) \in \{1,\dots,12\}$ for $j = 1,2,3$, and $I_i(\omega_2) \neq I_j(\omega_2)$ if $i \neq j$, $i, j = 1,2,3$,

(ii) $c(I_j(\omega_2), \omega_2)$, $j = 1,2,3$, are uniformly linearly independent, where we denote by $c(k, \omega_2)$ the k-th column vector of $\partial(x_1)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2)$, $k = 1,\dots,12$.

Let $\varepsilon, T > 0$. Write $\Omega_1 = \Omega_1(\mathbf{r}, T)$ as the set of all vectors of the form (6.5), where $x_4 \in \Omega$, $|\eta_j| \leqq \mathbf{r}$, $\eta_j \in \mathbb{R}^3$, $0 \leqq t_j \leqq T$, $j = 1,\dots,4$. Noting that if $\omega_j \in \Omega_j(\varepsilon, \mathbf{r}, T)$, then $\omega_{j+1} \in \Omega_{j+1}(\varepsilon, \mathbf{r}, T)$, $j=2,3$, we can decompose $\Omega_1(\mathbf{r}, T)$ into four disjoint subsets as follows: $\Omega_1(\mathbf{r}, T) \equiv N(\varepsilon, \mathbf{r}, T) \cup (\bigcup_{j=2}^{4} S_j(\varepsilon, \mathbf{r}, T))$, where

$$S_4(\varepsilon, \mathbf{r}, T) \equiv \{\omega_1 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \in \Omega_1(\mathbf{r}, T);\ \omega_4 = (x_4, \eta_4, t_4) \notin \Omega_4(\varepsilon, \mathbf{r}, T)\},$$

$$S_3(\varepsilon, \mathbf{r}, T) \equiv \{\omega_1 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \in \Omega_1(\mathbf{r}, T);\ \omega_4 = (x_4, \eta_4, t_4) \in \Omega_4(\varepsilon, \mathbf{r}, T),\ \omega_3 = (x_4, \eta_4, t_4, \eta_3, t_3) \notin \Omega_3(\varepsilon, \mathbf{r}, T)\},$$

$$S_2(\varepsilon, \mathbf{r}, T) \equiv \{\omega_1 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \in \Omega_1(\mathbf{r}, T);\ \omega_3 = (x_4, \eta_4, t_4, \eta_3, t_3) \in \Omega_3(\varepsilon, \mathbf{r}, T),\ \omega_2 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2) \notin \Omega_2(\varepsilon, \mathbf{r}, T)\},$$

$$N(\varepsilon, \mathbf{r}, T) \equiv \{\omega_1 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \in \Omega_1(\mathbf{r}, T);\ \omega_2 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2) \in \Omega_2(\varepsilon, \mathbf{r}, T)\}.$$

We denote by $\eta_{j,k}$ the k-th component of η_j, $k=1,2,3$, $j=1,\dots,4$. Write $\zeta(i)$, $i = 1,\dots,15$, as $\eta_{j,k}$, $j = 1,\dots,4$, $k = 1,2,3$, t_ℓ, $\ell = 2,3,4$, respectively. We do not need t_1.

Lemma 8.4. (i) Let $j = 2,3,4$. There exists a positive constant $c_{8.4}$ such that, for each ε, $T > 0$, x_4, t_k, $k \in \{1,\dots,4\}$, and η_i, $i \in \{1,\dots,4\} \setminus \{j\}$,

$$\text{meas}\,\{\eta_j \in \mathbb{R}^3; \omega_1 = (x_4, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \in S_j(\varepsilon, \mathbf{r}, T)\} \leqq c_{8.4} \mathbf{r}^{4-j} \varepsilon^{j-1},$$

where we denote the Lebesgue measure of a set $Y \subseteq \mathbb{R}^3$ by **meas** Y.

(ii) Let ε, $T > 0$. Then, there exist Lebesgue measurable, integer-valued functions $K_i = K_i(\omega_1)$, $i = 1,\dots,6$, such that, for almost all $\omega_1 \in N(\varepsilon, \mathbf{r}, T)$,

(1) $K_i(\omega_1) \in \{1,\dots,15\}$ for $i = 1,\dots,6$, and $K_i(\omega_1) \neq K_j(\omega_1)$ if $i \neq j$, $i, j = 1,\dots,6$,

(2) $\operatorname*{ess\,inf}_{\omega_1 \in N(\varepsilon,\mathbf{r},T)} |\det(\partial(x_0, \xi_0)/\partial(\zeta(K_1(\omega_1)),\dots,\zeta(K_6(\omega_1))))| > 0.$

§ 9. PROOF OF (G.1-2)

Decompose $\mathbb{G}(\mu, T)$ as follows (see (6.2-3)): $\mathbb{G}(\mu, T) = \sum_{k=1}^{4} \mathbb{G}_k(\varepsilon, \mu, \mathbf{r}, T)$, where

$$(\mathbb{G}_k(\varepsilon, \mu, \mathbf{r}, T)u(\cdot,\cdot))(x_4, \xi_4) \equiv \int_{0 \leqq t_j \leqq T, |\eta_j| \leqq \mathbf{r},\, j=1,\dots,4} \psi_k G u(x_0, \xi_0) dt d\eta,$$

$k = 1,\dots,4$. By $\psi_k = \psi_k(\omega_1)$, $k = 1,\dots,4$, we denote the characteristic functions of $N(\varepsilon, \mathbf{r}, T)$, $S_k(\varepsilon, \mathbf{r}, T)$, $k = 2,3,4$, respectively. The lemma below implies that $\mathbb{G}(\mu, T)$ is decomposed into the principal part $\mathbb{G}_1(\varepsilon, \mu, \mathbf{r}, T)$ and the negligible parts $\mathbb{G}_k(\varepsilon, \mu, \mathbf{r}, T)$, $k = 2,3,4$. Applying this lemma and Theorem VI.12, (a), p. 200[16], we can easily obtain (G.1-2).

Lemma 9.1. (i) $|||\mathbb{G}_k(\varepsilon, \mu, \mathbf{r}, T)||| \leqq c_{9.1}\, \varepsilon^{k-1}$, $k = 2,3,4$, for each $\varepsilon > 0$, $\mu \in \mathbb{D}$, and $T \in [1, +\infty)$, where $c_{9.1}$ is a positive constant which depends on $\mathbf{r}$, but is independent of ε, μ, and T.

(ii) If $1 \leqq T < +\infty$ and $\varepsilon > 0$, then

(1) $\mathbb{G}_1(\varepsilon, \mu, \mathbf{r}, T) \in \mathbb{C}(L^2(\Omega \times \mathbb{R}^3))$ for each $\mu \in \mathbb{D}$,

(2) $|||\mathbb{G}_1(\varepsilon, \mu, \mathbf{r}, T)||| \to 0$ as $|\mu| \to +\infty$, $\mu \in \mathbb{D}$.

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