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Kyoto University
CONSTRUCTING LOW-DISCREPANCY SEQUENCES
BY USING $\beta$-ADIC TRANSFORMATIONS

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Abstract. A new class of low-discrepancy sequences is constructed by the use of $\beta$-adic transformations. Here, $\beta$ is a real number greater than 1. When $\beta$ is an integer greater than 2, this sequence becomes the generalized van der Corput sequence in base $\beta$. It is also shown that for some special $\beta$, the discrepancy of this sequence decreases in the fastest order.

0. Introduction and background

First, we recall the notions of a uniformly distributed sequence and the discrepancy of points ([Niederreiter 1]). A sequence $x_1, x_2, \ldots$ in the $s$-dimensional unit cube $I^s = \prod_{i=1}^{s}[0,1)$ is said to be uniformly distributed in $I^s$ when

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_J(x_n) = \lambda_s(J)
$$

holds for all subintervals $J \subset I^s$, where $c_J$ is the characteristic function of $J$, and $\lambda_s$ is the $s$-dimensional Lebesgue measure. If $x_1, x_2, \ldots \in I^s$ is a uniformly distributed sequence, the formula

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{I^s} f(x) \, dx
$$

holds for any Riemann integrable function on $I^s$. The discrepancy of the point set $P = \{x_1, x_2, \ldots, x_N\}$ in $I^s$ is defined as follows:

$$
D_N(B; P) = \sup_{B \subset \varphi(I^s)} \left| \frac{A(B; P)}{N} - \lambda_s(B) \right|
$$

where $B \subset \varphi(I^s)$ is a non-empty family of Lebesgue measurable subsets and $A(B; P)$ is the counting function that indicates the number of $n$, where $1 \leq n \leq N$, for which $x_n \in B$. When $J^* = \{\prod_{i=1}^{s}[0, u_i), 0 \leq u_i < 1\}$, the star discrepancy $D_N^*(P)$ is defined by $D_N^*(P) = D_N(J^*; P)$. When $S = \{x_1, x_2, \ldots\}$ is a sequence in $I^s$, we

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define $D^\ast_N(S)$ as $D^\ast_N(S_N)$, where $S_N$ is the point set $\{x_1, x_2, \ldots, x_N\}$. Let $S$ be a sequence in $I^s$. It is known that the following two conditions are equivalent:

(a) $S$ is uniformly distributed in $I^s$;
(b) $\lim_{N \to \infty} D^\ast_N(S) = 0$.

The following classical theorem shows the importance of the notion of discrepancy.

**Theorem 0.1 (Koksma-Hlawka)[1].** If $f$ has bounded variation $V(f)$ on $I^s$ in the sense of Hardy and Krause, then for any $x_1, x_2, \ldots, x_N \in I^s$, we have

$$\left| \frac{1}{N} \sum_{n=1}^N f(x_n) - \int_{I^s} f(x) \, dx \right| \leq V(f) D^\ast_N(x_1, \ldots, x_N).$$

Schmidt [4] showed that, when $s = 1, 2$, there exists a positive constant $C$ that depends only on $s$, and the following inequality holds for an arbitrary point set $P$ consisting of $N$ elements:

(0.3) $$D^\ast_N(P) \geq C \frac{(\log N)^{s-1}}{N}.$$

If (0.3) holds, then there exists a positive constant $C$ that depends only on $s$, and any sequence $S \subset I^s$ satisfies

(0.4) $$D^\ast_N(S) \geq C \frac{(\log N)^s}{N}$$

for infinitely many $N$. Taking account of (0.3) and (0.4), we define a low-discrepancy sequence for the one-dimensional case as follows:

**Definition 0.1.** Let $S$ be an one-dimensional sequence in $[0,1)$. If $S$ satisfies

$$\lim_{N \to \infty} \frac{N D^\ast_N(S)}{\log N} = C \text{ (const)},$$

then $S$ is called a low-discrepancy sequence.

Hereafter we consider only the case where $s = 1$. We now introduce the classical van der Corput sequence [1].

**Definition 0.2.** Let $p \geq 2$ be an integer. Every integer $n \geq 0$ has a unique digit expansion

$$n = \sum_{j=0}^{\infty} a_j(n)p^j, \quad a_j(n) \in \{0, 1, \ldots, p-1\} \text{ for all } j \geq 0,$$

in base $p$. Then, the radical-inverse function $\phi_p$ is defined by

$$\phi_p(n) = \sum_{j=0}^{\infty} \tau_j(a_j(n)) p^{-j-1} \quad \text{for all integers} \quad n \geq 0,$$

where $\tau_j$ is a permutation of $\{0, 1, \ldots, p-1\}$. The van der Corput sequence in base $p$ is the sequence $V_p = \{\phi_p(n)\}_{n=0}^{\infty} \subset [0,1)$. 
**Theorem 0.2** [1]. For an arbitrary integer \( p \geq 2 \), \( V_p \) is a low-discrepancy sequence.

In the following part of this paper, the author defines a class of sequences by the use of \( \beta \)-adic transformation ([Rény 3], [Parry 2]) and shows that any member of this class is a low-discrepancy sequence when \( \beta = (L + \sqrt{L^2 + 4K})/2 \), where \( L \) and \( K \) are integers greater than 1 and satisfy \( K \leq L \). When \( \beta \) is an integer greater than 2, the sequence becomes \( V_\beta \).

### 1. \( \beta \)-adic transformation

In this section we define the fibred system and the \( \beta \)-adic transformation, following [Schweiger 5] and [Takahashi 6].

\( \mathbb{R} \), \( \mathbb{Z} \), and \( \mathbb{N} \) are the sets of all real numbers, all integers, and all natural numbers, respectively. For \( x \in \mathbb{R} \), \([x]\) denotes the integer part of \( x \).

**Definition 1.1.** Let \( B \) be a set and \( T : B \to B \) be a map. The pair \((B, T)\) is called a fibred system if the following conditions are satisfied:

(a) There is a finite countable set \( A \).

(b) There is a map \( k : B \to A \), and the sets

\[
B(i) = k^{-1}(\{i\}) = \{x \in B : k(x) = i\}
\]

form a partition of \( B \).

(c) For an arbitrary \( i \in A \), \( T|_{B(i)} \) is injective.

**Definition 1.2.** Let \( \Omega = A^\mathbb{N} \) and \( \sigma : \Omega \to \Omega \) be the one-sided shift operator. Let \( k_j(x) = k(T^{j-1}x) \). We derive a canonical map \( \varphi : B \to \Omega \) from

\[
\varphi(x) = (k_j(x))_{n=1}^\infty.
\]

\( \varphi \) is called the representation map.

We have the following commutative diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{T} & B \\
\varphi \downarrow & & \downarrow \varphi \\
\Omega & \xrightarrow{\sigma} & \Omega
\end{array}
\]

**Definition 1.3.** If a representation map \( \varphi \) is injective, \( \varphi \) is called a valid representation.

**Definition 1.4.** Let \( \omega \in \Omega \). If \( \omega \in \text{Im}(\varphi) \), \( \omega \) is called an admissible sequence.

**Definition 1.5.** The cylinder of rank \( n \) defined by \( a_1, a_2, \ldots, a_n \in A \) is the set

\[
B(a_1, a_2, \ldots, a_n) = B(a_1) \cap T^{-1}B(a_2) \cap \cdots \cap T^{-n+1}B(a_n).
\]

We define \( B \) to be a cylinder of rank 0.
Definition 1.6. Let $\beta > 1$ and $\beta \in \mathbb{R}$. Let $f_\beta : [0,1) \to [0,1)$ be a function defined by
\[ f_\beta(x) = \beta x - [\beta x]. \]
Let $A = \mathbb{Z} \cap [0, \beta)$. Then, we have the following fibred system $([0,1), f_\beta)$:
\[
\begin{array}{ccc}
[0,1) & \xrightarrow{f_\beta} & [0,1) \\
\varphi & \downarrow & \varphi \\
\Omega & \xrightarrow{\sigma} & \Omega
\end{array}
\]
(1.1)

The representation map $\varphi$ of this fibred system is defined by
\[ x = \sum_{n=0}^{\infty} \frac{a_n}{\beta^{n+1}} \iff \varphi(x) = (a_0, a_1, \ldots, a_n, \ldots) \in \Omega. \]

This fibred system $([0,1), f_\beta)$ is called a $\beta$-adic transformation. In this situation, we define $\zeta_\beta \in \Omega$ by
\[ \zeta_\beta = \lim_{x \nearrow 1} \varphi(x). \]
(1.2)

We also define $X_\beta \subset \Omega$ to be the set of all admissible sequences.

For a sequence $a \in \Omega$, we write the $i$-th element of $a$ as $a(i)$, that is, $a = (a(1), a(2), \ldots)$. We remark that $\varphi$ is not a valid representation at this point, because $(a_1, a_2, \ldots, a_n, 0, 0, \ldots)$ and $(a_1, a_2, \ldots, a_n - 1, \zeta_\beta(1), \zeta_\beta(2), \ldots)$ are two different representations of the same $x = \sum_{i=1}^{n} a_i \beta^{-i}$. In this paper we adopt the former representation and make $\varphi$ valid. We derive the following propositions directly from this definition.

Proposition 1.1.
\[ X_\beta = \{ \omega \in \Omega \mid \forall n \in \mathbb{Z}_{\geq 0} \quad \sigma^n \omega \prec \zeta_\beta \}, \]
where $\omega \prec \psi$ means that $\omega$ precedes $\psi$ in lexicographical order.

Proposition 1.2. For an arbitrary $i \in A$,
\[ B(i) = \begin{cases} \left[ \frac{i}{\beta}, \frac{i+1}{\beta} \right), & 0 \leq i < [\beta] \\ \left[ \frac{[\beta]}{\beta}, 1 \right), & \text{otherwise} \end{cases} \]
holds.

Let $\rho_\beta(x) = \sum_{n=0}^{\infty} a_n \beta^{-n-1}$; then, we have
\[ \rho_\beta(X_\beta) = [0,1] \]
and the following commutative diagram:
\[
\begin{array}{ccc}
[0,1) & \xrightarrow{f_\beta} & [0,1) \\
\varphi & \downarrow \uparrow_{\rho_\beta} & \varphi \\
\Omega & \xrightarrow{\rho} & \Omega
\end{array}
\]
(1.3)
2. Constructing the sequence

In this section, a sequence $N_\beta \subset [0,1)$ is defined by the use of $\beta$-adic transformation. Let $\beta \in \mathbb{R}_{>1}$ and let $([0,1], f_\beta)$ be a fibred system (1.3). Let $B = [0,1)$, and $\Lambda, \Omega, (X_\beta, \sigma, \rho_\beta, \varphi, \zeta_\beta, B(a_1, \ldots, a_n)$ be the same as in the previous section.

**Definition 2.1.** For an arbitrary $n \in \mathbb{Z}_{\geq 0}$, $X_\beta(n), Y_\beta(n) \subset X_\beta, F_\beta(n) \in \mathbb{Z}$, and $G_\beta(n) \in \mathbb{Z}$ are defined as follows:

$$X_\beta(n) = \begin{cases} \{(0,0, \ldots)\}, & n = 0 \\ \{\omega \in X_\beta \mid \sigma^n \omega \neq (0,0, \ldots) \text{ and } \sigma^n \omega = (0,0, \ldots)\}, & n \neq 0 \end{cases}$$

$$Y_\beta(n) = \bigcup_{i=0}^{n} X_\beta(i)$$

$$F_\beta(n) = \# X_\beta(n)$$

$$G_\beta(n) = \sum_{i=0}^{n} F_\beta(i) = \# Y_\beta(n)$$

It is apparent that

$$F_\beta(n) \leq ([\beta] + 1)^{n-1}.$$  

**Definition 2.2.** For an arbitrary $n \in \mathbb{N}$, define $l_n \in \mathbb{N}$ to satisfy $G_\beta(l_n) < n \leq G_\beta(l_n + 1)$. Define $\tau_n : X_\beta(n) \to \mathbb{N}^n_{\geq 1}$ by $\tau_n((k_1, \ldots, k_n)) = (k_n, \ldots, k_1)$. Induce the right-to-left lexicographical or reverse right-to-left lexicographical order to $X_\beta(l_n + 1) = \{\omega_1, \omega_2, \ldots, \omega_{F_\beta(l_n + 1)}\}$; that is to say, for all $i < j$, $\tau_n(\omega_i) < \tau_n(\omega_j)$ or $\tau_n(\omega_j) < \tau_n(\omega_i)$ holds, respectively. In this situation, the sequence $N_\beta$ is defined as follows:

$$N_\beta = \{\rho_\beta(\omega_{n-l_n})\}_{n=1}^\infty$$

In this paper, we assume that the elements of $X_\beta(l_n + 1)$ are arranged in right-to-left lexicographical order.

From this definition, we immediately have the following proposition:

**Proposition 2.1.** If $\beta \in \mathbb{Z}_{\geq 2}$ then $N_\beta$ is $V_\beta$.

From this proposition, we see that, if $\beta \in \mathbb{Z}_{\geq 2}$, $N_\beta$ is a low-discrepancy sequence. We also have the following theorem:

**Theorem 2.1.** Let $L, K \in \mathbb{N}$ and $K \leq L$. If $\beta = (L + \sqrt{L^2 + 4K})/2$, then $N_\beta$ is a low-discrepancy sequence.

To prove this theorem, we provide several lemmas, propositions, and definitions. We use the following notation for periodic sequences:

$$(a_1, a_2, \ldots, a_n, \ldots) = (a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}, a_n, a_{n+1}, \ldots, a_{n+m}, \ldots)$$

Let $\beta \in \mathbb{R}_{>1}$. 

Lemma 2.1. If $\zeta_\beta = (a_1, a_2, \ldots, (a_m - 1))$, then $\{F_\beta(n)\}_{n=1}^\infty$ and $\{G_\beta(n)\}_{n=1}^\infty$ satisfy the following linear recurrent equations:

$$F_\beta(n + m) - \sum_{i=1}^{m} a_i F_\beta(n + m - i) = 0 \quad \text{for all } n \geq 1 - m, \ n \neq 0$$

(2.1.F)

$$F_\beta(m) - \sum_{i=1}^{m} a_i F_\beta(m - i) + 1 = 0$$

(2.1.G)

$$G_\beta(n + m) - \sum_{i=1}^{m} a_i G_\beta(n + m - i) = 0 \quad \text{for all } n > 0.$$  

Here we extend the definition of $F_\beta(n)$ to $F_\beta(-n) = 0 \ (n > 0)$.

Proof. It is apparent from the definition of $\beta$-adic transformation that

(2.2.a) $a_1 = \begin{cases} [\beta], & \beta \notin \mathbb{Z} \\ \beta - 1, & \beta \in \mathbb{Z} \end{cases}$

and

(2.2.b) $a_1 \geq \begin{cases} a_j, & j = 1, \ldots, m - 1 \\ a_m - 1, & \end{cases}$

hold. From Proposition 1.1, we have

$$X_\beta(n + m) = \{(x, \omega_1) \mid x \in \{0, \ldots, a_1 - 1\}, \ \omega_1 \in X_\beta(n + m - 1)\}$$

$$\cup \{(a_1, x, \omega_2) \mid x \in \{0, \ldots, a_2 - 1\}, \ \omega_2 \in X_\beta(n + m - 2)\}$$

$$\vdots$$

$$\cup \{(a_1, \ldots, a_{m-1}, x, \omega_m) \mid x \in \{0, \ldots, a_m - 1\}, \ \omega_m \in X_\beta(n)\}$$

for all $n \geq 1$, and

$$X_\beta(0) = \{(0)\}$$

$$X_\beta(1) = \{(x, 0) \mid x \in \{1, \ldots, a_1\}\}$$

$$X_\beta(2) = \{(x, \omega_1) \mid x \in \{0, \ldots, a_1 - 1\}, \ \omega_1 \in X_\beta(1)\}$$

$$\cup \{(a_1, x, 0) \mid x \in \{1, \ldots, a_2\}\}$$

$$\vdots$$

$$X_\beta(m - 1) = \{(x, \omega_{m-2}) \mid x \in \{0, \ldots, a_1 - 1\}, \ \omega_{m-2} \in X_\beta(m - 2)\}$$

$$\cup \{(a_1, x, \omega_{m-3}) \mid x \in \{0, \ldots, a_2 - 1\}, \ \omega_{m-3} \in X_\beta(m - 3)\}$$

$$\vdots$$

$$\cup \{(a_1, \ldots, a_{m-2}, x, 0) \mid x \in \{1, \ldots, a_{m-1}\}\}$$

$$X_\beta(m) = \{(x, \omega_{m-1}) \mid x \in \{0, \ldots, a_1 - 1\}, \ \omega_{m-1} \in X_\beta(m - 1)\}$$

$$\cup \{(a_1, x, \omega_{m-2}) \mid x \in \{0, \ldots, a_2 - 1\}, \ \omega_{m-2} \in X_\beta(m - 2)\}$$

$$\vdots$$

$$\cup \{(a_1, \ldots, a_{m-1}, x, 0) \mid x \in \{1, \ldots, a_m - 1\}\}.$$
In the above expressions, we set \( \{0, \ldots, a_i - 1\} = \emptyset \) when \( a_i = 0 \). Remark \( a_1, a_m \geq 1 \). Then (2.1.F) holds. From Definition 2.1, (2.1.F), and

\[
F_{\beta}(m) + F_{\beta}(0) = \sum_{i=1}^{m} a_i F_{\beta}(m - i),
\]

we have

\[
G_{\beta}(n + m) = F_{\beta}(n + m) + F_{\beta}(n + m - 1) + \cdots + F_{\beta}(0)
\]

\[
= a_1 F_{\beta}(n + m - 1) + a_2 F_{\beta}(n + m - 2) + \cdots + a_m F_{\beta}(n)
\]

\[
+ a_1 F_{\beta}(n + m - 2) + a_2 F_{\beta}(n + m - 3) + \cdots + a_m F_{\beta}(n - 1)
\]

\[
+ \cdots
\]

\[
+ a_1 F_{\beta}(m) + a_2 F_{\beta}(m - 1) + \cdots + a_m F_{\beta}(1)
\]

\[
+ a_1 F_{\beta}(m - 1) + a_2 F_{\beta}(m - 2) + \cdots + a_{m-1} F_{\beta}(0)
\]

\[
+ a_1 F_{\beta}(m - 2) + a_2 F_{\beta}(m - 3) + \cdots + a_{m-2} F_{\beta}(0)
\]

\[
+ \cdots
\]

\[
= a_1 G_{\beta}(n + m - 1) + a_2 G_{\beta}(n + m - 2) + \cdots + a_m G_{\beta}(n).
\]

Thus (2.1.G) holds.

**Definition 2.3.** For \((k_1, k_2, \ldots, k_n) \in X_\beta(n)\), define

\[
d(k_1, k_2, \ldots, k_n) = \min\{\max\{0, n - m\} \leq d \leq n \mid 1 \in B(\sigma^d(k_1, \ldots, k_n))\}.
\]

**Lemma 2.2.** Let \((k_1, \ldots, k_n) \in Y_\beta(n)\). When \((k_1, \ldots, k_n) \in X_\beta(l)\) and \(l < n\), we set \(k_{l+1} = \cdots = k_n = 0\). If \(\zeta_{\beta} = (\hat{a}_1, a_2, \ldots, (a_m - 1))\), then

\[
\lambda(B(k_1, \ldots, k_n)) = \begin{cases} 
\frac{1}{\beta^d} \sum_{i=n-d+1}^{m} \frac{a_i}{\beta^i}, & \text{when } d > n - m \\
\frac{1}{\beta^n}, & \text{when } d = n - m
\end{cases}
\]

where \(d = d(k_1, \ldots, k_n)\) and \(\lambda\) is a one-dimensional Lebesgue measure.

**Proof.** From \(\zeta_{\beta} = (\hat{a}_1, a_2, \ldots, (a_m - 1))\) we have

\[
1 - \sum_{i=1}^{m} \frac{a_i}{\beta^i} = 0 \tag{2.3.a}
\]

\[
1 - \sum_{i=1}^{m} \frac{\zeta_{\beta}(i)}{\beta^i} = \frac{1}{\beta^{m-1}} \tag{2.3.b}
\]
where \( l \) is an arbitrary positive integer. If \( \beta \in \mathbb{N}_{\geq 2} \), this lemma is trivial. We assume that \( \beta \neq N \). We prove the lemma by induction on \( n \). Consider the case in which \( n = 1 \). From the definition of \( f_{\beta} \), (2.2), and (2.3.a), we have

\[
\lambda(B(0)) = \lambda(B(1)) = \cdots = \lambda(B(a_1 - 1)) = \frac{1}{\beta}
\]

and

\[
\lambda(B(a_1)) = \sum_{i=2}^{m} \frac{a_i}{\beta^i}.
\]

This means that the lemma's statement holds when \( n = 1 \). We show that this statement holds for \((k_1, \ldots, k_n, k_{n+1}) \in \bigcup_{i=1}^{n+1} \mathcal{X}_{\beta}(i) \) under the induction hypothesis. For any \( n \geq 1 \) and \( J \subset [0,1) \),

\[
(2.4) \quad f_{\beta}(f_{\beta}^{-n}(J)) = f_{\beta}^{-n+1}(J)
\]

holds from \( f_{\beta} \)'s surjectivity. Consider the case in which \( k_1 = 0,1, \ldots, a_1 - 1 \), that is to say, the case in which \( d = d(k_1, \ldots, k_{n+1}) \geq 1 \) and \( d(k_{2}, \ldots, k_{n+1}) = d - 1 \). In this case, \( f_{\beta}(B(k_1)) = [0,1) \) holds; therefore, considering (2.4), we have

\[
(2.5) \quad f_{\beta}(B(k_1, \ldots, k_{n+1})) = B(k_2, \ldots, k_{n+1})
\]

and

\[
(2.6) \quad \lambda(f_{\beta}(J)) = \beta \lambda(J)
\]

for an arbitrary \( J \subset B(k_1) \). From the induction hypothesis,

\[
\lambda(B(k_2, \ldots, k_{n+1})) = \begin{cases} 
\frac{1}{\beta^{d-1}} \sum_{i=n-d}^{m} \frac{a_i}{\beta^i}, & \text{when } d - 1 > n - m \\
\frac{1}{\beta^n}, & \text{when } d - 1 = n - m
\end{cases}
\]

holds. Therefore, from (2.5) and (2.6), this lemma's statement holds. When \( d = 0 \), the statement follows from (2.3.a) and (2.3.b).

For a sequence \( S \), \( S[N] \) denotes the point set consisting of the first \( N \) elements of \( S \), and \( S[N;M] = S[N + M] \setminus S[N] \).

**Lemma 2.3.** For an arbitrary \((k_1, \ldots, k_n) \in Y_{\beta}(n)\), we have

\[
A(B(k_1, \ldots, k_n); N_{\beta}[G_{\beta}(m + d + l)])
\]

\[
= \begin{cases} 
\sum_{i=1}^{m-n+d} a_{n-d+i}G_{\beta}(m + d + l - n - i), & \text{when } d > n - m \\
G_{\beta}(l), & \text{when } d = n - m
\end{cases}
\]
where \( d = d(k_1, \ldots, k_n) \) and \( l \in \mathbb{Z}_{\geq 0} \).

**Proof.** When \( d = n - m \) holds, it is trivial. Assume that \( d > n - m \). Let \( K = (k_1, \ldots, k_n) \). From Proposition 1.1,

\[
\begin{align*}
\{\omega \in \bigcup_{i=0}^{m-1} X_{\beta}(i) \mid \rho_{\beta}(\omega) \in B(k_1, \ldots, k_n)\} \\
= \{(K, x, \omega_1) \mid x \in \{0, \ldots, a_{n-d+1} - 1\}, \omega_1 \in Y_{\beta}(m + d + l - n - 1)\} \\
\cup \{(K, a_{n-d+1}, x, \omega_2) \mid x \in \{0, \ldots, a_{n-d+2} - 1\}, \omega_2 \in Y_{\beta}(m + d + l - n - 2)\} \\
\cup \{(K, a_{n-d+1}, \ldots, a_{m-1}, x, \omega_{m-n+d}) \mid x \in \{0, \ldots, a_{m} - 1\}, \omega_{m-n+d} \in Y_{\beta}(l)\}
\end{align*}
\]

holds. In the above expressions, we set \( \{0, \ldots, a_i - 1\} = \emptyset \) when \( a_i = 0 \). Therefore, we have

\[
A(B(k_1, \ldots, k_n); N_{\beta}(G_{\beta}(n + l))) \\
= \sum_{i=1}^{m-n+d-1} a_{n-d+i} \beta^{m+i+1} + a_m \beta^{m+n+l}
\]

**Proof of Theorem 2.1.** From the conditions of the theorem,

\[
(2.7) \quad \zeta_{\beta} = (\hat{L}, (K - 1))
\]

holds. Let \( \alpha = (L - \sqrt{L^2 + 4K})/2 \). Then we have

\[
(2.8.\text{F}) \quad F_{\beta}(n) = \begin{cases} 1, & n = 0 \\ \frac{1}{\beta - \alpha} (\beta^{n-1} (\beta^2 - 1) - \alpha^{n-1} (\alpha^2 - 1)), & n \geq 1 \end{cases}
\]

\[
(2.8.\text{G}) \quad G_{\beta}(n) = \begin{cases} 1, & n = 0 \\ \frac{1}{\beta - \alpha} (\beta^n (\beta + 1) - \alpha^n (\alpha + 1)), & n \geq 1 \end{cases}
\]

from (2.7) and Lemma 2.1. Define \( Z_{\beta}(n) \) and \( H_{\beta}(n) \) as follows:

\[
Z_{\beta}(n) = \{\omega \in Y_{\beta}(n) \mid \omega(n) \neq L\}
\]

\[
H_{\beta}(n) = \#Z_{\beta}(n)
\]

The following partitionings of \( Y_{\beta}(n) \) and \( Z_{\beta}(n) \) hold.

\[
(2.9.\text{Y}) \quad Y_{\beta}(n + 1) = \{(\omega, x) \mid x \in \{0, 1, \ldots, K - 1\}, \omega \in Y_{\beta}(n)\} \\
\cup \{(\omega, x) \mid x \in \{K, K + 1, \ldots, L\}, \omega \in Z_{\beta}(n)\}
\]
\[ Z_\beta(n + 1) = \{(\omega, x) | x \in \{0, 1, \ldots, K - 1\}, \omega \in Y_\beta(n)\} \]
\[ \cup \{(\omega, x) | x \in \{K, K + 1, \ldots, L - 1\}, \omega \in Z_\beta(n)\} \]

Then we have
\[ H_\beta(n + 1) = KG_\beta(n) + (L - K)H_\beta(n) \]
\[ G_\beta(n + 1) = KG_\beta(n) + (L - K - 1)H_\beta(n). \]

From (2.10) and Lemma 2.1, we have
\[ H_\beta(n + 2) - LH_\beta(n + 1) - KH_\beta(n) = 0, \quad n \geq 1. \]

From the same discussion as in the proof of Lemma 2.3,
\[ A(B(k_1, \ldots, k_n); \rho_\beta(Z_\beta(2 + d + l))) = \begin{cases} H_\beta(l), & d = n - 2 \\ KH_\beta(l), & d = n - 1 \\ H_\beta(l + 2), & d = n \end{cases} \]

holds for an arbitrary \((k_1, \ldots, k_n) \in Y_\beta(n)\). Define
\[ \Delta(B; P) = A(B; P) - M\lambda(B), \]
where \(B\) is an interval in \([0, 1)\) and \(P = \{x_1, x_2, \ldots, x_M\} \subset [0, 1)\). For any set of points \(P, S\) in \([0, 1)\), and any interval \(B \subset [0, 1)\),
\[ \Delta(B; P \cup S) = \Delta(B; P) + \Delta(B; S) \]
holds. Considering the order of \(N_\beta\) that we gave in Definition 2.2, we have
\[ N_\beta[H_\beta(n)] = \rho_\beta(Z_\beta(n)). \]

From Lemma 2.2, Lemma 2.3, (2.8.G), (2.11) and (2.12), we have
\[ \Delta(B(k_1, \ldots, k_n); N_\beta[G_\beta(2 + d + l)]) = \begin{cases} \alpha + 1 \left(\left(\frac{\alpha}{\beta}\right)^n - 1\right) \alpha^l, & d = n - 2 \\ K(\alpha + 1) \left(\left(\frac{\alpha}{\beta}\right)^{n+1} - 1\right) \alpha^l, & d = n - 1 \\ \alpha + 1 \left(\left(\frac{\alpha}{\beta}\right)^n - 1\right) \alpha^{l+2}, & d = n \end{cases} \]

and
\[ \Delta(B(k_1, \ldots, k_n); N_\beta[H_\beta(2 + d + l)]) = \begin{cases} \frac{1}{\beta - \alpha} \left(\left(\frac{\alpha}{\beta}\right)^n - 1\right) \alpha^{l+1}, & d = n - 2 \\ K \frac{1}{\beta - \alpha} \left(\left(\frac{\alpha}{\beta}\right)^{n+1} - 1\right) \alpha^{l+1}, & d = n - 1 \\ \frac{1}{\beta - \alpha} \left(\left(\frac{\alpha}{\beta}\right)^n - 1\right) \alpha^{l+3}, & d = n \end{cases} \]
where \((k_1, \ldots, k_n) \in Y_\beta(n), l \in \mathbb{Z}\) and \(d = d(k_1, \ldots, k_n)\). Define the truncating operator \(r_k : X_\beta \to Y_\beta(k)\) as follows:

\[
r_k(\omega) = \begin{cases} 
\omega, & \text{when } \omega \in X_\beta(j), \ j \leq k \\
(\omega(1), \ldots, \omega(k)) & \text{otherwise}
\end{cases}
\]

For any \(i, j \in \mathbb{Z}\) and any cylinder \(B\) of rank less than \(k\),

\[
A(B; N_\beta[i; j]) = A(B; r_k(N_\beta[i; j]))
\]

holds. Let \((k_1, \ldots, k_n) \in Y_\beta(n)\), let \(d = d(k_1, \ldots, k_n)\), and let \(M\) be an arbitrary integer greater than \(G_\beta(2 + d)\). Let \(l\) be an integer satisfying

\[
G_\beta(2 + d + l) \leq M < G_\beta(2 + d + l + 1).
\]

Applying partitioning (2.9.Y) and (2.9.Z) recursively for \(Y_\beta(2 + d + l + 1)\), we obtain the following partitioning of \(N_\beta[G_\beta(2 + d + l + 1)]\):

\[
N_\beta[G_\beta(2 + d + l + 1)] = N_\beta[G_\beta(2 + d + l)] \\
\quad \quad \quad \quad \cup N_\beta[G_\beta(2 + d + l); G_\beta(2 + d + l)] \\
\quad \quad \quad \quad \quad \vdots \\
\quad \quad \quad \quad \cup N_\beta[(K - 1)G_\beta(2 + d + l); G_\beta(2 + d + l)] \\
\quad \quad \quad \quad \cup N_\beta[KG_\beta(2 + d + l); H_\beta(2 + d + l)] \\
\quad \quad \quad \quad \quad \vdots \\
\quad \quad \quad \quad \cup N_\beta[KG_\beta(2 + d + l) + (L - K - 1)H_\beta(2 + d + l); H_\beta(2 + d + l)] \\
\quad \quad \quad \quad \cup N_\beta[KG_\beta(2 + d + l) + (L - K)H_\beta(2 + d + l); G_\beta(2 + d + l - 1)] \\
\quad \quad \quad \quad \quad \vdots \\
\quad \quad \quad \quad \cup N_\beta[KG_\beta(2 + d + l) + (L - K)H_\beta(2 + d + l) + KG_\beta(2 + d + l - 1) \\
\quad \quad \quad \quad \quad \quad \quad \; ; H_\beta(2 + d + l - 1)] \\
\quad \quad \quad \quad \quad \cup \\
\quad \quad \quad \quad \quad \quad \vdots
\]

Partition \(N_\beta[M]\) in the same way as (2.16); then, from (2.15), the additivity of \(\Delta\),
(2.9.Y), (2.9.Z), and the order we induced to \( N_\beta \), we have

\[
\begin{align*}
\Delta(B; N_\beta[M]) &\leq K |\Delta(B; N_\beta[G_\beta(2 + d + l)])| + (L - K) |\Delta(B; N_\beta[H_\beta(2 + d + l)])| \\
&\quad + K |\Delta(B; N_\beta[G_\beta(1 + d + l)])| + (L - K - 1) |\Delta(B; N_\beta[H_\beta(1 + d + l)])|
\end{align*}
\]

\[+ K |\Delta(B; N_\beta[G_\beta(d + l)])| + (L - K - 1) |\Delta(B; N_\beta[H_\beta(d + l)])| \]

\[+ K |\Delta(B; N_\beta[G_\beta(2 + d + l)])| + (L - K - 1) |\Delta(B; N_\beta[H_\beta(2 + d)])|
\]

\[+ (L - K) \sum_{i=0}^{l} |\Delta(B; N_\beta[G_\beta(2 + d + i)])|
\]

\[+ (L - K) \sum_{i=0}^{l} |\Delta(B; N_\beta[H_\beta(2 + d + i)])|
\]

where \( B = B(k_1, \ldots, k_n) \). From (2.13), (2.14), (2.17) and the fact that \(|\alpha| < 1 < |\beta|\), there exists a constant \( C_1 \) that satisfies the following inequality (2.18) for any cylinder \( B(k_1, \ldots, k_n) \) of any rank \( n \) and any integer \( M > G_{\beta}(2 + d) \).

\[
|\Delta(B(k_1, \ldots, k_n); N_\beta[M])| < C_1
\]

Choose an arbitrary \( u \in [0,1) \). Let \( M \in \mathbb{N} \) and \( l \) be an integer that satisfies

\( G_\beta(l) \leq M < G_\beta(l+1) \).

Let \( B(u_1, \ldots, u_l) \) be a cylinder of rank \( l \) that satisfies \( u \in B(u_1, \ldots, u_l) \). Then we have

\[
[0, u) = B_{t_1} \cup B_{t_2} \cup \cdots \cup B_{t_k} \cup R
\]

\[
0 \leq t_1 < t_2 < \cdots < t_k = l
\]

where \( B_{t_i} \) is a cylinder of rank \( t_i \) and \( \lambda(R) < \beta^{-l} \). From (2.8.G), there exist constants \( C_2 \) and \( C_3 \) that satisfy \( l < C_2 \log M \) and \( M \beta^{-l} < C_3 \). Then, from (2.18) and (2.19), we have

\[
|\Delta([0, u); N_\beta[M])| < C_1 C_2 \log M + C_3.
\]

The theorem follows from this.
References


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