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A Study on Substructural Logics with Restricted Exchange Rules

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Abstract

We consider several conditions that restrict applications of the exchange rule of Gentzen's sequent calculus LJ. (For example, we do not permit exchanges of atomic formulas.) We investigate theorem-equivalency and the cut-elimination property among the systems that are obtained from the implicational fragment of LJ by imposing the conditions on the exchange rule and deleting the other structural rules.

1 Introduction

Gentzen-type sequent calculi whose structural rules are restricted are called *substructural logics* (see, e.g., [3]). This paper is intended to study an interesting family of substructural logics, on which no thorough investigations have been made.

The rule

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma}$$

of inference of the system LJ (Gentzen's sequent calculus for intuitionistic logic) is called the *exchange rule*. We consider the following conditions, which restrict applications of this rule.

(C1) α must be of the form $\psi{\rightarrow}\phi$.

(C2) β must be of the form $\psi{\rightarrow}\phi$.

(C3) γ must be of the form $\psi{\rightarrow}\phi$ if Δ is empty.

Then we obtain eight restricted exchange rules by imposing arbitrary combinations of these three conditions. (Among the eight rules, one is the ordinary (i.e., non-restricted) exchange rule.) Let $\mathrm{FL}_{\rightarrow}$ be the system obtained from the implicational fragment of LJ by deleting all the structural rules except *cut* (i.e., deleting the *weakening*, the *contraction*, and the exchange rules). This paper investigates all the systems that are obtained by adding arbitrary combinations of the eight restricted exchange rules to $FL_{\rightarrow}$. We give a complete and nontrivial solution to the problem: Among these systems, which pair is theorem-equivalent, and which one enjoys the cut-elimination theorem?

The conditions (C1) (or (C2), or (C3)) corresponds to the restriction "α (or β, or γ) must be of the form $\psi\rightarrow\phi$" on the axiom scheme C: $(\alpha\rightarrow(\beta\rightarrow\gamma))\rightarrow(\beta\rightarrow(\alpha\rightarrow\gamma))$ in Hilbert-type systems. Such restrictions on the schemes C and K: $\alpha\rightarrow(\beta\rightarrow\alpha)$ appear in axiomatizations of the well-known relevance logics E and S4 (see, e.g., [1]). Indeed the corresponding restrictions on the exchange and weakening rules make cut-eliminable systems for the implicational fragments of E and S4. This fact and our nontrivial results show, the authors think, that such kind of substructural logics has certain significance and is worth studying.

This paper treats the systems of which some restricted exchange rules and the cut rule are the only structural rules and implication is the only logical connective. Investigations of the other structural rules and the other connectives are left for further research.

2 Definitions and an overview of the results

We assume that the reader is familiar with the basic knowledge of Gentzen-type sequent calculi (see, e.g., [4]). In this paper, *formulas* are constructed from the propositional variables and $\rightarrow$(implication). Greek small letters $\alpha, \beta, \ldots$ are used for formulas, and Greek capital letters $\Gamma, \Delta, \ldots$ are used for finite (possibly empty) sequences of formulas. A *sequent* is an expression of the form $\Gamma \Rightarrow \alpha$. By the symbol $\equiv$, we mean the equality as sequences of symbols. We adopt the convention of association to the right for omitting parentheses. For example, $(\alpha\rightarrow\beta\rightarrow\gamma)\rightarrow\beta\rightarrow\alpha\rightarrow\gamma \equiv (\alpha\rightarrow(\beta\rightarrow\gamma))\rightarrow(\beta\rightarrow(\alpha\rightarrow\gamma))$.

We say that a sequence $(\alpha_1, \ldots, \alpha_n)$ of formulas is a *single atom* if and only if $n = 1$ and α_1 is a propositional variable. If a formula α is of the form $\alpha_1\rightarrow\alpha_2$, then α is said to be an *implication* and denoted by $\overrightarrow{\alpha}$. Moreover, if (Γ, α) is not a single atom, then the sequent $\Gamma \Rightarrow \alpha$ is said to be an *implication* and denoted by $\overrightarrow{\Gamma \Rightarrow \alpha}$. Note that a sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ is an implication if and only if the formula $\alpha_1\rightarrow\cdots\rightarrow\alpha_n\rightarrow\beta$ is an implication.

We will introduce several systems of sequent calculi. If a sequent $\Gamma \Rightarrow \alpha$ is provable in a system S, then we write $S \vdash \Gamma \Rightarrow \alpha$. We say that two systems S_1 and S_2 are *theorem-equivalent* if $\{\Gamma \Rightarrow \alpha \mid S_1 \vdash \Gamma \Rightarrow \alpha\} = \{\Gamma \Rightarrow \alpha \mid S_2 \vdash \Gamma \Rightarrow \alpha\}$. A rule R of inference is said to be *admissible in* a system S if the following condition is satisfied: For any instance

$$\frac{\Gamma_1 \Rightarrow \alpha_1 \quad \cdots \quad \Gamma_n \Rightarrow \alpha_n}{\Delta \Rightarrow \beta}$$

of R, if $S \vdash \Gamma_i \Rightarrow \alpha_i$ for all i, then $S \vdash \Delta \Rightarrow \beta$. Moreover, R is said to be *derivable in* S if there is a derivation from $\Gamma_1 \Rightarrow \alpha_1$, ..., $\Gamma_n \Rightarrow \alpha_n$ to $\Delta \Rightarrow \beta$ in S. Note that derivability implies admissibility, and that R is admissible in S if and only if the two systems S and $S + R$ are theorem-equivalent. We say that two rules R_1 and R_2 are *rule-equivalent over* a system S if R_i is derivable in the system $S + R_j$ for $(i, j) = (1, 2), (2, 1)$. Note that, if rules R_1 and R_2 are rule-equivalent over a system S, and if S^+ is a system obtained by adding some rules to S, then the systems $S^+ + R_1$ and $S^+ + R_2$ are theorem-equivalent.

We give a precise definition of the system $FL_\rightarrow$, which is the implicational fragment of the system FL (full Lambek logic) in [2]. The initial sequents of $FL_\rightarrow$ are of the form

$$\alpha \Rightarrow \alpha.$$

The rules of inferences of $FL_\rightarrow$ are as follows.

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \beta}{\Delta, \Gamma, \Sigma \Rightarrow \beta,} \text{(cut)}$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta, \Sigma \Rightarrow \gamma}{\Delta, \alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \gamma,} (\rightarrow\text{left}) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta.} (\rightarrow\text{right})$$

The eight variants of the exchange rule explained in Section 1 are named $(\mathrm{e}^{x_1 x_2 x_3})$, where

$$\{x_1, x_2, x_3\} \subseteq \{0, 1\},$$

and

$$x_i = 1 \text{ if and only if the condition } (\mathrm{C}i) \text{ is imposed, for } i = 1, 2, 3.$$

For example, (e^{000}) is the ordinary exchange rule, and (e^{101}) is of the form

$$\frac{\Gamma, \alpha_1 \rightarrow \alpha_2, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha_1 \rightarrow \alpha_2, \Delta \Rightarrow \gamma}$$

where (Δ, γ) is not a single atom. This rule is also written as

$$\frac{\Gamma, \overrightarrow{\alpha}, \beta, \overrightarrow{\Delta \Rightarrow \gamma}}{\Gamma, \beta, \overrightarrow{\alpha}, \overrightarrow{\Delta \Rightarrow \gamma}.} (\mathrm{e}^{101})$$

We will investigate the 2^8 systems, which are obtained by adding arbitrary combinations of the eight variants of the exchange rule to $FL_\rightarrow$. But some of them are redundant. For example, we consider $FL_\rightarrow + (\mathrm{e}^{100}) + (\mathrm{e}^{101})$ to be the same system as $FL_\rightarrow + (\mathrm{e}^{100})$ because (e^{101}) is an instance of (e^{100}). By this consideration, the number of the systems is reduced to twenty. They are obtained by adding the sets of rules displayed in Figure 1 to $FL_\rightarrow$.

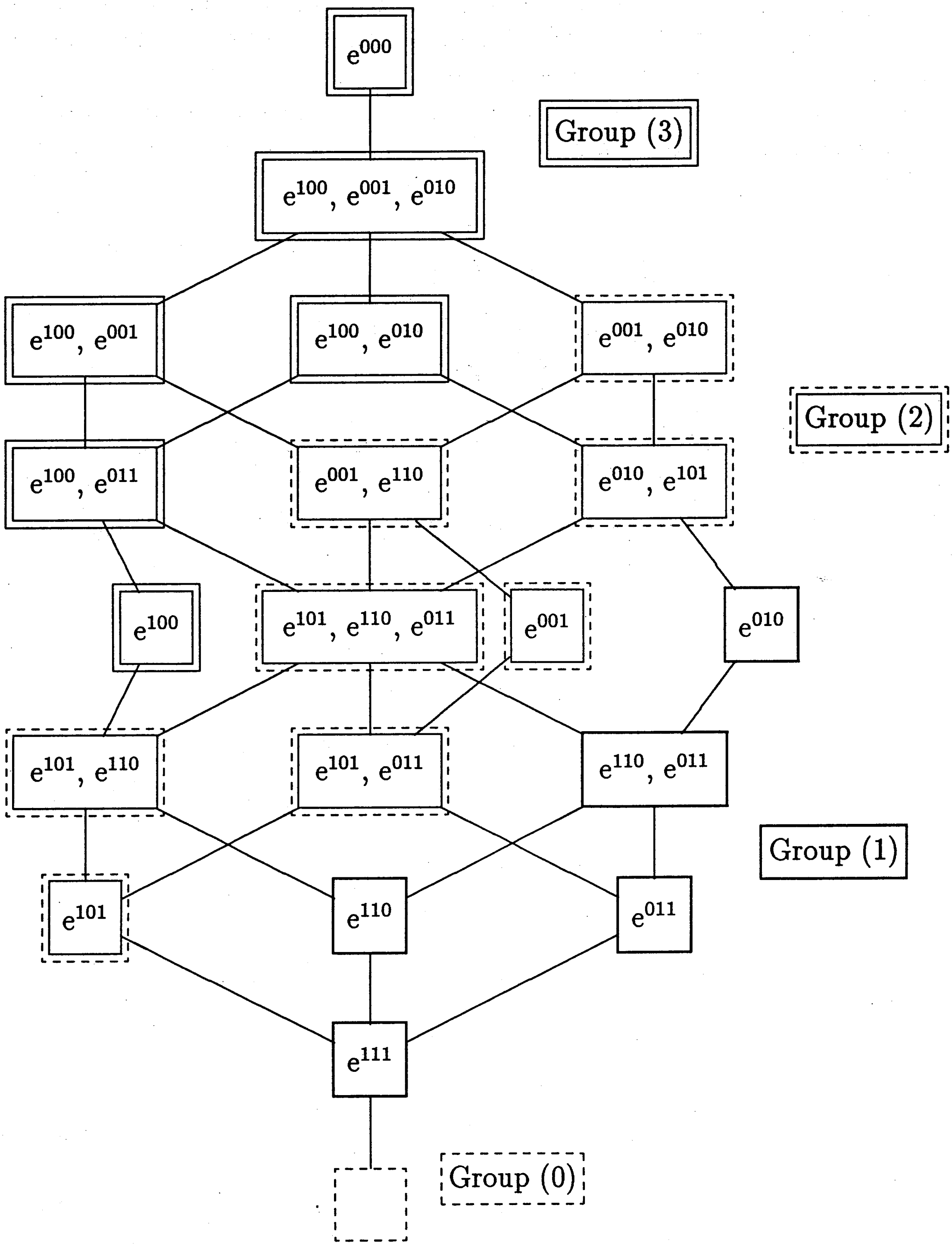
e^{000}
Group (3)
e^{100}, e^{001}, e^{010}
e^{100}, e^{001}
e^{100}, e^{010}
e^{001}, e^{010}
Group (2)
e^{100}, e^{011}
e^{001}, e^{110}
e^{010}, e^{101}
e^{100}
e^{101}, e^{110}, e^{011}
e^{001}
e^{010}
e^{101}, e^{110}
e^{101}, e^{011}
e^{110}, e^{011}
Group (1)
e^{101}
e^{110}
e^{011}
e^{111}
Group (0)

Figure 1

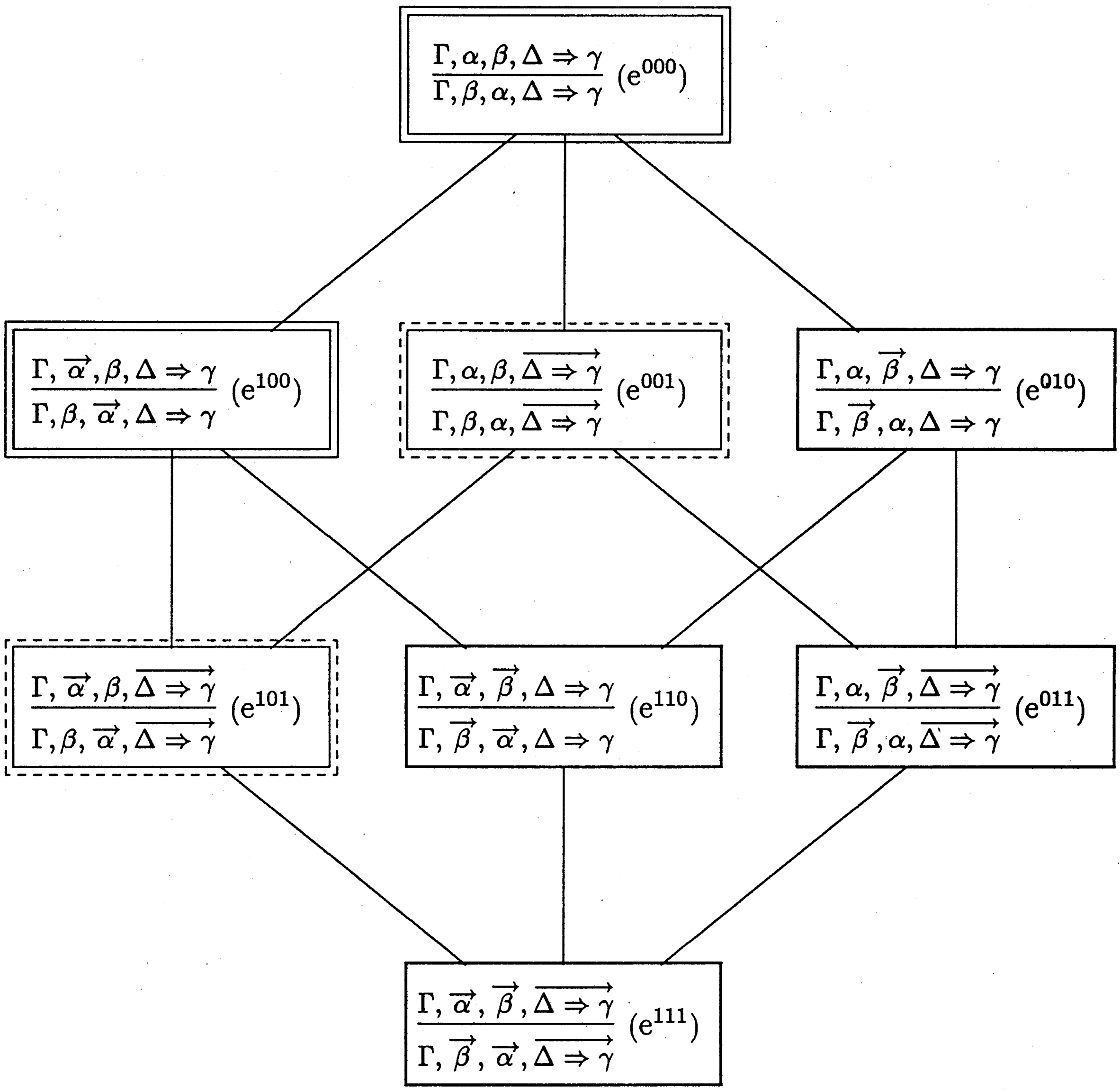

Figure 2

Our results on the equivalency between the variants of the exchange rule are as follows (Figure 2).

> The four rules (e^{111}), (e^{110}), (e^{011}), and (e^{010}) are rule-equivalent over $FL_{\rightarrow}$ each other.
>
> The rules (e^{101}) and (e^{001}) are rule-equivalent over $FL_{\rightarrow}$.
>
> The rules (e^{100}) and (e^{000}) are rule-equivalent over $FL_{\rightarrow}$.

Therefore the twenty systems collapse, modulo theorem-equivalence, into the four groups (Figure 1) :

(0) $FL_{\rightarrow}$.

(1) $FL_{\rightarrow}$+ (e^{111}), $FL_{\rightarrow}$+ (e^{110}), $FL_{\rightarrow}$+ (e^{011}), $FL_{\rightarrow}$+ (e^{110}) + (e^{011}), $FL_{\rightarrow}$+ (e^{010}).

(2) $FL_{\rightarrow}$+ (e^{101}), $FL_{\rightarrow}$+ (e^{101}) + (e^{110}), $FL_{\rightarrow}$+ (e^{101}) + (e^{011}), $FL_{\rightarrow}$+ (e^{101}) + (e^{110}) + (e^{011}), $FL_{\rightarrow}$+ (e^{001}), $FL_{\rightarrow}$+ (e^{001}) + (e^{110}), $FL_{\rightarrow}$+ (e^{010}) + (e^{101}), $FL_{\rightarrow}$+ (e^{001}) + (e^{010}).

(3) $FL_{\rightarrow}$+ (e^{100}), $FL_{\rightarrow}$+ (e^{100}) + (e^{011}), $FL_{\rightarrow}$+ (e^{100}) + (e^{001}), $FL_{\rightarrow}$+ (e^{100}) + (e^{010}), $FL_{\rightarrow}$+ (e^{100}) + (e^{001}) + (e^{010}), $FL_{\rightarrow}$+ (e^{000}).

Each group contains a cut-eliminable system (with an additional rule for (2)), and it is shown by these cut-elimination theorems that the systems in the group (i) are strictly weaker than those in (i+1), for $i = 0, 1, 2$.

The group (1) is investigated in Section 3. We prove the cut-elimination theorem for the three systems $FL_{\rightarrow}$+(e^{110}), $FL_{\rightarrow}$+(e^{110})+(e^{011}), and $FL_{\rightarrow}$+(e^{010}) by using an extra lemma to deal with the restricted exchange rules. The other systems in this group do not enjoy the cut-elimination theorem.

The group (2) is investigated in Section 4. The cut-elimination theorem does not hold for any systems in this group. Then we introduce a new rule of inference named (→left 2), which is admissible in the systems in this group; and we prove the cut-elimination theorem for the system $FL_{\rightarrow}$+ (e^{001}) + (e^{010}) + (→left 2).

The group (3) is investigated in Section 5. The cut-elimination theorem for $FL_{\rightarrow}$+(e^{000}) is well-known, and this implies the cut-elimination theorems for $FL_{\rightarrow}$+ (e^{100}) + (e^{001}) and for $FL_{\rightarrow}$+ (e^{100}) + (e^{001}) + (e^{010}). The other systems in this group do not enjoy the theorem.

3 Systems with e^{*1*}

In this section, we investigate the systems with the rules (e^{111}), (e^{110}), (e^{011}), and/or (e^{010}).

Lemma 3.1 $FL_{\rightarrow}$+ (e^{111}) $\vdash \psi\rightarrow\phi, \phi\rightarrow\phi \Rightarrow \psi\rightarrow\phi$.

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi \quad \phi \Rightarrow \phi}{\psi\to\phi,\psi \Rightarrow \phi}\,(\to\text{left}) \qquad \phi \Rightarrow \phi}{\phi\to\phi,\psi\to\phi,\psi \Rightarrow \phi}\,(\to\text{left})}{\phi\to\phi,\psi\to\phi \Rightarrow \psi\to\phi}\,(\to\text{right})}{\psi\to\phi,\phi\to\phi \Rightarrow \psi\to\phi.}\,(\mathrm{e}^{111})$$

∎

Theorem 3.2 (1) $\mathrm{FL}_{\to}+(\mathrm{e}^{111}) \vdash \overrightarrow{\beta}, \overrightarrow{\beta}\to\gamma \Rightarrow \gamma$.
(2) $\mathrm{FL}_{\to}+(\mathrm{e}^{111}) \vdash \alpha\to\overrightarrow{\beta}\to\gamma, \overrightarrow{\beta}, \alpha \Rightarrow \gamma$.
(3) *The rule* (e^{010}) *is derivable in* $\mathrm{FL}_{\to}+(\mathrm{e}^{111})$. *(Therefore the four rules* (e^{111}), (e^{110}), (e^{011}), *and* (e^{010}) *are rule-equivalent over* $\mathrm{FL}_{\to}$ *each other (see Figure 2).)*

Proof (1) Let $\overrightarrow{\beta} \equiv \beta_1\to\beta_2$. We have a proof of $\overrightarrow{\beta}, \overrightarrow{\beta}\to\gamma \Rightarrow \gamma$ in $\mathrm{FL}_{\to}+(\mathrm{e}^{111})$:

$$\dfrac{\dfrac{\beta_2 \Rightarrow \beta_2}{\Rightarrow \beta_2\to\beta_2}\,(\to\text{right}) \qquad \dfrac{\dfrac{\overset{\vdots\ \text{Lemma 3.1}}{\overrightarrow{\beta},\beta_2\to\beta_2 \Rightarrow \overrightarrow{\beta}} \qquad \dfrac{\overrightarrow{\beta} \Rightarrow \overrightarrow{\beta} \quad \gamma \Rightarrow \gamma}{\overrightarrow{\beta}\to\gamma, \overrightarrow{\beta} \Rightarrow \gamma}\,(\to\text{left})}{\overrightarrow{\beta}\to\gamma, \overrightarrow{\beta}, \beta_2\to\beta_2 \Rightarrow \gamma}\,(\text{cut})}{\overrightarrow{\beta}, \overrightarrow{\beta}\to\gamma, \beta_2\to\beta_2 \Rightarrow \gamma}\,(\mathrm{e}^{111})}{\overrightarrow{\beta}, \overrightarrow{\beta}\to\gamma \Rightarrow \gamma.}\,(\text{cut})$$

(2)

$$\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \overset{\vdots\ (1)}{\overrightarrow{\beta}, \overrightarrow{\beta}\to\gamma \Rightarrow \gamma}}{\overrightarrow{\beta}, \alpha\to\overrightarrow{\beta}\to\gamma, \alpha \Rightarrow \gamma}\,(\to\text{left})}{\alpha\to\overrightarrow{\beta}\to\gamma, \overrightarrow{\beta}, \alpha \Rightarrow \gamma.}\,(\mathrm{e}^{111})$$

(3) The derivability of

$$\dfrac{\Gamma, \alpha, \overrightarrow{\beta}, \Delta \Rightarrow \delta}{\Gamma, \overrightarrow{\beta}, \alpha, \Delta \Rightarrow \delta}\,(\mathrm{e}^{010})$$

is shown as follows. Let $\gamma \equiv \delta_n\to\cdots\to\delta_1\to\delta$ where $(\delta_n, ..., \delta_1) \equiv \Delta$. Then we have

$$\dfrac{\dfrac{\overset{\Gamma, \alpha, \overrightarrow{\beta}, \Delta \Rightarrow \delta}{\vdots\ (\to\text{right})}}{\Gamma \Rightarrow \alpha\to\overrightarrow{\beta}\to\gamma} \qquad \overset{\vdots\ (2)}{\alpha\to\overrightarrow{\beta}\to\gamma, \overrightarrow{\beta}, \alpha \Rightarrow \gamma}}{\Gamma, \overrightarrow{\beta}, \alpha \Rightarrow \gamma}\,(\text{cut}) \qquad \overset{\dfrac{\delta_2 \Rightarrow \delta_2 \qquad \dfrac{\delta_1 \Rightarrow \delta_1 \quad \delta \Rightarrow \delta}{\delta_1\to\delta, \delta_1 \Rightarrow \delta}\,(\to\text{left})}{\delta_2\to\delta_1\to\delta, \delta_2, \delta_1 \Rightarrow \delta}\,(\to\text{left})}{\vdots}$$

$$\dfrac{\Gamma, \overrightarrow{\beta}, \alpha \Rightarrow \gamma \qquad \gamma, \Delta \Rightarrow \delta}{\Gamma, \overrightarrow{\beta}, \alpha, \Delta \Rightarrow \delta.}\,(\text{cut})$$

∎

Corollary 3.3 *Among the twenty systems in Figure 1, the five systems* $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{111}$), $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$), $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{011}$), $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$) + ($\mathrm{e}^{011}$), *and* $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{010}$) *are theorem-equivalent each other.*

Proof By Theorem 3.2 (3). ∎

These five systems will be called *e^{*1*}-systems.* (The symbol $*$ represents an arbitrary sign in $\{0,1\}$.)

Theorem 3.4 *Each e^{*1*}-system is strictly stronger than* $\mathrm{FL}_{\rightarrow}$.

Proof It is well-known that the cut-elimination theorem holds for $\mathrm{FL}_{\rightarrow}$. Then we can easily verify that the sequent $\beta_1 \rightarrow \beta_2, (\beta_1 \rightarrow \beta_2) \rightarrow \gamma \Rightarrow \gamma$ is not (cut-free) provable in $\mathrm{FL}_{\rightarrow}$ if β_1, β_2 and γ are mutually distinct propositional variables, while it is provable in e^{*1*}-systems shown in Theorem 3.2 (1). ∎

Theorem 3.5 *The cut-elimination theorem holds for the three e^{*1*}-systems* $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$), $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$) + ($\mathrm{e}^{011}$), *and* $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{010}$)*; but does not hold for the other e^{*1*}-systems.*

Proof The cut-elimination theorem for $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$) will be given below.

The theorems for $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$) + ($\mathrm{e}^{011}$) and for $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{010}$) are implied from that for $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$): Suppose we have a proof of a sequent in $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$) + ($\mathrm{e}^{011}$) (or in $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{010}$)). Then, by Corollary 3.3 and the cut-elimination theorem for $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$), we get a cut-free proof of the same sequent in $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$). This is also a cut-free proof in $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$) + ($\mathrm{e}^{011}$) and in $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{010}$).

The fact that cut-free parts of $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{111}$) and $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{011}$) are strictly weaker than e^{*1*}-systems is shown by the same argument as Theorem 3.4. ∎

The rest of this section is devoted to prove the cut-elimination theorem for $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$).

Our cut-elimination proof is a standard one: Given a proof of a sequent in $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$), we transform it by "lifting up" the applications of the cut rule step by step, and finally we get a cut-free proof of the same sequent in $\mathrm{FL}_{\rightarrow}$+ ($\mathrm{e}^{110}$). But a standard transformation does not work in the following situation. We wish to transform the proof

$$\dfrac{\begin{array}{c}\vdots \\ \Phi \Rightarrow \overrightarrow{\psi}\end{array} \qquad \dfrac{\begin{array}{c}\vdots \\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\psi}, \Delta \Rightarrow \beta\end{array}}{\Gamma, \overrightarrow{\psi}, \overrightarrow{\alpha}, \Delta \Rightarrow \beta}\,(\mathrm{e}^{110})}{\Gamma, \Phi, \overrightarrow{\alpha}, \Delta \Rightarrow \beta}\,(\mathrm{cut})$$

into the proof

$$\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\\ \Phi \Rightarrow \overrightarrow{\psi}\end{array} \quad \begin{array}{c}\vdots\\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\psi}, \Delta \Rightarrow \beta\end{array}}{\Gamma, \overrightarrow{\alpha}, \Phi, \Delta \Rightarrow \beta}\,(\mathrm{cut})\\
\vdots\ (\mathrm{e}^{110})\\
\Gamma, \Phi, \overrightarrow{\alpha}, \Delta \Rightarrow \beta.
\end{array}$$

However, these applications of (e^{110}) are not permitted if Φ contains an atomic formula. Our strategy to overcome this difficulty is as follows. For example, given a proof of the form

$$\dfrac{\dfrac{p \Rightarrow p \quad \begin{array}{c}\vdots\\ \overrightarrow{\phi} \Rightarrow \overrightarrow{\psi}\end{array}}{p \to \overrightarrow{\phi}, p \Rightarrow \overrightarrow{\psi}}\,(\to\mathrm{left}) \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\psi}, \Delta \Rightarrow \beta\end{array}}{\Gamma, \overrightarrow{\psi}, \overrightarrow{\alpha}, \Delta \Rightarrow \beta}\,(\mathrm{e}^{110})}{\Gamma, p \to \overrightarrow{\phi}, p, \overrightarrow{\alpha}, \Delta \Rightarrow \beta}\,(\mathrm{cut})$$

where p is a propositional variable, we transform it into the proof

$$\dfrac{p \Rightarrow p \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \overrightarrow{\phi} \Rightarrow \overrightarrow{\psi}\end{array} \quad \begin{array}{c}\vdots\\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\psi}, \Delta \Rightarrow \beta\end{array}}{\Gamma, \overrightarrow{\alpha}, \overrightarrow{\phi}, \Delta \Rightarrow \beta}\,(\mathrm{cut})}{\Gamma, \overrightarrow{\phi}, \overrightarrow{\alpha}, \Delta \Rightarrow \beta}\,(\mathrm{e}^{110})}{\Gamma, p \to \overrightarrow{\phi}, p, \overrightarrow{\alpha}, \Delta \Rightarrow \beta.}\,(\to\mathrm{left})$$

In a general case, given a proof

$$\dfrac{\begin{array}{c}\vdots\ P\\ \Phi \Rightarrow \overrightarrow{\psi}\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\psi}, \Delta \Rightarrow \beta\end{array}}{\Gamma, \overrightarrow{\psi}, \overrightarrow{\alpha}, \Delta \Rightarrow \beta}\,(\mathrm{e}^{110})}{\Gamma, \Phi, \overrightarrow{\alpha}, \Delta \Rightarrow \beta,}\,(\mathrm{cut})$$

we first transform the subproof P into a proof P^- of $\Phi^- \Rightarrow \overrightarrow{\psi}$ where Φ^- contains no atomic formula and P^- is obtained by dropping some inferences from P. Then we lift up the cut rule as

$$\dfrac{\begin{array}{c}\vdots\ P^-\\ \Phi^- \Rightarrow \overrightarrow{\psi}\end{array} \quad \begin{array}{c}\vdots\\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\psi}, \Delta \Rightarrow \beta\end{array}}{\begin{array}{c}\Gamma, \overrightarrow{\alpha}, \Phi^-, \Delta \Rightarrow \beta\\ \vdots\ (e^{110})\\ \Gamma, \Phi^-, \overrightarrow{\alpha}, \Delta \Rightarrow \beta\\ \vdots\ (*)\\ \Gamma, \Phi, \overrightarrow{\alpha}, \Delta \Rightarrow \beta\end{array}}\ (\text{cut})$$

where $(*)$ is a series of inferences corresponding to the dropped inferences in P. The precise way to obtain P^- and $(*)$ is described in the following lemma.

Lemma 3.6 *Let P be a cut-free proof of $\Phi, \Psi \Rightarrow \psi$ in $\mathrm{FL}_{\rightarrow}+(\mathrm{e}^{110})$. If the sequence (Ψ, ψ) is not a single atom, then we get a sequence Φ^- and a proof P^- which satisfy the following conditions.*

(1) P^- is a cut-free proof of $\Phi^-, \Psi \Rightarrow \psi$ in $\mathrm{FL}_{\rightarrow}+(\mathrm{e}^{110})$.

(2) Φ^- contains no atomic formula (i.e., Φ^- is of the form $(\overrightarrow{\alpha_1}, ..., \overrightarrow{\alpha_n})$ and $n \geq 0$).

(3) The rule of inference

$$\frac{\Gamma, \Phi^-, \Delta \Rightarrow \alpha}{\Gamma, \Phi, \Delta \Rightarrow \alpha}\ (\mathcal{R}^{\Phi^-}_{\Phi})$$

is derivable in the cut-free part of $\mathrm{FL}_{\rightarrow}+(\mathrm{e}^{110})$. That is, for any sequence (Γ, Δ, α), there is a cut-free derivation from $\Gamma, \Phi^-, \Delta \Rightarrow \alpha$ to $\Gamma, \Phi, \Delta \Rightarrow \alpha$ in $\mathrm{FL}_{\rightarrow}+(\mathrm{e}^{110})$.

(The sequences Φ and (Ψ, ψ), which are components of the last sequent of the given proof P, will be called respectively a redex and an invariant.)

Proof By induction on P. We distinguish cases according to the last inference in P and to the partition between the redex and invariant.

(Case 1-1): P is an initial sequent $\psi \Rightarrow \psi$, and the redex Φ is empty. In this case, Φ^- is also empty, and $P^- \equiv P$. Derivability of $\mathcal{R}^{\Phi^-}_{\Phi}$ is obvious.

(Case 1-2): P is an initial sequent $\psi \Rightarrow \psi$, and the redex Φ is ψ. In this case, ψ is an implication because of the condition that the invariant is not a single atom. Then $\Phi^- \equiv \Phi \equiv \psi$, and $P^- \equiv P$. Derivability of $\mathcal{R}^{\Phi^-}_{\Phi}$ is obvious.

(Case 2-1): P is of the form

$$\frac{\begin{array}{c}\vdots\ Q\\ \Lambda \Rightarrow \beta\end{array} \quad \begin{array}{c}\vdots\ R\\ \Pi_1, \Pi_2, \gamma, \Sigma \Rightarrow \psi\end{array}}{\Pi_1, \Pi_2, \beta \rightarrow \gamma, \Lambda, \Sigma \Rightarrow \psi,}\ (\rightarrow\text{left})$$

and the redex Φ is Π_1. We apply the induction hypothesis to R in which the redex is Π_1; and we get a sequence Π_1^- and a proof R^-. Then the required sequence Φ^- is Π_1^-, and the required proof P^- is

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Lambda \Rightarrow \beta\end{matrix} \qquad \begin{matrix}\vdots\, R^- \\ \Pi_1^-, \Pi_2, \gamma, \Sigma \Rightarrow \psi\end{matrix}}{\Pi_1^-, \Pi_2, \beta\to\gamma, \Lambda, \Sigma \Rightarrow \psi.}\;(\to\text{left})$$

Derivability of $\mathcal{R}_{\Phi}^{\Phi^-}$ is obvious because of the induction hypothesis.

(Case 2-2): P is of the form

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Lambda_1, \Lambda_2 \Rightarrow \beta\end{matrix} \qquad \begin{matrix}\vdots\, R \\ \Pi, \gamma, \Sigma \Rightarrow \psi\end{matrix}}{\Pi, \beta\to\gamma, \Lambda_1, \Lambda_2, \Sigma \Rightarrow \psi,}\;(\to\text{left})$$

the redex Φ is $(\Pi, \beta\to\gamma, \Lambda_1)$, and Λ_2 is not empty. (The case that Λ_2 is empty is included in (Case 2-3).) We apply the induction hypothesis to Q in which the redex is Λ_1, and apply the induction hypothesis to R in which the redex is Π. (Note that neither (Λ_2, β) nor (γ, Σ, ψ) is a single atom.) Then we get the sequences Λ_1^- and Π^-, and the proofs Q^- and R^-; and the required sequence Φ^- is $(\Pi^-, \beta\to\gamma, \Lambda_1^-)$, and the required proof P^- is

$$\dfrac{\begin{matrix}\vdots\, Q^- \\ \Lambda_1^-, \Lambda_2 \Rightarrow \beta\end{matrix} \qquad \begin{matrix}\vdots\, R^- \\ \Pi^-, \gamma, \Sigma \Rightarrow \psi\end{matrix}}{\Pi^-, \beta\to\gamma, \Lambda_1^-, \Lambda_2, \Sigma \Rightarrow \psi.}\;(\to\text{left})$$

Derivability of $\mathcal{R}_{\Pi,\beta\to\gamma,\Lambda_1}^{\Pi^-,\beta\to\gamma,\Lambda_1^-}$ is shown as follows:

$$\dfrac{\dfrac{\Gamma, \Pi^-, \beta\to\gamma, \Lambda_1^-, \Delta \Rightarrow \alpha}{\Gamma, \Pi, \beta\to\gamma, \Lambda_1^-, \Delta \Rightarrow \alpha}\;(\mathcal{R}_{\Pi}^{\Pi^-})\text{ (ind. hyp.)}}{\Gamma, \Pi, \beta\to\gamma, \Lambda_1, \Delta \Rightarrow \alpha.}\;(\mathcal{R}_{\Lambda_1}^{\Lambda_1^-})\text{ (ind. hyp.)}$$

(Case 2-3): P is of the form

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Lambda \Rightarrow \beta\end{matrix} \qquad \begin{matrix}\vdots\, R \\ \Pi, \gamma, \Sigma_1, \Sigma_2 \Rightarrow \psi\end{matrix}}{\Pi, \beta\to\gamma, \Lambda, \Sigma_1, \Sigma_2 \Rightarrow \psi,}\;(\to\text{left})$$

and the redex Φ is $(\Pi, \beta\to\gamma, \Lambda, \Sigma_1)$. We apply the induction hypothesis to R in which the redex is $\Omega \equiv (\Pi, \gamma, \Sigma_1)$; and we get a sequence Ω^- and a proof R^-. Then the required sequence Φ^- is Ω^-, and the required proof P^- is

$$\begin{matrix}\vdots\, R^- \\ \Omega^-, \Sigma_2 \Rightarrow \psi.\end{matrix}$$

Derivability of $\mathcal{R}^{\Omega^-}_{\Pi,\beta\to\gamma,\Lambda,\Sigma_1}$ is shown as follows:

$$\dfrac{\dfrac{\vdots\, Q}{\Lambda \Rightarrow \beta} \quad \dfrac{\Gamma, \Omega^-, \Delta \Rightarrow \alpha}{\Gamma, \Pi, \gamma, \Sigma_1, \Delta \Rightarrow \alpha}\,(\mathcal{R}^{\Omega^-}_{\Pi,\gamma,\Sigma_1})\text{ (ind. hyp.)}}{\Gamma, \Pi, \beta\to\gamma, \Lambda, \Sigma_1, \Delta \Rightarrow \alpha.}\,(\to\text{left})$$

(Case 3): P is of the form

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Phi, \Sigma, \gamma \Rightarrow \delta\end{matrix}}{\Phi, \Sigma \Rightarrow \gamma\to\delta,}\,(\to\text{right})$$

and the redex is Φ. We apply the induction hypothesis to Q in which the redex is Φ; and we get a sequence Φ^- and a proof Q^-. Then the required sequence is Φ^-, and the required proof P^- is

$$\dfrac{\begin{matrix}\vdots\, Q^- \\ \Phi^-, \Sigma, \gamma \Rightarrow \delta\end{matrix}}{\Phi^-, \Sigma \Rightarrow \gamma\to\delta.}\,(\to\text{right})$$

Derivability of $\mathcal{R}^{\Phi^-}_{\Phi}$ is obvious because of the induction hypothesis.

(Case 4-1): P is of the form

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Pi_1, \Pi_2, \overrightarrow{\beta}, \overrightarrow{\gamma}, \Sigma \Rightarrow \psi\end{matrix}}{\Pi_1, \Pi_2, \overrightarrow{\gamma}, \overrightarrow{\beta}, \Sigma \Rightarrow \psi,}\,(\text{e}^{110})$$

and the redex Φ is Π_1. We apply the induction hypothesis to Q in which the redex is Π_1; and we get a sequence Π_1^- and a proof Q^-. Then the required sequence Φ^- is Π_1^-, and the required proof P^- is

$$\dfrac{\begin{matrix}\vdots\, Q^- \\ \Pi_1^-, \Pi_2, \overrightarrow{\beta}, \overrightarrow{\gamma}, \Sigma \Rightarrow \psi\end{matrix}}{\Pi_1^-, \Pi_2, \overrightarrow{\gamma}, \overrightarrow{\beta}, \Sigma \Rightarrow \psi.}\,(\text{e}^{110})$$

Derivability of $\mathcal{R}^{\Phi^-}_{\Phi}$ is obvious because of the induction hypothesis.

(Case 4-2): P is of the form

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Pi, \overrightarrow{\beta}, \overrightarrow{\gamma}, \Sigma \Rightarrow \psi\end{matrix}}{\Pi, \overrightarrow{\gamma}, \overrightarrow{\beta}, \Sigma \Rightarrow \psi,}\,(\text{e}^{110})$$

and the redex Φ is $(\Pi, \vec{\gamma})$. We apply the induction hypothesis to Q in which the redex is Π; and we get a sequence Π^- and a proof Q^-. Then the required sequence Φ^- is $(\Pi^-, \vec{\gamma})$, and the required proof P^- is

$$\dfrac{\begin{matrix} \vdots\, Q^- \\ \Pi^-, \vec{\beta}, \vec{\gamma}, \Sigma \Rightarrow \psi \end{matrix}}{\Pi^-, \vec{\gamma}, \vec{\beta}, \Sigma \Rightarrow \psi.}\ (\mathrm{e}^{110})$$

Derivability of $\mathcal{R}^{\Pi^-,\vec{\gamma}}_{\Pi,\vec{\gamma}}$ is shown as follows:

$$\dfrac{\Gamma, \Pi^-, \vec{\gamma}, \Delta \Rightarrow \alpha}{\Gamma, \Pi, \vec{\gamma}, \Delta \Rightarrow \alpha.}\ (\mathcal{R}^{\Pi^-}_{\Pi})\ (\text{ind. hyp.})$$

(Case 4-3): P is of the form

$$\dfrac{\begin{matrix} \vdots\, Q \\ \Pi, \vec{\beta}, \vec{\gamma}, \Sigma_1, \Sigma_2 \Rightarrow \psi \end{matrix}}{\Pi, \vec{\gamma}, \vec{\beta}, \Sigma_1, \Sigma_2 \Rightarrow \psi,}\ (\mathrm{e}^{110})$$

and the redex Φ is $(\Pi, \vec{\gamma}, \vec{\beta}, \Sigma_1)$. We apply the induction hypothesis to Q in which the redex is $\Theta \equiv (\Pi, \vec{\beta}, \vec{\gamma}, \Sigma_1)$; and we get a sequence Θ^- and a proof Q^-. Then the required sequence Φ^- is Θ^-, and the required proof P^- is

$$\begin{matrix} \vdots\, Q^- \\ \Theta^-, \Sigma_2 \Rightarrow \psi. \end{matrix}$$

Derivability of $\mathcal{R}^{\Theta^-}_{\Pi,\vec{\gamma},\vec{\beta},\Sigma_1}$ is shown as follows:

$$\dfrac{\dfrac{\Gamma, \Theta^-, \Delta \Rightarrow \alpha}{\Gamma, \Pi, \vec{\beta}, \vec{\gamma}, \Sigma_1, \Delta \Rightarrow \alpha}\ (\mathcal{R}^{\Theta^-}_{\Pi,\vec{\beta},\vec{\gamma},\Sigma_1})\ (\text{ind. hyp.})}{\Gamma, \Pi, \vec{\gamma}, \vec{\beta}, \Sigma_1, \Delta \Rightarrow \alpha.}\ (\mathrm{e}^{110})$$

∎

We need one more lemma.

Lemma 3.7 *If* $\Gamma \Rightarrow \alpha \to \beta$ *is cut-free provable in* $\mathrm{FL}_{\to}+$ (e^{110}), *then also* $\Gamma, \alpha \Rightarrow \beta$ *is cut-free provable in this system.*

Proof By induction on the cut-free proof of $\Gamma \Rightarrow \alpha \to \beta$. ∎

Now we can prove the cut-elimination theorem for $\mathrm{FL}_{\to}+$ (e^{110}).

Theorem 3.8 *The cut rule is admissible in the cut-free part of* $\mathrm{FL}_{\to}+$ (e^{110}).

Proof Let P be a proof

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \phi\end{array} \quad \begin{array}{c}\vdots\, R\\ \Psi_1, \phi, \Psi_2 \Rightarrow \psi\end{array}}{\Psi_1, \Phi, \Psi_2 \Rightarrow \psi}\ (\mathrm{cut})$$

where Q and R are cut-free proofs in $\mathrm{FL}_{\to}+$ (e^{110}). We define the grade g of this cut to be the length of the formula ϕ, and the rank r of this cut to be the length of the proof R. We prove, by double induction on the grade and rank of this cut (i.e., by transfinite induction on $\omega \cdot g + r$), that there is a cut-free proof of $\Psi_1, \Phi, \Psi_2 \Rightarrow \psi$ in $\mathrm{FL}_{\to}+$ (e^{110}). We distinguish cases according to the form of R.

(Case 1): R is an initial sequent. In this case, P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \phi\end{array} \quad \phi \Rightarrow \phi}{\Phi \Rightarrow \phi,}\ (\mathrm{cut})$$

and Q is the required proof.

(Case 2-1): The last inference of R is ($\to$left), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \phi\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \Delta_1, \phi, \Delta_2, \beta, \Sigma \Rightarrow \psi\end{array}}{\Delta_1, \phi, \Delta_2, \alpha \to \beta, \Gamma, \Sigma \Rightarrow \psi}\ (\to\mathrm{left})}{\Delta_1, \Phi, \Delta_2, \alpha \to \beta, \Gamma, \Sigma \Rightarrow \psi.}\ (\mathrm{cut})$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{array} \quad \dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \phi\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \Delta_1, \phi, \Delta_2, \beta, \Sigma \Rightarrow \psi\end{array}}{\Delta_1, \Phi, \Delta_2, \beta, \Sigma \Rightarrow \psi}\ (\mathrm{cut})}{\Delta_1, \Phi, \Delta_2, \alpha \to \beta, \Gamma, \Sigma \Rightarrow \psi.}\ (\to\mathrm{left})$$

(The rank of this cut is less than r.)

(Case 2-2): The last inference of R is ($\to$left), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \alpha \to \beta\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \Delta, \beta, \Sigma \Rightarrow \psi\end{array}}{\Delta, \alpha \to \beta, \Gamma, \Sigma \Rightarrow \psi}\ (\to\mathrm{left})}{\Delta, \Phi, \Gamma, \Sigma \Rightarrow \psi.}\ (\mathrm{cut})$$

Then consider the proof

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, Q'\\ \Phi, \alpha \Rightarrow \beta\end{array}}{\Phi, \Gamma \Rightarrow \beta}\ (\mathrm{cut}) \quad \begin{array}{c}\vdots\, R_2\\ \Delta, \beta, \Sigma \Rightarrow \psi\end{array}}{\Delta, \Phi, \Gamma, \Sigma \Rightarrow \psi}\ (\mathrm{cut})$$

where Q' is the proof obtained from Q by applying Lemma 3.7. The grades of these cuts are less than g, and we get the required proof by the induction hypotheses.

(Case 2-3): The last inference of R is ($\rightarrow$left), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \phi\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma_1, \phi, \Gamma_2 \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \Delta, \beta, \Sigma \Rightarrow \psi\end{array}}{\Delta, \alpha\rightarrow\beta, \Gamma_1, \phi, \Gamma_2, \Sigma \Rightarrow \psi}\,(\rightarrow\text{left})}{\Delta, \alpha\rightarrow\beta, \Gamma_1, \Phi, \Gamma_2, \Sigma \Rightarrow \psi.}\,(\text{cut})$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \phi\end{array} \quad \begin{array}{c}\vdots\, R_1\\ \Gamma_1, \phi, \Gamma_2 \Rightarrow \alpha\end{array}}{\Gamma_1, \Phi, \Gamma_2 \Rightarrow \alpha}\,(\text{cut}) \quad \begin{array}{c}\vdots\, R_2\\ \Delta, \beta, \Sigma \Rightarrow \psi\end{array}}{\Delta, \alpha\rightarrow\beta, \Gamma_1, \Phi, \Gamma_2, \Sigma \Rightarrow \psi.}\,(\rightarrow\text{left})$$

(The rank of this cut is less than r.)

(Case 2-4): The last inference of R is ($\rightarrow$left), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \phi\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \Delta, \beta, \Sigma_1, \phi, \Sigma_2 \Rightarrow \psi\end{array}}{\Delta, \alpha\rightarrow\beta, \Gamma, \Sigma_1, \phi, \Sigma_2 \Rightarrow \psi}\,(\rightarrow\text{left})}{\Delta, \alpha\rightarrow\beta, \Gamma, \Sigma_1, \Phi, \Sigma_2 \Rightarrow \psi.}\,(\text{cut})$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{array} \quad \dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \phi\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \Delta, \beta, \Sigma_1, \phi, \Sigma_2 \Rightarrow \psi\end{array}}{\Delta, \beta, \Sigma_1, \Phi, \Sigma_2 \Rightarrow \psi}\,(\text{cut})}{\Delta, \alpha\rightarrow\beta, \Gamma, \Sigma_1, \Phi, \Sigma_2 \Rightarrow \psi.}\,(\rightarrow\text{left})$$

(The rank of this cut is less than r.)

(Case 3): The last inference of R is ($\rightarrow$right). In this case, P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \phi\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_0\\ \Gamma, \phi, \Delta, \alpha \Rightarrow \beta\end{array}}{\Gamma, \phi, \Delta \Rightarrow \alpha\rightarrow\beta}\,(\rightarrow\text{right})}{\Gamma, \Phi, \Delta \Rightarrow \alpha\rightarrow\beta,}\,(\text{cut})$$

and the required proof is obtained from the proof

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \phi\end{array} \quad \begin{array}{c}\vdots\, R_0\\ \Gamma, \phi, \Delta, \alpha \Rightarrow \beta\end{array}}{\Gamma, \Phi, \Delta, \alpha \Rightarrow \beta}\,(\text{cut})}{\Gamma, \Phi, \Delta \Rightarrow \alpha\rightarrow\beta}\,(\rightarrow\text{right})$$

by the induction hypothesis. (The rank of this cut is less than r.)

(Case 4-1): The last inference of R is (e^{110}), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi\Rightarrow\phi\end{array}\quad \dfrac{\begin{array}{c}\vdots\, R_0\\ \Gamma_1,\phi,\Gamma_2,\overrightarrow{\alpha},\overrightarrow{\beta},\Delta\Rightarrow\psi\end{array}}{\Gamma_1,\phi,\Gamma_2,\overrightarrow{\beta},\overrightarrow{\alpha},\Delta\Rightarrow\psi}\,(e^{110})}{\Gamma_1,\Phi,\Gamma_2,\overrightarrow{\beta},\overrightarrow{\alpha},\Delta\Rightarrow\psi.}\,(\mathrm{cut})$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi\Rightarrow\phi\end{array}\quad \begin{array}{c}\vdots\, R_0\\ \Gamma_1,\phi,\Gamma_2,\overrightarrow{\alpha},\overrightarrow{\beta},\Delta\Rightarrow\psi\end{array}}{\Gamma_1,\Phi,\Gamma_2,\overrightarrow{\alpha},\overrightarrow{\beta},\Delta\Rightarrow\psi}\,(\mathrm{cut})}{\Gamma_1,\Phi,\Gamma_2,\overrightarrow{\beta},\overrightarrow{\alpha},\Delta\Rightarrow\psi.}\,(e^{110})$$

(The rank of this cut is less than r.)

(Case 4-2): The last inference of R is (e^{110}), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi\Rightarrow\overrightarrow{\beta}\end{array}\quad \dfrac{\begin{array}{c}\vdots\, R_0\\ \Gamma,\overrightarrow{\alpha},\overrightarrow{\beta},\Delta\Rightarrow\psi\end{array}}{\Gamma,\overrightarrow{\beta},\overrightarrow{\alpha},\Delta\Rightarrow\psi}\,(e^{110})}{\Gamma,\Phi,\overrightarrow{\alpha},\Delta\Rightarrow\psi.}\,(\mathrm{cut})$$

We apply Lemma 3.6 to Q in which the redex is Φ; and we get a sequence Φ^- of non-atomic formulas, a cut-free proof Q^- of $\Phi^-\Rightarrow\overrightarrow{\beta}$, and cut-free derivability of the rule $\mathcal{R}^{\Phi^-}_{\Phi}$. Then consider the proof

$$\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\, Q^-\\ \Phi^-\Rightarrow\overrightarrow{\beta}\end{array}\quad \begin{array}{c}\vdots\, R_0\\ \Gamma,\overrightarrow{\alpha},\overrightarrow{\beta},\Delta\Rightarrow\psi\end{array}}{\Gamma,\overrightarrow{\alpha},\Phi^-,\Delta\Rightarrow\psi}\,(\mathrm{cut})\\ \vdots\,(e^{110})\\ \Gamma,\Phi^-,\overrightarrow{\alpha},\Delta\Rightarrow\psi\end{array}}{\Gamma,\Phi,\overrightarrow{\alpha},\Delta\Rightarrow\psi.}\,(\mathcal{R}^{\Phi^-}_{\Phi})$$

The rank of this cut is less than r, and we get the required proof by the induction hypothesis.

(Case 4-3): The last inference of R is (e^{110}), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\ Q\\ \Phi \Rightarrow \overrightarrow{\alpha}\end{array} \quad \dfrac{\begin{array}{c}\vdots\ R_0\\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\beta}, \Delta \Rightarrow \psi\end{array}}{\Gamma, \overrightarrow{\beta}, \overrightarrow{\alpha}, \Delta \Rightarrow \psi}\,(e^{110})}{\Gamma, \overrightarrow{\beta}, \Phi, \Delta \Rightarrow \psi.}\,(\mathrm{cut})$$

We apply Lemma 3.6 to Q in which the redex is Φ; and we get a sequence Φ^- of non-atomic formulas, a cut-free proof Q^- of $\Phi^- \Rightarrow \overrightarrow{\alpha}$, and cut-free derivability of the rule $\mathcal{R}_{\Phi}^{\Phi^-}$. Then consider the proof

$$\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\ Q^-\\ \Phi^- \Rightarrow \overrightarrow{\alpha}\end{array} \quad \begin{array}{c}\vdots\ R_0\\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\beta}, \Delta \Rightarrow \psi\end{array}}{\Gamma, \Phi^-, \overrightarrow{\beta}, \Delta \Rightarrow \psi}\,(\mathrm{cut})\\ \vdots\ (e^{110})\\ \Gamma, \overrightarrow{\beta}, \Phi^-, \Delta \Rightarrow \psi\end{array}}{\Gamma, \overrightarrow{\beta}, \Phi, \Delta \Rightarrow \psi.}\,(\mathcal{R}_{\Phi}^{\Phi^-})$$

The rank of this cut is less than r, and we get the required proof by the induction hypothesis.

(Case 4-4): The last inference of R is (e^{110}), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\ Q\\ \Phi \Rightarrow \phi\end{array} \quad \dfrac{\begin{array}{c}\vdots\ R_0\\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\beta}, \Delta_1, \phi, \Delta_2 \Rightarrow \psi\end{array}}{\Gamma, \overrightarrow{\beta}, \overrightarrow{\alpha}, \Delta_1, \phi, \Delta_2 \Rightarrow \psi}\,(e^{110})}{\Gamma, \overrightarrow{\beta}, \overrightarrow{\alpha}, \Delta_1, \Phi, \Delta_2 \Rightarrow \psi.}\,(\mathrm{cut})$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ Q\\ \Phi \Rightarrow \phi\end{array} \quad \begin{array}{c}\vdots\ R_0\\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\beta}, \Delta_1, \phi, \Delta_2 \Rightarrow \psi\end{array}}{\Gamma, \overrightarrow{\alpha}, \overrightarrow{\beta}, \Delta_1, \Phi, \Delta_2 \Rightarrow \psi}\,(\mathrm{cut})}{\Gamma, \overrightarrow{\beta}, \overrightarrow{\alpha}, \Delta_1, \Phi, \Delta_2 \Rightarrow \psi.}\,(e^{110})$$

(The rank of this cut is less than r.) ∎

4 Systems with e^{*01}

In this section, we investigate the systems with the rule (e^{101}) and/or (e^{001}).

Theorem 4.1 (1) $\mathrm{FL}_{\rightarrow}+$ $(\mathrm{e}^{101}) \vdash \beta, \beta\rightarrow\vec{\gamma} \Rightarrow \vec{\gamma}$.
(2) $\mathrm{FL}_{\rightarrow}+$ $(\mathrm{e}^{101}) \vdash \alpha\rightarrow\beta\rightarrow\vec{\gamma}, \beta, \alpha \Rightarrow \vec{\gamma}$.
(3) *The rule* (e^{001}) *is derivable in* $\mathrm{FL}_{\rightarrow}+$ (e^{101}). *(Therefore the rules* (e^{101}) *and* (e^{001}) *are rule-equivalent over* $\mathrm{FL}_{\rightarrow}$ *(see Figure 2).)*

Proof (1)

$$\dfrac{\dfrac{\beta \Rightarrow \beta \quad \vec{\gamma} \Rightarrow \vec{\gamma}}{\beta\rightarrow\vec{\gamma}, \beta \Rightarrow \vec{\gamma}}\,(\rightarrow\text{left})}{\beta, \beta\rightarrow\vec{\gamma} \Rightarrow \vec{\gamma}.}\,(\mathrm{e}^{101})$$

(2)

$$\dfrac{\dfrac{\begin{matrix}\vdots\,(1)\\ \beta, \beta\rightarrow\vec{\gamma} \Rightarrow \vec{\gamma}\end{matrix}}{\beta \Rightarrow (\beta\rightarrow\vec{\gamma})\rightarrow\vec{\gamma}}\,(\rightarrow\text{right}) \qquad \dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \quad \beta\rightarrow\vec{\gamma} \Rightarrow \beta\rightarrow\vec{\gamma}}{\alpha\rightarrow\beta\rightarrow\vec{\gamma}, \alpha \Rightarrow \beta\rightarrow\vec{\gamma}}\,(\rightarrow\text{left}) \qquad \vec{\gamma} \Rightarrow \vec{\gamma}}{(\beta\rightarrow\vec{\gamma})\rightarrow\vec{\gamma}, \alpha\rightarrow\beta\rightarrow\vec{\gamma}, \alpha \Rightarrow \vec{\gamma}}\,(\rightarrow\text{left})}{\alpha\rightarrow\beta\rightarrow\vec{\gamma}, (\beta\rightarrow\vec{\gamma})\rightarrow\vec{\gamma}, \alpha \Rightarrow \vec{\gamma}}\,(\mathrm{e}^{101})}{\alpha\rightarrow\beta\rightarrow\vec{\gamma}, \beta, \alpha \Rightarrow \vec{\gamma}.}\,(\text{cut})$$

(3): Similar to Theorem 3.2 (3). ∎

Corollary 4.2 *Among the twenty systems in Figure 1, the eight systems* $\mathrm{FL}_{\rightarrow}+$ (e^{101}), $\mathrm{FL}_{\rightarrow}+$ (e^{101}) + (e^{110}), $\mathrm{FL}_{\rightarrow}+$ (e^{101}) + (e^{011}), $\mathrm{FL}_{\rightarrow}+$ (e^{101}) + (e^{110}) + (e^{011}), $\mathrm{FL}_{\rightarrow}+$ (e^{001}), $\mathrm{FL}_{\rightarrow}+$ (e^{001}) + (e^{110}), $\mathrm{FL}_{\rightarrow}+$ (e^{010}) + (e^{101}), *and* $\mathrm{FL}_{\rightarrow}+$ (e^{001}) + (e^{010}) *are theorem-equivalent each other.*

Proof By Theorem 3.2 (3) and Theorem 4.1 (3). ∎

These eight systems will be called e^{*01}*-systems.*

Theorem 4.3 *Each* e^{*01}*-system is strictly stronger than* e^{*1*}*-systems.*

Proof We can easily verify that the sequent $\beta, \beta\rightarrow\gamma_1\rightarrow\gamma_2 \Rightarrow \gamma_1\rightarrow\gamma_2$ is not cut-free provable in $\mathrm{FL}_{\rightarrow}+$ (e^{110}) if β, γ_1 and γ_2 are mutually distinct propositional variables. Therefore the cut-elimination theorem for $\mathrm{FL}_{\rightarrow}+$ (e^{110}) (Theorem 3.8) and Theorem 4.1(1) imply this theorem. ∎

Theorem 4.4 *The cut-elimination theorem does not hold for any* e^{*01}*-systems.*

Proof We have a proof in e^{*01}-systems:

$$\dfrac{\dfrac{\beta\Rightarrow\beta}{\Rightarrow\beta\to\beta}\,(\to\text{right}) \qquad \dfrac{\dfrac{\alpha\Rightarrow\alpha \quad \dfrac{\beta\to\beta\Rightarrow\beta\to\beta \quad \gamma\Rightarrow\gamma}{(\beta\to\beta)\to\gamma,\beta\to\beta\Rightarrow\gamma}\,(\to\text{left})}{\alpha\to(\beta\to\beta)\to\gamma,\alpha,\beta\to\beta\Rightarrow\gamma}\,(\to\text{left})}{\alpha,\alpha\to(\beta\to\beta)\to\gamma,\beta\to\beta\Rightarrow\gamma}\,(e^{*01})}{\alpha,\alpha\to(\beta\to\beta)\to\gamma\Rightarrow\gamma,}\,(\text{cut})$$

while it is easily shown that $FL_{\to}$+ (e^{001}) + (e^{010}) and other e^{*01}-systems have no cut-free proof of this sequent if α and γ are propositional variables. ∎

Then we wish to obtain a system which is theorem-equivalent to e^{*01}-systems and enjoys the cut-elimination theorem. For this purpose, we introduce a rule of inference

$$\dfrac{\Gamma\Rightarrow\alpha \quad \Delta,\beta,\Sigma\Rightarrow\gamma}{\{\Delta,\alpha\to\beta,\Gamma\},\Sigma\Rightarrow\gamma}\,(\to\text{left 2})$$

where either α or β is an implication, and $\{\Delta,\alpha\to\beta,\Gamma\}$ is an arbitrary permutation of the sequence $(\Delta,\alpha\to\beta,\Gamma)$. We define a system L to be $FL_{\to}$+ (e^{001}) + (e^{010}) + ($\to$left 2), and L^- to be the cut-free part of L. We will show that L, L^-, and e^{*01}-systems are theorem-equivalent each other. Hence L is a system we wish to obtain.

Theorem-equivalency of L and e^{*01}-systems is proved by the following theorem.

Theorem 4.5 *The rule ($\to$left 2) is derivable in any e^{*01}-systems.*

Proof We show that ($\to$left 2) is derivable in $FL_{\to}$+ (e^{001}). (Then the derivability in e^{*01}-systems is obvious by Theorem 4.1 (3).) When $\alpha\equiv\alpha_1\to\alpha_2$, we have

$$\dfrac{\dfrac{\alpha_2\Rightarrow\alpha_2}{\Rightarrow\alpha_2\to\alpha_2}\,(\to\text{right}) \qquad \begin{array}{c}\dfrac{\dfrac{\Gamma\Rightarrow\alpha \quad \begin{array}{c}\vdots\ \text{Lemma 3.1}\\ \alpha,\alpha_2\to\alpha_2\Rightarrow\alpha\end{array}}{\Gamma,\alpha_2\to\alpha_2\Rightarrow\alpha}\,(\text{cut}) \quad \Delta,\beta,\Sigma\Rightarrow\gamma}{\Delta,\alpha\to\beta,\Gamma,\alpha_2\to\alpha_2,\Sigma\Rightarrow\gamma}\,(\to\text{left})\\ \vdots\ (e^{001})\\ \{\Delta,\alpha\to\beta,\Gamma\},\alpha_2\to\alpha_2,\Sigma\Rightarrow\gamma\end{array}}{\{\Delta,\alpha\to\beta,\Gamma\},\Sigma\Rightarrow\gamma.}\,(\text{cut})$$

When $\beta\equiv\beta_1\to\beta_2$, we have

$$\dfrac{\dfrac{\beta_2\Rightarrow\beta_2}{\Rightarrow\beta_2\to\beta_2}\,(\to\text{right}) \qquad \begin{array}{c}\dfrac{\Gamma\Rightarrow\alpha \quad \dfrac{\begin{array}{c}\vdots\ \text{Lemma 3.1}\\ \beta,\beta_2\to\beta_2\Rightarrow\beta\end{array} \quad \Delta,\beta,\Sigma\Rightarrow\gamma}{\Delta,\beta,\beta_2\to\beta_2,\Sigma\Rightarrow\gamma}\,(\text{cut})}{\Delta,\alpha\to\beta,\Gamma,\beta_2\to\beta_2,\Sigma\Rightarrow\gamma}\,(\to\text{left})\\ \vdots\ (e^{001})\\ \{\Delta,\alpha\to\beta,\Gamma\},\beta_2\to\beta_2,\Sigma\Rightarrow\gamma\end{array}}{\{\Delta,\alpha\to\beta,\Gamma\},\Sigma\Rightarrow\gamma.}\,(\text{cut})$$

∎

Theorem-equivalency of L and L^- (i.e., the cut-elimination theorem for L) will be shown in Theorem 4.10. For the proof of this theorem, we give some lemmas.

Lemma 4.6 *If* $\Gamma \Rightarrow \alpha \rightarrow \beta$ *is provable in* L^-, *then also* $\Gamma, \alpha \Rightarrow \beta$ *is provable in* L^-.

Proof By induction on the proof of L^-. ∎

Let ϕ be a formula. Then the rule

$$\frac{\Gamma \Rightarrow \phi \quad \Delta, \phi, \Sigma \Rightarrow \alpha}{\Delta, \Gamma, \Sigma \Rightarrow \alpha}$$

is called *ϕ-cut.* (ϕ-cut is the cut rule whose cut-formula is ϕ.)

Lemma 4.7 *The rule p-cut is admissible in* L^- *for any atomic formula p.*

Proof Let P be a proof

$$\frac{\overset{\vdots\, Q}{\Gamma \Rightarrow p} \quad \overset{\vdots\, R}{\Delta, p, \Sigma \Rightarrow \alpha}}{\Delta, \Gamma, \Sigma \Rightarrow \alpha} \ (p\text{-cut})$$

where Q and R are proofs in L^-. We prove, by induction on Q, that there is a proof of $\Delta, \Gamma, \Sigma \Rightarrow \alpha$ in L^-. We distinguish cases according to the form of Q.

(Case 1): The last inference of Q is (e^{001}). In this case, P is of the form

$$\frac{\dfrac{\overset{\vdots\, Q_0}{\Gamma_1, \beta, \gamma, \Gamma_2 \Rightarrow p}}{\Gamma_1, \gamma, \beta, \Gamma_2 \Rightarrow p} \ (e^{001}) \quad \overset{\vdots\, R}{\Delta, p, \Sigma \Rightarrow \alpha}}{\Delta, \Gamma_1, \gamma, \beta, \Gamma_2, \Sigma \Rightarrow \alpha} \ (p\text{-cut})$$

where Γ_2 is not empty, and the required proof is obtained from the proof

$$\dfrac{\dfrac{\overset{\vdots\, Q_0}{\Gamma_1, \beta, \gamma, \Gamma_2 \Rightarrow p} \quad \overset{\vdots\, R}{\Delta, p, \Sigma \Rightarrow \alpha}}{\Delta, \Gamma_1, \beta, \gamma, \Gamma_2, \Sigma \Rightarrow \alpha} \ (p\text{-cut})}{\Delta, \Gamma_1, \gamma, \beta, \Gamma_2, \Sigma \Rightarrow \alpha} \ (e^{001})$$

by the induction hypothesis.

(Case 2): The last inference of Q is ($\rightarrow$left 2). In this case, P is of the form

$$\frac{\dfrac{\overset{\vdots\, Q_1}{\Pi \Rightarrow \beta} \quad \overset{\vdots\, Q_2}{\Lambda, \gamma, \Theta \Rightarrow p}}{\{\Lambda, \beta \rightarrow \gamma, \Pi\}, \Theta \Rightarrow p} \ (\rightarrow\text{left } 2) \quad \overset{\vdots\, R}{\Delta, p, \Sigma \Rightarrow \alpha}}{\Delta, \{\Lambda, \beta \rightarrow \gamma, \Pi\}, \Theta, \Sigma \Rightarrow \alpha,} \ (p\text{-cut})$$

and the required proof is obtained from the proof

$$\dfrac{\begin{array}{c}\vdots\, Q_1\\ \Pi \Rightarrow \beta\end{array} \quad \dfrac{\begin{array}{c}\vdots\, Q_2\\ \Lambda, \gamma, \Theta \Rightarrow p\end{array} \quad \begin{array}{c}\vdots\, R\\ \Delta, p, \Sigma \Rightarrow \alpha\end{array}}{\Delta, \Lambda, \gamma, \Theta, \Sigma \Rightarrow \alpha}\,(p\text{-cut})}{\Delta, \{\Lambda, \beta \rightarrow \gamma, \Pi\}, \Theta, \Sigma \Rightarrow \alpha}\,(\rightarrow\text{left } 2)$$

by the induction hypothesis.

The other cases are similar. ∎

Lemma 4.8 *Suppose there are proofs Q of $\Phi \Rightarrow \overrightarrow{\phi}$ and R of $\Psi, \overrightarrow{\phi} \Rightarrow p$ in L^- where $\overrightarrow{\phi}$ is an implication and p is an atomic formula. If ϕ'-cut is admissible in L^- for any proper subformula ϕ' of $\overrightarrow{\phi}$, then the sequence (Φ, Ψ) contains at least one implication.*

Proof We define the grade of the pair $\langle Q, R\rangle$ of proofs to be the length of the formula $\overrightarrow{\phi}$, and the rank of $\langle Q, R\rangle$ to be the sum of the lengths of the proofs Q and R. We prove, by double induction on the grade and rank of $\langle Q, R\rangle$, that there is an implication in (Φ, Ψ). We distinguish cases according to the forms of Q and R.

(Case 1): Q is an initial sequent, or the last inference of Q is ($\rightarrow$left), (e^{010}), or ($\rightarrow$left 2). In this case, Φ contains an implication.

(Case 2): The last inference of R is ($\rightarrow$left) or ($\rightarrow$left 2) by which an implication is introduced in Ψ; or the last inference of R is (e^{010}). In this case, Ψ contains an implication.

(Case 3): The last inference of Q is (e^{001}). Let Q be of the form

$$\dfrac{\begin{array}{c}\vdots\, Q_0\\ \Phi_0 \Rightarrow \overrightarrow{\phi}\end{array}}{\Phi \Rightarrow \overrightarrow{\phi}.}\,(\mathrm{e}^{001})$$

Then Φ_0 is a permutation of Φ, and (Φ_0, Ψ) contains an implication by the induction hypothesis for $\langle Q_0, R\rangle$. (The rank is decreased.)

(Case 4): The last inference of R is (e^{001}). In this case, R is of the form

$$\dfrac{\begin{array}{c}\vdots\, R_0\\ \Psi_0, \overrightarrow{\phi} \Rightarrow p\end{array}}{\Psi, \overrightarrow{\phi} \Rightarrow p}\,(\mathrm{e}^{001})$$

and Ψ_0 is a permutation of Ψ. Then (Φ, Ψ_0) contains an implication by the induction hypothesis for $\langle Q, R_0\rangle$. (The rank is decreased.)

(Case 5): The last inference of Q is ($\rightarrow$right), and the last inference of R is ($\rightarrow$left) or ($\rightarrow$left 2) by which $\overrightarrow{\phi}$ is introduced. Let $\overrightarrow{\phi} \equiv \phi_1 \rightarrow \phi_2$. Then Q is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q_0 \\ \Phi, \phi_1 \Rightarrow \phi_2\end{array}}{\Phi \Rightarrow \overrightarrow{\phi},}\ (\rightarrow\text{right})$$

and R is of the form

$$\dfrac{\begin{array}{cc}\vdots\, R_1 & \vdots\, R_2 \\ \Psi_1 \Rightarrow \phi_1 & \Psi_2, \phi_2 \Rightarrow p\end{array}}{\Psi, \overrightarrow{\phi} \Rightarrow p}\ x$$

where x is either ($\rightarrow$left) or ($\rightarrow$left 2), and (Ψ_1, Ψ_2) is a permutation of Ψ. We show that either ϕ_1 or ϕ_2 is an implication: Suppose ϕ_1 is atomic. Then Ψ_1 is not empty since R_1 is a proof in L^-. This implies that the rule x is ($\rightarrow$left 2), and therefore ϕ_2 is an implication. Now we consider two cases.

(Subcase 5-1): ϕ_1 is an implication. Consider the proof

$$\dfrac{\begin{array}{cc}\vdots\, Q_0 & \vdots\, R_2 \\ \Phi, \phi_1 \Rightarrow \phi_2 & \Psi_2, \phi_2 \Rightarrow p\end{array}}{\Psi_2, \Phi, \phi_1 \Rightarrow p.}\ (\phi_2\text{-cut})$$

Then, by the admissibility of ϕ_2-cut, we get a proof P of $\Psi_2, \Phi, \phi_1 \Rightarrow p$ in L^-; and (Ψ_1, Ψ_2, Φ) contains an implication by the induction hypothesis for $\langle R_1, P\rangle$. (The grade is decreased, and the admissibility of the cut rules for proper subformulas of ϕ_1 is implied by that of ϕ.)

(Subcase 5-2): ϕ_2 is an implication. Consider the proof

$$\dfrac{\begin{array}{cc}\vdots\, R_1 & \vdots\, Q_0 \\ \Psi_1 \Rightarrow \phi_1 & \Phi, \phi_1 \Rightarrow \phi_2\end{array}}{\Phi, \Psi_1 \Rightarrow \phi_2.}\ (\phi_1\text{-cut})$$

Then, by the admissibility of ϕ_1-cut, we get a proof P of $\Phi, \Psi_1 \Rightarrow \phi_2$ in L^-; and (Φ, Ψ_1, Ψ_2) contains an implication by the induction hypothesis for $\langle P, R_2\rangle$. ∎

We introduce a new rule

$$\dfrac{\Phi \Rightarrow \overrightarrow{\phi} \quad \Psi_1, \overrightarrow{\phi}, \Psi_2 \Rightarrow \psi}{\{\Psi_1, \Phi\}, \Psi_2 \Rightarrow \psi}\ (\text{cut 2})$$

where $\{\Psi_1, \Phi\}$ denotes an arbitrary permutation of (Ψ_1, Φ). Note that ϕ-cut is an instance of (cut 2) if ϕ is an implication. Now the following lemma is the main part of our cut-elimination proof for L.

Lemma 4.9 *The rule (cut 2) is admissible in L^-.*

Proof Let P be a proof

$$\dfrac{\begin{array}{c}\vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\phi}\end{array} \quad \begin{array}{c}\vdots\, R \\ \Psi_1, \overrightarrow{\phi}, \Psi_2 \Rightarrow \psi\end{array}}{\{\Psi_1, \Phi\}, \Psi_2 \Rightarrow \psi}\ (\text{cut } 2)$$

where Q and R are proofs in L^-. We define the grade g of this (cut 2) to be the length of the formula $\overrightarrow{\phi}$, and the rank r of this (cut 2) to be the length of the proof R. We prove, by double induction on the grade and rank of this (cut 2), that there is a proof of $\{\Psi_1, \Phi\}, \Psi_2 \Rightarrow \psi$ in L^-. We distinguish cases according to the form of R.

(Case 1): R is an initial sequent. In this case, P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\phi}\end{array} \quad \overrightarrow{\phi} \Rightarrow \overrightarrow{\phi}}{\{\Phi\} \Rightarrow \overrightarrow{\phi},}\ (\text{cut } 2)$$

and the required proof is

$$\begin{array}{c}\vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\phi} \\ \vdots\, (\mathrm{e}^{001}) \\ \{\Phi\} \Rightarrow \overrightarrow{\phi}.\end{array}$$

(Case 2-1): The last inference of R is ($\rightarrow$left), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\phi}\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_1 \\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R_2 \\ \Delta_1, \overrightarrow{\phi}, \Delta_2, \beta, \Sigma \Rightarrow \psi\end{array}}{\Delta_1, \overrightarrow{\phi}, \Delta_2, \alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \psi}\ (\rightarrow\text{left})}{\{\Delta_1, \Phi\}, \Delta_2, \alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \psi.}\ (\text{cut } 2)$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\begin{array}{c}\vdots\, R_1 \\ \Gamma \Rightarrow \alpha\end{array} \quad \dfrac{\begin{array}{c}\vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\phi}\end{array} \quad \begin{array}{c}\vdots\, R_2 \\ \Delta_1, \overrightarrow{\phi}, \Delta_2, \beta, \Sigma \Rightarrow \psi\end{array}}{\{\Delta_1, \Phi\}, \Delta_2, \beta, \Sigma \Rightarrow \psi}\ (\text{cut } 2)}{\{\Delta_1, \Phi\}, \Delta_2, \alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \psi.}\ (\rightarrow\text{left})$$

(The rank of this (cut 2) is less than r.)

(Case 2-2): The last inference of R is ($\rightarrow$left), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q \\ \Phi \Rightarrow \alpha \rightarrow \beta\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_1 \\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R_2 \\ \Delta, \beta, \Sigma \Rightarrow \psi\end{array}}{\Delta, \alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \psi}\ (\rightarrow\text{left})}{\{\Delta, \Phi\}, \Gamma, \Sigma \Rightarrow \psi.}\ (\text{cut } 2)$$

Then consider the proof

$$\dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\, Q'\\ \Phi,\alpha\Rightarrow\beta\end{array}\quad \begin{array}{c}\vdots\, R_2\\ \Delta,\beta,\Sigma\Rightarrow\psi\end{array}}{\Delta,\Phi,\alpha,\Sigma\Rightarrow\psi}\,(\beta\text{-cut})\\ \vdots\,(\mathrm{e}^{001})\\ \{\Delta,\Phi\},\alpha,\Sigma\Rightarrow\psi\end{array}}{\{\Delta,\Phi\},\Gamma,\Sigma\Rightarrow\psi}\,(\alpha\text{-cut})$$

where Q' is the proof obtained from Q by applying Lemma 4.6. When the formula α (or β) is an implication, the admissibility of α-cut (or β-cut) is guaranteed by the induction hypothesis because it is an instance of (cut 2) with less grade than g. When the formula α (or β) is atomic, the admissibility of α-cut (or β-cut) is guaranteed by Lemma 4.7. Then we get the required proof.

(Case 2-3): The last inference of R is ($\rightarrow$left), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi\Rightarrow\overrightarrow{\phi}\end{array} \qquad \dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma_1,\overrightarrow{\phi},\Gamma_2\Rightarrow\alpha\end{array}\quad \begin{array}{c}\vdots\, R_2\\ \Delta,\beta,\Sigma\Rightarrow\psi\end{array}}{\Delta,\alpha\rightarrow\beta,\Gamma_1,\overrightarrow{\phi},\Gamma_2,\Sigma\Rightarrow\psi}\,(\rightarrow\text{left})}{\{\Delta,\alpha\rightarrow\beta,\Gamma_1,\Phi\},\Gamma_2,\Sigma\Rightarrow\psi.}\,(\text{cut 2})$$

(Subcase 2-3-1): α is an implication. In this case, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi\Rightarrow\overrightarrow{\phi}\end{array}\quad \begin{array}{c}\vdots\, R_1\\ \Gamma_1,\overrightarrow{\phi},\Gamma_2\Rightarrow\alpha\end{array}}{\Gamma_1,\Phi,\Gamma_2\Rightarrow\alpha}\,(\text{cut 2}) \qquad \begin{array}{c}\vdots\, R_2\\ \Delta,\beta,\Sigma\Rightarrow\psi\end{array}}{\{\Delta,\alpha\rightarrow\beta,\Gamma_1,\Phi\},\Gamma_2,\Sigma\Rightarrow\psi.}\,(\rightarrow\text{left 2})$$

(The rank of this (cut 2) is less than r.)

(Subcase 2-3-2): Γ_2 is not empty. In this case, by the induction hypothesis, the required proof is obtained from the proof

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi\Rightarrow\overrightarrow{\phi}\end{array}\quad \begin{array}{c}\vdots\, R_1\\ \Gamma_1,\overrightarrow{\phi},\Gamma_2\Rightarrow\alpha\end{array}}{\Gamma_1,\Phi,\Gamma_2\Rightarrow\alpha}\,(\text{cut 2}) \qquad \begin{array}{c}\vdots\, R_2\\ \Delta,\beta,\Sigma\Rightarrow\psi\end{array}}{\Delta,\alpha\rightarrow\beta,\Gamma_1,\Phi,\Gamma_2,\Sigma\Rightarrow\psi}\,(\rightarrow\text{left})\\ \vdots\,(\mathrm{e}^{001})\\ \{\Delta,\alpha\rightarrow\beta,\Gamma_1,\Phi\},\Gamma_2,\Sigma\Rightarrow\psi.\end{array}$$

(The rank of this (cut 2) is less than r.)

(Subcase 2-3-3): α is atomic, and Γ_2 is empty. In this case, we apply Lemma 4.8 to Q and R_1. (The admissibility of ϕ'-cut for each proper subformula ϕ' of $\overrightarrow{\phi}$ is implied by the induction hypothesis or Lemma 4.7.) Then we get the fact that (Φ, Γ_1) contains an implication, say $\overrightarrow{\gamma}$. Let Λ be a sequence obtained from (Φ, Γ_1) by deleting an occurrence of $\overrightarrow{\gamma}$, and let Θ_1 and Θ_2 be sequences such that

$$(\Theta_1, \overrightarrow{\gamma}, \Theta_2) \equiv \{\Delta, \alpha\to\beta, \Gamma_1, \Phi\}.$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\begin{array}{c}
\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\vdots\; Q}{\Phi \Rightarrow \overrightarrow{\phi}} \quad \genfrac{}{}{0pt}{}{\vdots\; R_1}{\Gamma_1, \overrightarrow{\phi} \Rightarrow \alpha}}{\Lambda, \overrightarrow{\gamma} \Rightarrow \alpha}\,(\text{cut } 2) \qquad \genfrac{}{}{0pt}{}{\vdots\; R_2}{\Delta, \beta, \Sigma \Rightarrow \psi}}{\Delta, \alpha\to\beta, \Lambda, \overrightarrow{\gamma}, \Sigma \Rightarrow \psi}\,(\to\text{left}) \\
\vdots\; (e^{001}) \\
\Theta_1, \Theta_2, \overrightarrow{\gamma}, \Sigma \Rightarrow \psi \\
\vdots\; (e^{010}) \\
\Theta_1, \overrightarrow{\gamma}, \Theta_2, \Sigma \Rightarrow \psi.
\end{array}$$

(The rank of this (cut 2) is less than r.)

(Case 2-4): The last inference of R is $(\to$left), and P is of the form

$$\dfrac{\genfrac{}{}{0pt}{}{\vdots\; Q}{\Phi \Rightarrow \overrightarrow{\phi}} \qquad \dfrac{\genfrac{}{}{0pt}{}{\vdots\; R_1}{\Gamma \Rightarrow \alpha} \quad \genfrac{}{}{0pt}{}{\vdots\; R_2}{\Delta, \beta, \Sigma_1, \overrightarrow{\phi}, \Sigma_2 \Rightarrow \psi}}{\Delta, \alpha\to\beta, \Gamma, \Sigma_1, \overrightarrow{\phi}, \Sigma_2 \Rightarrow \psi}\,(\to\text{left})}{\{\Delta, \alpha\to\beta, \Gamma, \Sigma_1, \Phi\}, \Sigma_2 \Rightarrow \psi.}\,(\text{cut } 2)$$

(Subcase 2-4-1): β is an implication. In this case, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\genfrac{}{}{0pt}{}{\vdots\; R_1}{\Gamma \Rightarrow \alpha} \qquad \dfrac{\genfrac{}{}{0pt}{}{\vdots\; Q}{\Phi \Rightarrow \overrightarrow{\phi}} \quad \genfrac{}{}{0pt}{}{\vdots\; R_2}{\Delta, \beta, \Sigma_1, \overrightarrow{\phi}, \Sigma_2 \Rightarrow \psi}}{\Delta, \Sigma_1, \Phi, \beta, \Sigma_2 \Rightarrow \psi}\,(\text{cut } 2)}{\{\Delta, \alpha\to\beta, \Gamma, \Sigma_1, \Phi\}, \Sigma_2 \Rightarrow \psi.}\,(\to\text{left } 2)$$

(The rank of this (cut 2) is less than r.)

(Subcase 2-4-2): (Σ_2, ψ) is not a single atom. In this case, by the induction hypothesis, the required proof is obtained from the proof

$$\begin{array}{c}
\dfrac{\genfrac{}{}{0pt}{}{\vdots\; R_1}{\Gamma \Rightarrow \alpha} \qquad \dfrac{\genfrac{}{}{0pt}{}{\vdots\; Q}{\Phi \Rightarrow \overrightarrow{\phi}} \quad \genfrac{}{}{0pt}{}{\vdots\; R_2}{\Delta, \beta, \Sigma_1, \overrightarrow{\phi}, \Sigma_2 \Rightarrow \psi}}{\Delta, \beta, \Sigma_1, \Phi, \Sigma_2 \Rightarrow \psi}\,(\text{cut } 2)}{\Delta, \alpha\to\beta, \Gamma, \Sigma_1, \Phi, \Sigma_2 \Rightarrow \psi}\,(\to\text{left}) \\
\vdots\; (e^{001}) \\
\{\Delta, \alpha\to\beta, \Gamma, \Sigma_1, \Phi\}, \Sigma_2 \Rightarrow \psi.
\end{array}$$

(The rank of this (cut 2) is less than r.)

(Subcase 2-4-3): β is atomic, and (Σ_2, ψ) is a single atom. In this case, we apply Lemma 4.8 to Q and R_2. Then the Lemma and the condition that β is atomic imply the fact that (Φ, Δ, Σ_1) contains an implication, say $\overrightarrow{\gamma}$. Let Λ be a sequence obtained from (Φ, Δ, Σ_1) by deleting an occurrence of $\overrightarrow{\gamma}$, and let Θ_1 and Θ_2 be sequences such that

$$(\Theta_1, \overrightarrow{\gamma}, \Theta_2) \equiv \{\Delta, \alpha{\to}\beta, \Gamma, \Sigma_1, \Phi\}.$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\begin{matrix} \vdots\, R_1 \\ \Gamma \Rightarrow \alpha \end{matrix} \quad \dfrac{\begin{matrix} \vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \begin{matrix} \vdots\, R_2 \\ \Delta, \beta, \Sigma_1, \overrightarrow{\phi} \Rightarrow \psi \end{matrix}}{\beta, \Lambda, \overrightarrow{\gamma} \Rightarrow \psi}\,(\text{cut } 2)}{\begin{matrix} \alpha{\to}\beta, \Gamma, \Lambda, \overrightarrow{\gamma} \Rightarrow \psi \\ \vdots\, (\mathrm{e}^{001}) \\ \Theta_1, \Theta_2, \overrightarrow{\gamma} \Rightarrow \psi \\ \vdots\, (\mathrm{e}^{010}) \\ \Theta_1, \overrightarrow{\gamma}, \Theta_2 \Rightarrow \psi. \end{matrix}}\,(\to\text{left})$$

(The rank of this (cut 2) is less than r.)

(Case 3): The last inference of R is ($\to$right). In this case, P is of the form

$$\dfrac{\begin{matrix} \vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \dfrac{\begin{matrix} \vdots\, R_0 \\ \Gamma, \overrightarrow{\phi}, \Delta, \alpha \Rightarrow \beta \end{matrix}}{\Gamma, \overrightarrow{\phi}, \Delta \Rightarrow \alpha{\to}\beta}\,(\to\text{right})}{\{\Gamma, \Phi\}, \Delta \Rightarrow \alpha{\to}\beta,}\,(\text{cut } 2)$$

and the required proof is obtained from the proof

$$\dfrac{\dfrac{\begin{matrix} \vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \begin{matrix} \vdots\, R_0 \\ \Gamma, \overrightarrow{\phi}, \Delta, \alpha \Rightarrow \beta \end{matrix}}{\{\Gamma, \Phi\}, \Delta, \alpha \Rightarrow \beta}\,(\text{cut } 2)}{\{\Gamma, \Phi\}, \Delta \Rightarrow \alpha{\to}\beta}\,(\to\text{right})$$

by the induction hypothesis. (The rank of this (cut 2) is less than r.)

(Case 4-1): The last inference of R is (e^{001}), and P is of the form

$$\dfrac{\begin{matrix} \vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\phi} \end{matrix} \quad \dfrac{\begin{matrix} \vdots\, R_0 \\ \Gamma_1, \overrightarrow{\phi}, \Gamma_2, \alpha, \beta, \overrightarrow{\Delta \Rightarrow \psi} \end{matrix}}{\Gamma_1, \overrightarrow{\phi}, \Gamma_2, \beta, \alpha, \overrightarrow{\Delta \Rightarrow \psi}}\,(\mathrm{e}^{001})}{\{\Gamma_1, \Phi\}, \Gamma_2, \beta, \alpha, \overrightarrow{\Delta \Rightarrow \psi}.}\,(\text{cut } 2)$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\dfrac{\begin{matrix}\vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\phi}\end{matrix} \qquad \begin{matrix}\vdots\, R_0 \\ \Gamma_1, \overrightarrow{\phi}, \Gamma_2, \alpha, \beta, \overrightarrow{\Delta \Rightarrow \psi}\end{matrix}}{\{\Gamma_1, \Phi\}, \Gamma_2, \alpha, \beta, \overrightarrow{\Delta \Rightarrow \psi}}\,\text{(cut 2)}}{\{\Gamma_1, \Phi\}, \Gamma_2, \beta, \alpha, \overrightarrow{\Delta \Rightarrow \psi}.}\,(\mathrm{e}^{001})$$

(The rank of this (cut 2) is less than r.)

(Case 4-2): The last inference of R is (e^{001}), and P is of the form

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\beta}\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\, R_0 \\ \Gamma, \alpha, \overrightarrow{\beta}, \overrightarrow{\Delta \Rightarrow \psi}\end{matrix}}{\Gamma, \overrightarrow{\beta}, \alpha, \overrightarrow{\Delta \Rightarrow \psi}}\,(\mathrm{e}^{001})}{\{\Gamma, \Phi\}, \alpha, \overrightarrow{\Delta \Rightarrow \psi}.}\,\text{(cut 2)}$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\beta}\end{matrix} \qquad \begin{matrix}\vdots\, R_0 \\ \Gamma, \alpha, \overrightarrow{\beta}, \overrightarrow{\Delta \Rightarrow \psi}\end{matrix}}{\{\Gamma, \Phi\}, \alpha, \overrightarrow{\Delta \Rightarrow \psi}.}\,\text{(cut 2)}$$

(The rank of this (cut 2) is less than r.)

(Case 4-3): The last inference of R is (e^{001}), and P is of the form

$$\dfrac{\begin{matrix}\vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\alpha}\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\, R_0 \\ \Gamma, \overrightarrow{\alpha}, \beta, \overrightarrow{\Delta \Rightarrow \psi}\end{matrix}}{\Gamma, \beta, \overrightarrow{\alpha}, \overrightarrow{\Delta \Rightarrow \psi}}\,(\mathrm{e}^{001})}{\{\Gamma, \beta, \Phi\}, \overrightarrow{\Delta \Rightarrow \psi}.}\,\text{(cut 2)}$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\begin{matrix}\dfrac{\begin{matrix}\vdots\, Q \\ \Phi \Rightarrow \overrightarrow{\alpha}\end{matrix} \qquad \begin{matrix}\vdots\, R_0 \\ \Gamma, \overrightarrow{\alpha}, \beta, \overrightarrow{\Delta \Rightarrow \psi}\end{matrix}}{\Gamma, \Phi, \beta, \overrightarrow{\Delta \Rightarrow \psi}}\,\text{(cut 2)} \\ \vdots\, (\mathrm{e}^{001}) \\ \{\Gamma, \beta, \Phi\}, \overrightarrow{\Delta \Rightarrow \psi}.\end{matrix}$$

(The rank of this (cut 2) is less than r.)

(Case 4-4): The last inference of R is (e^{001}), and P is of the form

$$\dfrac{\begin{matrix}\vdots\ Q \\ \Phi \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \dfrac{\begin{matrix}\vdots\ R_0 \\ \Gamma, \alpha, \beta, \Delta_1, \overrightarrow{\phi}, \Delta_2 \Rightarrow \psi\end{matrix}}{\Gamma, \beta, \alpha, \Delta_1, \overrightarrow{\phi}, \Delta_2 \Rightarrow \psi}\,(\mathrm{e}^{001})}{\{\Gamma, \beta, \alpha, \Delta_1, \Phi\}, \Delta_2 \Rightarrow \psi.}\,(\text{cut } 2)$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\begin{matrix}\vdots\ Q \\ \Phi \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \begin{matrix}\vdots\ R_0 \\ \Gamma, \alpha, \beta, \Delta_1, \overrightarrow{\phi}, \Delta_2 \Rightarrow \psi\end{matrix}}{\{\Gamma, \beta, \alpha, \Delta_1, \Phi\}, \Delta_2 \Rightarrow \psi.}\,(\text{cut } 2)$$

(The rank of this (cut 2) is less than r.)

(Case 5-1): The last inference of R is (e^{010}), and P is of the form

$$\dfrac{\begin{matrix}\vdots\ Q \\ \Phi \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \dfrac{\begin{matrix}\vdots\ R_0 \\ \Gamma_1, \overrightarrow{\phi}, \Gamma_2, \alpha, \overrightarrow{\beta}, \Delta \Rightarrow \psi\end{matrix}}{\Gamma_1, \overrightarrow{\phi}, \Gamma_2, \overrightarrow{\beta}, \alpha, \Delta \Rightarrow \psi}\,(\mathrm{e}^{010})}{\{\Gamma_1, \Phi\}, \Gamma_2, \overrightarrow{\beta}, \alpha, \Delta \Rightarrow \psi.}\,(\text{cut } 2)$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\dfrac{\begin{matrix}\vdots\ Q \\ \Phi \Rightarrow \overrightarrow{\phi}\end{matrix} \quad \begin{matrix}\vdots\ R_0 \\ \Gamma_1, \overrightarrow{\phi}, \Gamma_2, \alpha, \overrightarrow{\beta}, \Delta \Rightarrow \psi\end{matrix}}{\{\Gamma_1, \Phi\}, \Gamma_2, \alpha, \overrightarrow{\beta}, \Delta \Rightarrow \psi}\,(\text{cut } 2)}{\{\Gamma_1, \Phi\}, \Gamma_2, \overrightarrow{\beta}, \alpha, \Delta \Rightarrow \psi.}\,(\mathrm{e}^{010})$$

(The rank of this (cut 2) is less than r.)

(Case 5-2): The last inference of R is (e^{010}), and P is of the form

$$\dfrac{\begin{matrix}\vdots\ Q \\ \Phi \Rightarrow \overrightarrow{\beta}\end{matrix} \quad \dfrac{\begin{matrix}\vdots\ R_0 \\ \Gamma, \alpha, \overrightarrow{\beta}, \Delta \Rightarrow \psi\end{matrix}}{\Gamma, \overrightarrow{\beta}, \alpha, \Delta \Rightarrow \psi}\,(\mathrm{e}^{010})}{\{\Gamma, \Phi\}, \alpha, \Delta \Rightarrow \psi.}\,(\text{cut } 2)$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\begin{matrix}\vdots\ Q \\ \Phi \Rightarrow \overrightarrow{\beta}\end{matrix} \quad \begin{matrix}\vdots\ R_0 \\ \Gamma, \alpha, \overrightarrow{\beta}, \Delta \Rightarrow \psi\end{matrix}}{\{\Gamma, \Phi\}, \alpha, \Delta \Rightarrow \psi.}\,(\text{cut } 2)$$

(The rank of this (cut 2) is less than r.)

(Case 5-3): The last inference of R is (e^{010}), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \overrightarrow{\alpha}\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_0\\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\beta}, \Delta \Rightarrow \psi\end{array}}{\Gamma, \overrightarrow{\beta}, \overrightarrow{\alpha}, \Delta \Rightarrow \psi}\,(\mathrm{e}^{010})}{\{\Gamma, \overrightarrow{\beta}, \Phi\}, \Delta \Rightarrow \psi.}\,(\text{cut } 2)$$

Let Θ_1 and Θ_2 be sequences such that

$$(\Theta_1, \overrightarrow{\beta}, \Theta_2) \equiv \{\Gamma, \overrightarrow{\beta}, \Phi\}.$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \overrightarrow{\alpha}\end{array} \quad \begin{array}{c}\vdots\, R_0\\ \Gamma, \overrightarrow{\alpha}, \overrightarrow{\beta}, \Delta \Rightarrow \psi\end{array}}{\Theta_1, \Theta_2, \overrightarrow{\beta}, \Delta \Rightarrow \psi}\,(\text{cut } 2)\\ \vdots\, (\mathrm{e}^{010})\\ \Theta_1, \overrightarrow{\beta}, \Theta_2, \Delta \Rightarrow \psi.\end{array}$$

(The rank of this (cut 2) is less than r.)

(Case 5-4): The last inference of R is (e^{010}), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \overrightarrow{\phi}\end{array} \quad \dfrac{\begin{array}{c}\vdots\, R_0\\ \Gamma, \alpha, \overrightarrow{\beta}, \Delta_1, \overrightarrow{\phi}, \Delta_2 \Rightarrow \psi\end{array}}{\Gamma, \overrightarrow{\beta}, \alpha, \Delta_1, \overrightarrow{\phi}, \Delta_2 \Rightarrow \psi}\,(\mathrm{e}^{010})}{\{\Gamma, \overrightarrow{\beta}, \alpha, \Delta_1, \Phi\}, \Delta_2 \Rightarrow \psi.}\,(\text{cut } 2)$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \overrightarrow{\phi}\end{array} \quad \begin{array}{c}\vdots\, R_0\\ \Gamma, \alpha, \overrightarrow{\beta}, \Delta_1, \overrightarrow{\phi}, \Delta_2 \Rightarrow \psi\end{array}}{\{\Gamma, \overrightarrow{\beta}, \alpha, \Delta_1, \Phi\}, \Delta_2 \Rightarrow \psi.}\,(\text{cut } 2)$$

(The rank of this (cut 2) is less than r.)

(Case 6-1): The last inference of R is ($\rightarrow$left 2), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \overrightarrow{\phi}\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \Delta_1, \overrightarrow{\phi}, \Delta_2, \beta, \Sigma \Rightarrow \psi\end{array}}{\{\Delta_1, \overrightarrow{\phi}, \Delta_2, \alpha \rightarrow \beta, \Gamma\}, \Sigma \Rightarrow \psi}\,(\rightarrow\text{left } 2)\\ |||\\ \Theta_1, \overrightarrow{\phi}, \Theta_2, \Sigma \Rightarrow \psi\end{array}}{\{\Theta_1, \Phi\}, \Theta_2, \Sigma \Rightarrow \psi.}\,(\text{cut } 2)$$

Then, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\begin{matrix}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\, Q\\ \Phi \Rightarrow \vec{\phi}\end{matrix} \quad \begin{matrix}\vdots\, R_2\\ \Delta_1, \vec{\phi}, \Delta_2, \beta, \Sigma \Rightarrow \psi\end{matrix}}{\Delta_1, \Phi, \Delta_2, \beta, \Sigma \Rightarrow \psi}\,(\text{cut } 2)}{\{\Theta_1, \Phi\}, \Theta_2, \Sigma \Rightarrow \psi.}\,(\rightarrow\text{left } 2)$$

(The rank of this (cut 2) is less than r.) Note that $(\{\Theta_1, \Phi\}, \Theta_2)$ is a permutation of $(\Delta_1, \Phi, \Delta_2, \alpha\rightarrow\beta, \Gamma)$.

(Case 6-2): The last inference of R is ($\rightarrow$left 2), and P is of the form

$$\dfrac{\begin{matrix}\vdots\, Q\\ \Phi \Rightarrow \alpha\rightarrow\beta\end{matrix} \qquad \begin{matrix}\dfrac{\begin{matrix}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{matrix} \quad \begin{matrix}\vdots\, R_2\\ \Delta, \beta, \Sigma \Rightarrow \psi\end{matrix}}{\{\Delta, \alpha\rightarrow\beta, \Gamma\}, \Sigma \Rightarrow \psi}\,(\rightarrow\text{left } 2)\\ |||\\ \Theta_1, \alpha\rightarrow\beta, \Theta_2, \Sigma \Rightarrow \psi\end{matrix}}{\{\Theta_1, \Phi\}, \Theta_2, \Sigma \Rightarrow \psi.}\,(\text{cut } 2)$$

(Subcase 6-2-1): α is an implication. Consider the proof

$$\dfrac{\begin{matrix}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\, Q'\\ \Phi, \alpha \Rightarrow \beta\end{matrix} \quad \begin{matrix}\vdots\, R_2\\ \Delta, \beta, \Sigma \Rightarrow \psi\end{matrix}}{\Delta, \Phi, \alpha, \Sigma \Rightarrow \psi}\,(\beta\text{-cut})}{\{\Theta_1, \Phi\}, \Theta_2, \Sigma \Rightarrow \psi}\,(\text{cut } 2)$$

where Q' is the proof obtained from Q by applying Lemma 4.6. Here note that $(\{\Theta_1, \Phi\}, \Theta_2)$ is a permutation of (Δ, Φ, Γ). The admissibility of β-cut is guaranteed by the induction hypothesis or Lemma 4.7, and the admissibility of this (cut 2) is guaranteed by the induction hypothesis (the grade is less than g). Then we get the required proof.

(Subcase 6-2-2): β is an implication. Consider the proof

$$\dfrac{\dfrac{\begin{matrix}\vdots\, R_1\\ \Gamma \Rightarrow \alpha\end{matrix} \quad \begin{matrix}\vdots\, Q'\\ \Phi, \alpha \Rightarrow \beta\end{matrix}}{\Phi, \Gamma \Rightarrow \beta}\,(\alpha\text{-cut}) \qquad \begin{matrix}\vdots\, R_2\\ \Delta, \beta, \Sigma \Rightarrow \psi\end{matrix}}{\{\Theta_1, \Phi\}, \Theta_2, \Sigma \Rightarrow \psi}\,(\text{cut } 2)$$

where Q' is the proof obtained from Q by applying Lemma 4.6. Then, by the similar argument to (6-2-1), we get the required proof.

(Case 6-3): The last inference of R is ($\to$left 2), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \vec{\phi}\end{array} \qquad \begin{array}{c} \dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma_1, \vec{\phi}, \Gamma_2 \Rightarrow \alpha\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \Delta,\beta,\Sigma\Rightarrow\psi\end{array}}{\{\Delta,\alpha\to\beta,\Gamma_1,\vec{\phi},\Gamma_2\},\Sigma\Rightarrow\psi}\,(\to\text{left } 2)\\ |||\\ \Theta_1,\vec{\phi},\Theta_2,\Sigma\Rightarrow\psi\end{array}}{\{\Theta_1,\Phi\},\Theta_2,\Sigma\Rightarrow\psi.}\,(\text{cut } 2)$$

In this case, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \vec{\phi}\end{array} \quad \begin{array}{c}\vdots\, R_1\\ \Gamma_1, \vec{\phi}, \Gamma_2 \Rightarrow \alpha\end{array}}{\Gamma_1,\Phi,\Gamma_2\Rightarrow\alpha}\,(\text{cut } 2) \qquad \begin{array}{c}\vdots\, R_2\\ \Delta,\beta,\Sigma\Rightarrow\psi\end{array}}{\{\Theta_1,\Phi\},\Theta_2,\Sigma\Rightarrow\psi.}\,(\to\text{left } 2)$$

(The rank of this (cut 2) is less than r.) Note that $(\{\Theta_1,\Phi\},\Theta_2)$ is a permutation of $(\Delta,\alpha\to\beta,\Gamma_1,\Phi,\Gamma_2)$.

(Case 6-4): The last inference of R is ($\to$left 2), and P is of the form

$$\dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \vec{\phi}\end{array} \qquad \dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma\Rightarrow\alpha\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \Delta,\beta,\Sigma_1,\vec{\phi},\Sigma_2\Rightarrow\psi\end{array}}{\{\Delta,\alpha\to\beta,\Gamma\},\Sigma_1,\vec{\phi},\Sigma_2\Rightarrow\psi}\,(\to\text{left } 2)}{\{\{\Delta,\alpha\to\beta,\Gamma\},\Sigma_1,\Phi\},\Sigma_2\Rightarrow\psi.}\,(\text{cut } 2)$$

In this case, by the induction hypothesis, the required proof is obtained from the proof

$$\dfrac{\begin{array}{c}\vdots\, R_1\\ \Gamma\Rightarrow\alpha\end{array} \qquad \dfrac{\begin{array}{c}\vdots\, Q\\ \Phi \Rightarrow \vec{\phi}\end{array} \quad \begin{array}{c}\vdots\, R_2\\ \Delta,\beta,\Sigma_1,\vec{\phi},\Sigma_2\Rightarrow\psi\end{array}}{\Delta,\Sigma_1,\Phi,\beta,\Sigma_2\Rightarrow\psi}\,(\text{cut } 2)}{\{\{\Delta,\alpha\to\beta,\Gamma\},\Sigma_1,\Phi\},\Sigma_2\Rightarrow\psi.}\,(\to\text{left } 2)$$

(The rank of this (cut 2) is less than r.) This completes the proof of Lemma 4.9. ∎

Theorem 4.10 *The cut rule is admissible in* L^-.

By Lemmas 4.7 and 4.9. ∎

5 Systems with e^{*00}

In this section, we investigate the systems with the rule (e^{100}) and/or (e^{000}).

Theorem 5.1 (1) $FL_{\rightarrow}+ (e^{100}) \vdash \beta, \beta{\rightarrow}\gamma \Rightarrow \gamma$.
(2) $FL_{\rightarrow}+ (e^{100}) \vdash \alpha{\rightarrow}\beta{\rightarrow}\gamma, \beta, \alpha \Rightarrow \gamma$.
(3) *The rule* (e^{000}) *is derivable in* $FL_{\rightarrow}+ (e^{100})$. *(Therefore the rules* (e^{100}) *and* (e^{000}) *are rule-equivalent over* $FL_{\rightarrow}$ *(see Figure 2).)*

Proof Similar to Theorem 4.1. ∎

Corollary 5.2 *Among the twenty systems in Figure 1, the six systems* $FL_{\rightarrow}+ (e^{100})$, $FL_{\rightarrow}+ (e^{100}) + (e^{011})$, $FL_{\rightarrow}+ (e^{100}) + (e^{001})$, $FL_{\rightarrow}+ (e^{100}) + (e^{010})$, $FL_{\rightarrow}+ (e^{100}) + (e^{001}) + (e^{010})$, *and* $FL_{\rightarrow}+ (e^{000})$ *are theorem-equivalent each other.*

Proof By Theorem 3.2 (3), Theorem 4.1 (3), and Theorem 5.1 (3). ∎

These six systems will be called *e^{*00}-systems.*

Theorem 5.3 *Each e^{*00}-system is strictly stronger than e^{*01}-systems.*

Proof We can easily verify that the sequent $\beta, \beta{\rightarrow}\gamma \Rightarrow \gamma$ is not provable in the system $L^- = (FL_{\rightarrow}+ (e^{001}) + (e^{010}) + ({\rightarrow}\text{left } 2) - (\text{cut}))$ if β and γ are propositional variables. Then the cut-elimination theorem for L (Theorem 4.10) and Theorem 5.1(1) imply this theorem. ∎

Lemma 5.4 *If a sequent $\Psi, p, \Phi \Rightarrow q$ is cut-free provable in* $FL_{\rightarrow}+ (e^{100}) + (e^{001})$, *and if p and q are propositional variables, then $\Psi, \Phi, p \Rightarrow q$ is cut-free provable in this system.*

Proof By induction on the cut-free proof of $\Psi, p, \Phi \Rightarrow q$. When the proof is of the form

$$\dfrac{\begin{matrix}\vdots\, P \\ \Gamma_1, p, \Gamma_2 \Rightarrow \alpha\end{matrix} \quad \begin{matrix}\vdots\, Q \\ \Delta, \beta, \Sigma \Rightarrow q\end{matrix}}{\Delta, \alpha{\rightarrow}\beta, \Gamma_1, p, \Gamma_2, \Sigma \Rightarrow q,}\ ({\rightarrow}\text{left})$$

the required proof is

$$\begin{matrix}\dfrac{\begin{matrix}\vdots\, P' \\ \Gamma_1, \Gamma_2, p \Rightarrow \alpha\end{matrix} \quad \begin{matrix}\vdots\, Q' \\ \Delta, \Sigma, \beta \Rightarrow q\end{matrix}}{\Delta, \Sigma, \alpha{\rightarrow}\beta, \Gamma_1, \Gamma_2, p \Rightarrow q}\ ({\rightarrow}\text{left}) \\ \vdots\, (e^{001}) \\ \Delta, \alpha{\rightarrow}\beta, \Gamma_1, \Gamma_2, \Sigma, p \Rightarrow q\end{matrix}$$

where P' is the proof obtained from P by either the induction hypothesis or applications of (e^{001}), and Q' is the proof obtained from Q by either the induction hypothesis or applications of (e^{100}). The other cases are easy by the induction hypotheses. ∎

Theorem 5.5 *The rule* (e^{000}) *is admissible in the cut-free part of the system* $FL_{\rightarrow}+ (e^{100}) + (e^{001})$.

Proof Suppose a sequent $\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma$ is cut-free provable in $FL_{\rightarrow}+ (e^{100}) + (e^{001})$. If either α or $\Delta \Rightarrow \gamma$ is an implication, then the two formulas α and β can be exchanged by the rule (e^{100}) or (e^{001}). If Δ is empty and both α and γ are atomic, then the previous Lemma 5.4 says that $\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma$ is also cut-free provable. ∎

Theorem 5.6 *The cut-elimination theorem holds for the three e^{*00}-systems* $FL_{\rightarrow}+ (e^{100}) + (e^{001})$, $FL_{\rightarrow}+ (e^{100}) + (e^{001}) + (e^{010})$, *and* $FL_{\rightarrow}+ (e^{000})$*; but does not hold for the other e^{*00}-systems.*

Proof It is well-known that the cut-elimination theorem holds for $FL_{\rightarrow}+ (e^{000})$. Now Theorem 5.5 implies that the three cut-free parts of $FL_{\rightarrow}+ (e^{100}) + (e^{001})$, $FL_{\rightarrow}+ (e^{100}) + (e^{001}) + (e^{010})$, and $FL_{\rightarrow}+ (e^{000})$ are theorem-equivalent each other. Therefore the cut-elimination theorem also holds for $FL_{\rightarrow}+ (e^{100}) + (e^{001})$ and for $FL_{\rightarrow}+ (e^{100}) + (e^{001}) + (e^{010})$.

The fact that the theorem does not hold for the other e^{*00}-systems is shown as follows. The sequent $((\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow \gamma) \rightarrow \delta \rightarrow \varepsilon, \alpha, \delta, \beta \Rightarrow \varepsilon$ is provable in $FL_{\rightarrow}+ (e^{000})$:

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ \alpha, \beta \Rightarrow (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow \gamma\end{array} \quad \begin{array}{c}\vdots \\ \delta \rightarrow \varepsilon, \delta \Rightarrow \varepsilon\end{array}}{((\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow \gamma) \rightarrow \delta \rightarrow \varepsilon, \alpha, \beta, \delta \Rightarrow \varepsilon}(\rightarrow\text{left})}{((\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow \gamma) \rightarrow \delta \rightarrow \varepsilon, \alpha, \delta, \beta \Rightarrow \varepsilon,}(e^{000})$$

while we can verify that this sequent is not cut-free provable in $FL_{\rightarrow}+ (e^{100}) + (e^{010})$ if $\alpha - \varepsilon$ are mutually distinct propositional variables. ∎

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