数理解析研究所講究録
1011巻 1997年 16-24

Constructions of Piecewise Linear Maps Generating Markov Sources

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1 Introduction

Nonlinear one-dimensional maps may be the simplest object that can produce chaos. There are several attempts to produce pseudo-random numbers by using one-dimensional maps displaying chaos.[1]-[3]

Pseudo-random sequences generated by dynamical systems can possess statistical properties of Markov or hidden Markov information sources.[4] On the other hand, we can consider the inverse problem of constructing a dynamical system which has a stochastic structure of an arbitrary prescribed Markov source. In this paper we give two solutions for this problem. Using piecewise-linear maps, we construct a dynamical system which has a structure of an arbitrary prescribed finite state Markov chain. A construction of a dynamical system having a statistical structure of an arbitrary prescribed reverse Markov source is also given.

A construction of piecewise-linear maps that realize discrete memoryless information sources was given by Kac[5] for uniform distributions and Kanaya[6] for non-uniform distributions, respectively. It is interesting to note that the proposed constructions of Markov and Reverse Markov sources contain the results of Kac and Kanaya, as special cases, respectively.

2 Dynamical Systems Realizing Markov Sources

We consider a Markov chain with a state $S=\{1,2,\cdots,n\}$. Let $p_{ij},\ i,j=1,2,\cdots,n$ be transition probabilities from the state $i\in S$ to the state $j\in S$. Let $P=[p_{ij}]$ denote by the stochastic matrix that has transition probabilities p_{ij} as the (i,j)-element for $i,j=1,2,\cdots,n$. We write an initial probability distribution as $\boldsymbol{p}=[p_1,p_2,\cdots,p_n]$. Markov source is defined by the stochastic process $\{X_N\}_{N=0}^{\infty}$ produced by the triple of state space S, stochastic matrix P and initial probability vector $\boldsymbol{p}$. We denote it by $\mathcal{M}(S,P,\boldsymbol{p})$. The generic probability of a string $a_0a_1\cdots a_{N-1}\in S^N$ is given by

$$Prob\{X_0=a_0,X_1=a_1,\cdots,X_{N-1}=a_{N-1}\}$$

$$= p_{a_0} p_{a_0 a_1} \cdots p_{a_{N-2} a_{N-1}} \,. \tag{1}$$

Our aim is to construct a dynamical system which is equivalent to an arbitrary prescribed Markov source. We first define a dynamical system equivalent to a given Markov source. Let I denote the closed interval $[0,1]$ and $\tau : I \to I$ be a nonlinear map. Consider the following decomposition of I:

$$0 = \Delta_0 \leq \Delta_1 \leq \Delta_2 \leq \cdots \leq \Delta_n = 1\,.$$

and set

$$\begin{aligned} I_k &= [\Delta_{k-1}, \Delta_k) \quad \text{for } k = 1, 2, \cdots, n-1 \\ I_n &= [\Delta_{n-1}, \Delta_n]\,. \end{aligned}$$

Define the quantizer σ by

$$\sigma(x) = k \quad \text{for } x \in I_k\,.$$

For any positive integer N and any string $a_0 a_1 \cdots a_{N-1} \in S^N$, let $A(a_0 a_1 \cdots a_{N-1})$ denote the set each element x of which produces $a_0 a_1 \cdots a_{N-1}$ by the random number generator $\left\{\sigma(\tau^k(x))\right\}_{k=0}^{N-1}$ and $\mu[a_0 a_1 \cdots a_{N-1}]$ denote the measure of the set $A(a_0 a_1 \cdots a_{N-1})$, that is,

$$\begin{aligned} & A(a_0 a_1 \cdots a_{N-1}) \\ = & \left\{x : \sigma(x) = a_0, \sigma(\tau(x)) = a_1, \cdots, \sigma(\tau^{N-1}(x)) = a_{N-1}\right\} \end{aligned} \tag{2}$$

$$\begin{aligned} & \mu[a_0 a_1 \cdots a_{N-1}] \\ = & \mu(A(a_0 a_1 \cdots a_{N-1}))\,. \end{aligned} \tag{3}$$

Let $\mathcal{F}(\sigma, \tau, \mu)$ denote the dynamical system that consists of the triple of quantizer σ, one-dimensional map τ and measure μ. We call $\mathcal{F}(\sigma, \tau, \mu)$ is equivalent to $\mathcal{M}(S, P, \boldsymbol{p})$ if

$$\mu[a_0 a_1 \cdots a_{N-1}] \tag{4}$$

$$= p_{a_0} p_{a_0 a_1} \cdots p_{a_{N-2} a_{N-1}} \tag{5}$$

for any N and any string $a_0 a_1 \cdots a_{N-1} \in S^N$.

3 The Construction of Markov Sources by Piecewise Linear Maps

In this section we construct a dynamical system that is equivalent to an arbitrarily prescribed Markov source $\mathcal{M}(S, P, \boldsymbol{p})$.

Let the decomposition of the interval $I = [0,1]$ be

$$\Delta_k = \frac{k}{n} \quad \text{for } k = 1, 2, \cdots, n$$

and $\tau_k(x)$, $k = 1, 2, \cdots, n$ be the restriction of the map $\tau(x)$ to subinterval I_k. Define $\tau_k(x)$, $k = 1, 2, \cdots, n$ by

$$\left.\begin{array}{l} \tau_k(x) = p_{kl}^{-1}(x - c_{kl-1}) + \dfrac{l-1}{n} \\ \qquad \text{for } c_{kl-1} \le x \le c_{kl} \text{ and } l = 1, 2, \cdots, n, \end{array}\right\} \tag{6}$$

where c_{kl} is defined by

$$\left.\begin{array}{ll} c_{k0} = \dfrac{k-1}{n}, & \\ c_{kl} = \dfrac{1}{n} \cdot \displaystyle\sum_{j=1}^{l} p_{kj} + \dfrac{k-1}{n} & \text{for } k, l = 1, 2, \cdots, n. \end{array}\right\} \tag{7}$$

Furthermore define μ by

$$\mu(x)dx = np_k dx \quad \text{for } x \in I_k. \tag{8}$$

For the dynamical system $\mathcal{F}(\sigma, \tau, \mu)$ constructed as above, the following theorem holds.

Theorem 1 The dynamical system $\mathcal{F}(\sigma, \tau, \mu)$ is equivalent to $\mathcal{M}(S, P, \boldsymbol{p})$.

Proof: To compute $\mu[a_0 a_1 \cdots a_{N-1}]$, let BV the set of bounded variation, and for any $f \in BV$, define the linear operator $P_\tau : BV \to BV$ called Perron-Frobenius operator (PF-operator) by

$$P_\tau(f(x)) = \sum_{y \in \tau^{-1}(x)} \frac{f(y)}{|\tau'(y)|}. \tag{9}$$

By the definition of the PF-operator

$$\int_0^1 f(x) g(\tau(x)) dx = \int_0^1 P_\tau(f(x))\, g(x) dx \tag{10}$$

can easily be verified. Next, let $y_k(x) : I_k \to I$, $k = 1, 2, \cdots, n$ be inverse branches of $\tau(x)$ and $\eta_k(x)$, $k = 1, 2, \cdots, n$ be indicator functions that take 1 on I_k and otherwise 0. By the definition (9) of PF-operator we have

$$\begin{aligned} P_\tau(\eta_k(x)\mu(x)) &= \sum_{j=1}^{n} \frac{\eta_k(y_j(x))\mu(y_j(x))}{|\tau'(y_j(x))|} \\ &= \frac{np_k}{|\tau'(y_k(x))|} \\ &= \sum_{j=1}^{n} np_k p_{kj} \eta_j(x) \end{aligned} \tag{11}$$

for $k = 1, 2 \cdots, n$. Furthermore, it follows from similar calculations that

$$P_\tau(\eta_k(x)) = \sum_{j=1}^{n} p_{kj} \eta_j(x) \tag{12}$$

for $k = 1, 2 \cdots, n$. The calculation of $\mu[a_0 a_1 a_2 \cdots a_{N-1}]$ by using (10), (11) and (12) yields the following chain of equalities:

$$
\begin{aligned}
& \mu[a_0 a_1 a_2 \cdots a_{N-1}] \\
= & \int_0^1 \eta_{a_0}(x)\eta_{a_1}(\tau(x))\eta_{a_2}(\tau^2(x)) \\
& \quad \cdots \eta_{a_{N-1}}(\tau^{N-1}(x))\mu(x)dx \\
= & \int_0^1 P_\tau\left(\eta_{a_0}(x)\mu(x)\right)\eta_{a_1}(x)\eta_{a_2}(\tau(x)) \\
& \quad \cdots \eta_{a_{N-1}}(\tau^{N-2}(x))dx \\
= & \int_0^1 \sum_{j=0}^{l} n p_{a_0} p_{a_0 j}\eta_j(x)\eta_{a_1}(x)\eta_{a_2}(\tau(x)) \\
& \quad \cdots \eta_{a_{N-1}}(\tau^{N-2}(x))dx \\
= & \; n p_{a_0} p_{a_0 a_1} \int_0^1 \eta_{a_1}(x)\eta_{a_2}(\tau(x)) \\
& \qquad \cdots \eta_{a_{N-1}}(\tau^{N-2}(x))dx \\
= & \; n p_{a_0} p_{a_0 a_1} \int_0^1 P_\tau\left(\eta_{a_1}(x)\right)\eta_{a_2}(x) \\
& \qquad \cdots \eta_{a_{N-1}}(\tau^{N-3}(x))dx \\
= & \; n p_{a_0} p_{a_0 a_1} \int_0^1 \sum_{j=0}^{l} p_{a_1 j}\eta_j(x)\eta_{a_2}(x) \\
& \qquad \cdots \eta_{a_{N-1}}(\tau^{N-3}(x))dx \\
= & \; n p_{a_0} p_{a_0 a_1} p_{a_0 a_2} \int_0^1 \eta_{a_2}(x) \cdots \eta_{a_{N-1}}(\tau^{N-3}(x))dx \\
\cdots & \\
= & \; n p_{a_0} p_{a_0 a_1} p_{a_1 a_2} \cdots p_{a_{N-2} a_{N-1}} \int_0^1 \eta_{a_{N-1}}(x)dx \\
= & \; p_{a_0} p_{a_0 a_1} p_{a_1 a_2} \cdots p_{a_{N-2} a_{N-1}}\,,
\end{aligned}
\tag{13}
$$

which coincides with (1). ∎

In general any hidden Markov chain can be considered as a functional process over a certain Markov chain. Accordingly, a dynamical system that realizes an arbitrary prescribed hidden Markov chain can easily obtained from a dynamical system equivalent to a suitable Markov chain a certain functional process over which coincides the given hidden Markov chain.

Kac[5] proved that for integer $g \geq 3$ the map $\tau(x) = gx \pmod 1$ generates independently uniformly distributed random sequence over the set of g-alphabets, $\{0, 1, \cdots, g-1\}$. Our way of construction contains the result of Kac as a special case when we choice $\tau_k(x)$, $k = 1, 2, \cdots, n$ as straight lines and $\mu(x) = 1$.

The state diagram for the Markov chain with state $S = \{1, 2, 3\}$ and stochastic matrix

$$
P = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \tag{14}
$$

is shown in Fig. 1. An example of realization of this Markov chain with a piece-wise linear map is shown in Fig. 2.

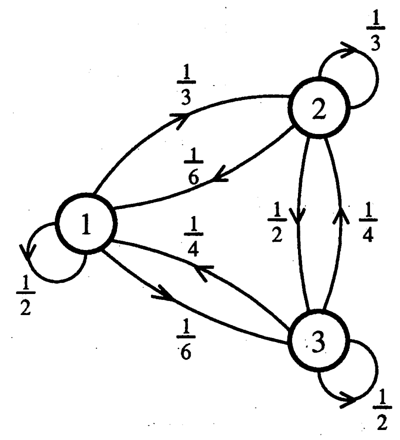

Fig. 1. The state diagram for the Markov source.

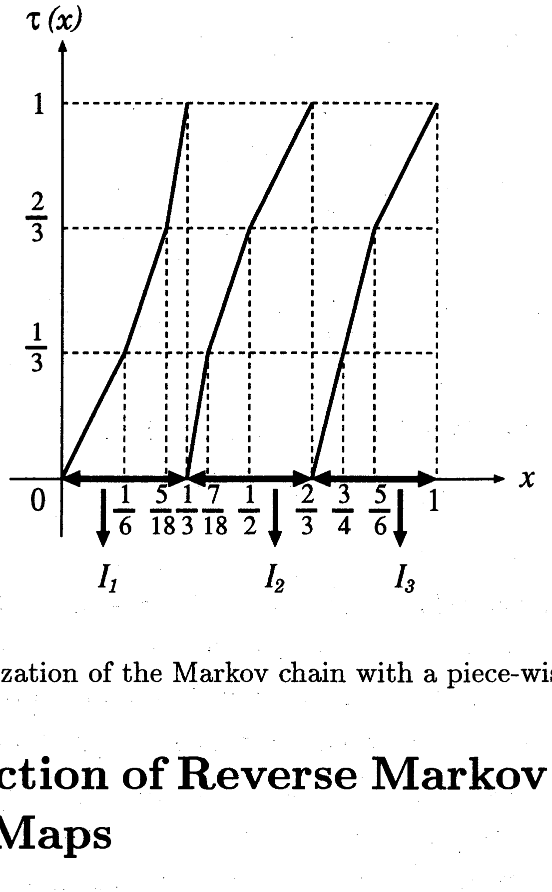

Fig. 2. A realization of the Markov chain with a piece-wise linear map.

4 The Construction of Reverse Markov Chains by Piece-wise Linear Maps

Consider the state S and stochastic matrix P on S. If the stochastic matrix P is irreducible, then there exists an unique provability vector $\boldsymbol{u} = [u_1, u_2, \cdots, u_n]$, $u_k > 0\,, k = 1, 2, \cdots, n$ such that

$$\boldsymbol{u} = \boldsymbol{u}P\,. \tag{15}$$

If the generic probability of any string $a_0a_1\cdots a_{N-1}\in S^N$ from the source defined by the stochastic process $\{X_N\}_{N=0}^{\infty}$ is given by

$$
\begin{aligned}
&Prob\,\{X_0=a_0, X_1=a_1,\cdots,X_{N-1}=a_{N-1}\}\\
&=u_{a_{N-1}}p_{a_{N-1}a_{N-2}}\cdots p_{a_1a_0}\,,
\end{aligned}
\tag{16}
$$

then we call the source $\{X_N\}_{N=0}^{\infty}$ the reverse Markov source with (S,P) and denote it by $\mathcal{M}^*(S,P)$.

Consider a dynamical system $\mathcal{F}(\sigma,\tau,\mu)$ with uniform measure μ on the interval I. In particular, let $\mathcal{F}^*(\sigma,\tau)$ denote a dynamical system with uniform measure μ on the interval I. In this section we show that for an arbitrary prescribed reverse Markov source $\mathcal{M}^*(S,P)$, we can construct a pair of quantizer σ and nonlinear map τ so that the dynamical system $\mathcal{F}^*(\sigma,\tau)$ is equivalent to $\mathcal{M}^*(S,P)$.

We first consider cumulative probabilities

$$
c_0=0\,,\;\; c_k=\sum_{j=1}^{k}u_j \quad \text{for } k=1,2,\cdots,n \tag{17}
$$

and define the decomposition of the interval $I=[0,1]$ by $\Delta_k=c_k\,,\; k=1,2,\cdots,n$. Next, for $k=1,2,\cdots,n$, define the map $\tau_k(x):I_k\to I$ by

$$
\left.
\begin{aligned}
&\tau_k(x)=p_{lk}^{-1}\left(x-c_{l-1k}^*\right)+c_{l-1}\\
&\qquad \text{for } c_{l-1k}^*\le x\le c_{lk}^* \text{ and } l=1,2,\cdots n\,,
\end{aligned}
\right\}
\tag{18}
$$

where c_{lk}^* is

$$
\left.
\begin{aligned}
&c_{0k}^*=c_{k-1}\,,\\
&c_{lk}^*=\sum_{j=1}^{l}u_jp_{jk}+c_{k-1} \quad \text{for } k,l=1,2,\cdots,n\,.
\end{aligned}
\right\}
\tag{19}
$$

For the dynamical system $\mathcal{F}^*(\sigma,\tau)$ constructed as above, the following theorem holds.

Theorem 2 The dynamical system $\mathcal{F}^*(\sigma,\tau)$ is equivalent to the reverse Markov model $\mathcal{M}^*(S,P)$.

Proof: To calculate the probability measure $\mu[a_0a_1\cdots a_{N-1}]$ consider the indicator function $\eta_k(x)\,,k=1,2,\cdots,n$ that take 1 on the interval I_k 1 and 0 on otherwise. From the definition of PF-operator (9), we have, for $k=1,2\cdots,n$,

$$
P_\tau(\eta_k(x))=\sum_{j=1}^{n}p_{jk}\eta_j(x)\,. \tag{20}
$$

Using the above formula and (10) we can calculate $\mu[a_0,a_1\cdots a_{N-1}]$ to verify that it coincides with (16). Since the argument is quite similar to that of the previous section, we omit the detail. ∎

Kanaya[6] provided a construction of a dynamical system which is equivalent to an arbitrary prescribed discrete memoryless source. Our way of construction contains that of Kanaya as a special case by letting $\tau_k(x)\,,k=1,2,\cdots,n$ be straight lines.

We consider an example that $S = \{1, 2, 3\}$ and stochastic matrix P is the same as specified (14) in the previous section. For this case the stationary probability is $\boldsymbol{u} = [\frac{3}{10}, \frac{3}{10}, \frac{2}{5}]$. An example of construction of the reverse Markov source with a piecewise linear map is shown in Fig. 3.

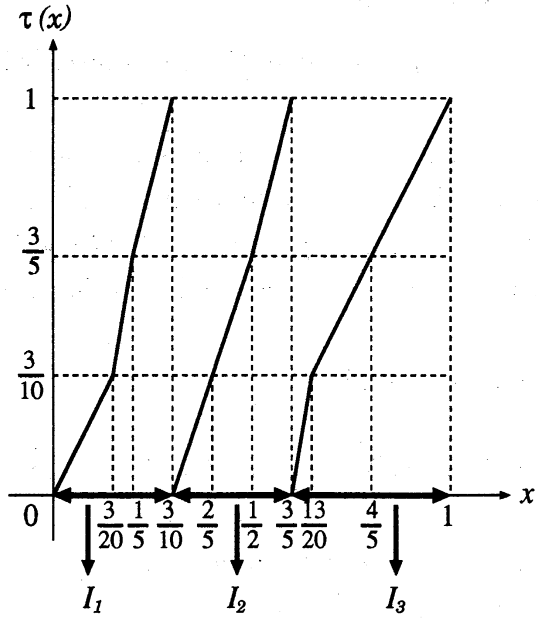

Fig. 3. A realization of the reverse Markov source with a piecewise linear map.

The next example is that $S = \{1, 2\}$ and stochastic matrix P is

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}.$$

The stationary probability for this case is $\boldsymbol{u} = [\frac{2}{3}, \frac{1}{3}]$. The state diagram for this reverse Markov model is shown in Fig. 4. An example of construction of the reverse Markov source with a piecewise linear map is shown in Fig. 5. It can be seen from the form of the map that this source has the structure of hidden Markov source as shown in Fig. 6.

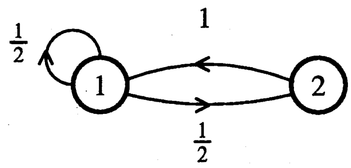

Fig. 4. The state diagram for the reverse Markov source.

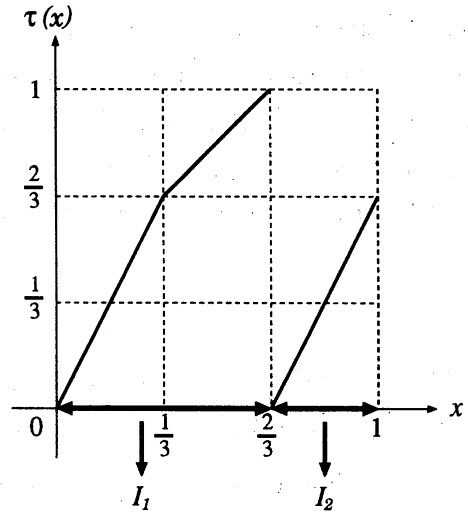

Fig. 5. A realization of the reverse Markov source with a piecewise linear map.

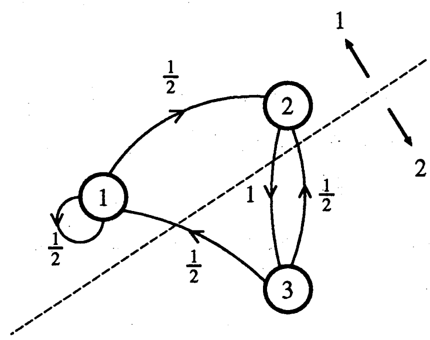

Fig. 6. A hidden Markov source equivalent to the reverse Markov source.

5 Conclusions

We have considered the construction problem of dynamical systems that realize arbitrary prescribed Markov (or reverse Markov) sources and provided solutions for this problem by giving two ways of construction with piecewise linear maps. The established representations of information sources by dynamical systems are useful not only for the design of pseudo-random number generator that has prescribed statistical properties but also the design of white chaos,[7] source coding problems, model estimations for sources.

One of further researches is the construction of dynamical systems that realize a certain wider family of information sources containing hidden Markov sources as a special case. The linearly dependent process is one of such families of information sources. In general the generic probability of any string emitted from the hidden Markov source can be expressed by the product of several sub-matrices of stochastic matrix.[8] On the other hand, the generic probability of any string emitted from the linearly dependent process can be expressed by the product of several sub-matrices of the matrix whose elements are not always non-negative.[8]

In our constructions, elements of the stochastic matrix correspond to absolute inverse values of the gradient of piecewise linear functions. On the other hand, in order to solve the construction problem of an arbitrary prescribed linearly dependent process, we should characterize, as parameters of a suitable dynamical system, not necessary non-negative elements appearing in the matrix that defines the linearly dependent process.

However, this problem is very hard and no key for resolutions has not been found so far.

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