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ON BLOCH FUNCTIONS AND THE CONTRACTION OF TEICHMÜLLER METRICS

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ABSTRACT. In this note, we consider the properties of Bloch functions determined by Beltrami coefficient. A sufficient condition for extremal quasiconformal mapping with nonexistence of degenerating sequence is obtained. As a result, we consider the contraction or preserved of Teichmüller metrics for the related Beltrami lines under the projection mapping $\pi$.

1. INTRODUCTION

Let $Q_I$ be the class of quasiconformal mappings $f$ of the unit disk $D = \{z| |z| \leq 1\}$ onto itself with $f(0) = f(1) - 1 = 0$, $\mu_f$ be the complex dilatation of $f$, $k_f = \|\mu_f\|_\infty = \operatorname{esssup}_{z \in D} |\mu_f|$, $k_0(f) = \inf_{g} k_g$, where $g \in Q_I$ with $g|_{\partial D} = f|_{\partial D}$. We say that $f(z)$ is extremal if $k_f = k_0(f)$, and the corresponding $\mu_f$ is called extremal.

We know that the universal Teichmüller space $T(1)$ can be represented as a quotient space of $QS$ by the Möbius group $PSL(2, R)$, where $QS$ is the group of all quasi-symmetric homeomorphisms of a circle, and the Teichmüller distance $d([f],[g])$, from a point $[g]$ to another point $[f]$ in $T(1)$, is equal to

$$d([f],[g]) = \frac{1}{2} \log \frac{1 + k_0(g \circ f^{-1})}{1 - k_0(g \circ f^{-1})}. \tag{1.1}$$

$QS$ contains another topological subgroup, which is much larger than $PSL(2, R)$, the subgroup $S$ of symmetric homeomorphisms. Gardiner-Sullivan [1] showed that $QSmodS$ also has a natural complex Banach manifold structure and a natural quotient metric $\overline{d}$, called the Teichmüller metric in $QSmodS$. Let $k_f = \inf_{U} \operatorname{esssup}_{x \in U} |\mu_f(z)|$, where $U$ moves all neighborhoods of $\partial D$ in $D$, $\overline{k}_f$ is called the boundary dilatation of $f$. Set $\overline{k}_0(f) = \inf_{g} \overline{k}_g$, where $g$ moves all quasiconformal mappings of $D$ with the same boundary values as $f$. If $\overline{k}_0(f) = \overline{k}_f$, then $f(z)$ is called extremal in $QSmodS$. The distance between two points $\pi[f]$ and $\pi[g]$ in $QSmodS$ is equal to

$$\overline{d}(\pi[f], \pi[g]) = \frac{1}{2} \log \frac{1 + \overline{k}_0(g \circ f^{-1})}{1 - \overline{k}_0(g \circ f^{-1})}. \tag{1.2}$$

Suppose $\mu(z)$ is a given Beltrami coefficient, we consider the Beltrami line $C_\mu = \{[f^t]| -1 \leq t \leq 1\}$ or $\pi C_\mu = \{\pi[f^t]| -1 \leq t \leq 1\}$, where $f^t = t \frac{\mu}{\|\mu\|_\infty}$. If $\mu$ is

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extremal in $T(1)$ or in $QSmodS$, then the natural mapping $t \mapsto t\|\mu\|_{\infty}$ from the open interval $(-1, 1)$ with the Poincaré metric onto $C_{\mu}$ or $\pi C_{\mu}$ with the Teichmüller metric is an isometry. Whether $\mu$ is extremal or not, such mapping is weakly contracting. The following problem is very interesting and considered by many authors(cf. [2],[3]):

For which points $[f] \in T(1)$, does the Teichmüller distance from 0 to $[f]$ in $QS$ strictly greater than the distance from 0 to $\pi[f]$ in $QSmodS$?

In this note, we will investigate some properties for Bloch functions determined by $\mu$ and partially solve the above problem.

2. MAIN RESULTS AND THEIR PROOFS

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $D$, $f(z)$ is called a Bloch function if

\begin{equation}
\|f\|_B = \sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty.
\end{equation}

The Bloch functions will be denoted by $B$. $B_0$ will be the subset of $B$ with

\begin{equation}
\|f\|_{B_0} = \limsup_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0.
\end{equation}

$A(D) = \{f(z)|f(z)\}$ analytic in $D$, $\|f(z)\|_1 = \frac{1}{\pi} \iint_{D} |f(z)| \, dx \, dy < \infty$. The quasi-conformal mapping $f$ from $D$ onto itself is called a Teichmüller mapping of finite type, if $\mu_f = \|\mu(z)\|_{\infty}\overline{\varphi}/|\varphi|, \varphi_0 \in A(D)$. From Reich’s example(cf.[4]), we know that even the point $[f]$ corresponds to a Teichmüller mapping of finite type, the distance from 0 to $[f]$ under the projection $\pi$ may not contract. However, if $[f] \in T(1)$, and $\overline{d}(0, \pi[f]) < d(0, [f])$, then $[f]$ contains a Teichmüller mapping of finite type. This makes the above problem more complicated.

Suppose $\kappa(z) \in L^\infty(D)$, the space of complex-valued bounded measurable functions in $D$ with $\|\kappa\|_\infty = \text{esssup}_{z \in D} |\kappa(z)|$, we consider a linear functional $L_\kappa$ on $A(D)$

\begin{equation}
L_\kappa(f) = \frac{1}{\pi} \iint_{D} \kappa(z)f(z) \, dx \, dy, \quad f(z) \in A(D),
\end{equation}

then

\begin{equation}
\|L_\kappa\| \leq \|\kappa\|_\infty.
\end{equation}

Hamilton, Reich and Streble [5, 6] showed that

**Theorem A.** A Beltrami coefficient $\mu$ is extremal if and only if one of the following statements holds:

1) There exist $\varphi \in A(D)$ and $k \in [0, 1)$ such that $\mu = k\varphi/|\varphi|$ for almost everywhere on $D$.

2) There is a degeneration sequence $\{\varphi_n\} \in A(D), \|\varphi_n\|_1 = 1$, converging to 0 locally uniformly in $D$, such that
\begin{equation}
\lim_{n \to \infty} \left| \iint_{D} \varphi_{n} \mu \, dx \, dy \right| = \| \mu \|_{\infty}.
\end{equation}

For a given Beltrami coefficient \( \mu(z) \), let

\begin{equation}
b_{n} = \frac{n + 2}{\pi} \iint_{D} \mu(z) z^{n} \, dx \, dy, \quad g(\zeta) = \sum_{n=0}^{\infty} b_{n} \zeta^{n},
\end{equation}

it is clearly that \( |b_{n}| \leq 2\| \mu(z) \|_{\infty} \) and \( g(\zeta) \) is analytic in \( D \). We call that the analytic function \( g(\zeta) \) is determined by \( \mu(z) \).

Let \( G(\zeta) = (g(\zeta)) \), Anderson proved in [7] the following

**Theorem B.** For a given \( \mu(z) \in L^{\infty}(D) \), then

\begin{equation}
\| L_{\mu} \| \leq \| G(\zeta) \|_{B} \leq 4 \| L_{\mu} \|,
\end{equation}

where \( G'(\zeta) = \frac{2}{\pi} \iint_{D} \frac{\mu(z)}{(1 - (z)^{3})} \, dx \, dy \).

**Theorem C.** If \( \mu(z) \) possesses a degenerating sequence, then

\begin{equation}
\| L_{\mu} \| \leq \lim_{|z| \to 1} \sup (1 - |z|^{2}) |G'(z)|,
\end{equation}

where \( G'(\zeta) = \frac{2}{\pi} \iint_{D} \frac{\mu(z)}{(1 - (z)^{3})} \, dx \, dy \). In particular, if

\begin{equation}
\iint_{D} \frac{\mu(z)}{(1 - (z)^{3})} \, dx \, dy = o(1 - |\zeta|^{2})^{-1} \quad (|\zeta| \to 1^{-}),
\end{equation}

then \( \mu(z) = \| \mu \|_{\infty} \frac{g(\zeta)}{|\varphi_{0}(\zeta)|} \), \( \varphi_{0} \in A(D) \), for almost all \( z \in D \).

Theorem C means that if \( \mu(z) \) is extremal and \( \lim_{|z| \to 1} \sup (1 - |z|^{2}) |G'(z)| = 0 \), then

\[ \mu(z) = \| \mu \|_{\infty} \frac{\varphi_{0}(z)}{|\varphi_{0}(z)|}, \quad \varphi_{0}(z) \in A(D), \]

for almost everywhere \( z \in D \). For an extremal quasiconformal mapping \( f^{\mu(z)} \in Q_{I} \), in what case, is it a finite type Teichmüller mapping or even has it no degenerating sequence? This problem is very interesting itself (cf. [8, 9] and the references cited there). First, we will prove the following

**Theorem 1.** Suppose \( \mu(z) \) is extremal, let \( g(z) \) be defined in (2.6), if there exists a \( \rho_{0}, 0 < \rho_{0} < 1 \), such that

\begin{equation}
\sup_{|z| < 1} (1 - |z|^{2}) |g'(z)| < 1,
\end{equation}

then there exists a \( \varphi_{0} \in A(D) \) with \( \mu(z) = \| \mu(z) \|_{\infty} \frac{\varphi_{0}(z)}{|\varphi_{0}(z)|} \) for almost all \( z \in D \). In particular, \( \mu(z) \) possesses no degenerating sequence.

**The proof of Theorem 1.** If \( \mu(z) \) is an extremal Beltrami coefficient, let \( g(\zeta) \) be defined in (2.6), if \( f(z) = \sum_{n=0}^{\infty} a_{n} z^{n} \in A(D), 0 < \rho < 1 \), we have

\[ L_{\mu}(f(\rho z)) = \sum_{n=0}^{\infty} a_{n} \rho^{n} L_{\mu}(z^{n}) = \sum_{n=0}^{\infty} \frac{a_{n} b_{n}}{n + 2} \rho^{n}. \]
Since $\|f(\rho z) - f(z)\|_1 \to 0$, when $\rho \to 1^-$, then we have

$$L_{\mu}(f) = \lim_{\rho \to 1^-} \sum_{n=0}^{\infty} \frac{a_n b_n}{n + 2} \rho^n.$$  

On the other hand, if $G(\zeta) = \zeta g(\zeta)$, then

$$\frac{1}{2\pi} \int_{0}^{2\pi} f(re^{i\theta}) G'(\zeta re^{-i\theta}) d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} (\sum_{n=0}^{\infty} a_n r^n e^{in\theta})(\sum_{n=0}^{\infty} (n + 1) b_n \zeta^n r^n e^{-in\theta}) d\theta = \sum_{n=0}^{\infty} (n + 1) a_n b_n \zeta^n r^{2n}.$$  

Thus, we have

$$(2.11) \quad \lim_{\rho \to 1^-} \sum_{n=0}^{\infty} \frac{a_n b_n}{n + 2} \rho^n = \frac{1}{\pi} \int_{0}^{1} \int_{0}^{2\pi} f(re^{i\theta}) G'(\zeta re^{-i\theta})(1 - r^2) r \, dr \, d\theta,$$  

for any $f(z) \in A(D)$. Since

$$g(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n = \sum_{n=0}^{\infty} \frac{n + 2}{\pi} \int_{D} x^n \mu(z) \, dx \, dy \zeta^n = \frac{1}{\pi} \int_{D} (\sum_{n=0}^{\infty} (n + 2) x^n \zeta^n \mu(z)) \, dx \, dy = \frac{1}{\pi} \int_{D} [\frac{2 - z\zeta}{(1 - z\zeta)^2}] \mu(z) \, dx \, dy,$$  

then,

$$(2.11) \quad |g(\zeta)| \leq \frac{3\|\mu\|_{\infty}}{\pi|\zeta|} \log \frac{1 + |\zeta|}{1 - |\zeta|} = o((1 - |\zeta|^2)^{-1}), \quad |\zeta| \to 1^-.$$  

If $\{f_n(z)\}$ is a degenerating sequence for $\mu(z)$ with $\|f_n\|_1 = 1$, by Theorem B and (2.11), we can choose a $\rho'$ with $\rho_0 < \rho' < 1$ such that

$$|L_{\mu}(f_n)| \leq \frac{4\|\mu\|_{\infty}}{\pi} \int_{|z| \leq \rho'} |f_n(re^{i\theta})| r \, dr \, d\theta + \sup_{\rho' < |z| < 1} (1 - |z|^2)|g(z)| \sup_{\rho' < |z| < 1} (1 - |z|^2)|g'(z)| < 1,$$  

which contradicts that $\{f_n(z)\}$ is a degenerating sequence. By Theorem A, Theorem 1 is proved.

The following example 1 shows that there is non-extremal Beltrami coefficient $\mu(z)$ with the bound $\sup_{\rho_0 < |z| < 1} (1 - |z|^2)|g'(z)| = \frac{2}{\pi}$.  

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Example 1. Set Beltrami coefficient
\[
\mu(z) = \begin{cases} 
1, & \text{for } \Re z \geq 0, |z| < 1 \\
0, & \text{for } \Re z < 0, |z| < 1.
\end{cases}
\]

Then by [8, Theorem 1], we see that \(\mu(z)\) is not extremal. In this case, by calculation, we have
\[
g'(z) = 2 + \frac{2i}{\pi} [z + \frac{1}{3} z^3 + \cdots + \frac{1}{2n-1} z^{2n-1} + \cdots]
\]
and \(\lim_{|z| \to 1} (1 - |z|^2)|g'(z)| = \frac{2}{\pi}\).

Next we will investigate the relationship between extremal Beltrami coefficient \(\mu\) and the coefficients of \(g(z)\) defined in (2.6).

From [11] and Theorem 1, we know that if \(\mu(z)\) is extremal and the determined analytic function \(g(z) \in B_0\), then \(\lim_{n \to \infty} |b_n| = 0\). However, we also know that even if \(f(z) \in B\) and \(\lim_{n \to \infty} |b_n| = 0\), one can not derive that \(f(z) \in B_0\). From this we will prove the following

Corollary 1. Suppose \(\mu(z)\) is extremal, and let \(g(z) = \sum_{n=0}^{\infty} b_n z^n\) be defined in (2.6), if there exist a positive number \(N_0\) and \(l, 0 < l < \frac{1}{2}\), such that
\[
|b_n| < \frac{l}{n}, \quad \text{for } n > N_0,
\]
then there exists a \(\varphi_0(z) \in A(D)\) with
\[
\mu(z) = ||\mu||_{\infty} \frac{\varphi_0}{|\varphi_0|}, \quad \text{for almost all } z \in D.
\]

The proof of Collary 1. If \(\mu(z)\) is extremal, and let \(g(z) = \sum_{n=0}^{\infty} b_n z^n\) be defined in (2.6), we have
\[
|g'(z)| \leq \sum_{n=0}^{N_0} nb_n z^n + \sum_{n=N_0+1}^{\infty} l |z^n|
\leq |\sum_{n=0}^{N_0} nb_n z^n| + l \frac{|z|^{N_0+1}}{1 - |z|},
\]
thus there exists a \(\rho_0 > 0\), such that \(\sup_{|z| < 1} (1 - |z|^2)|g'(z)| < 1\), by Theorem 1, we obtain the assertion.

Let \(\Pi\) denote the subset of \(T(1)\) consisting of elements of \([f]\) which correspond to Teichmüller mappings of finite type whose complex dilatations \(\mu = \mu_f\) satisfy the following condition: There exists a \(\rho_0, 0 < \rho_0 < 1\), such that \(\sup_{|z| < 1} (1 - |\zeta|^2)|g'(\zeta)| < 1\), where \(g(\zeta)\) is defined in (2.6). We will prove the following

Theorem 2. For \([f] \in \Pi\), then \(d(0, \pi([f])) < d(0, [f])\).

In order to prove Theorem 2, we need the following Theorem D due to Gardiner [2].
Theorem D. For every \([f] \in T(1)\), then \(k_f = k_0(f)\) if and only if

\[
\sup \limsup_{n \to \infty} \left| \int_D \varphi_n \mu_f \, dxdy \right| = k_f,
\]

where the supremum is taken over all degenerating sequences \(\{\varphi_n\}\) for \(\mu_f\) with \(\|\varphi_n\|_1 = 1\) in \(A(D)\).

The proof of Theorem 2. We use the same way as in [3] to prove Theorem 2. If \([f] \in \Pi\), then we conclude that \(k_0(f) = k_0(f)\). On the contrary, by Theorem D, we can find a degenerating sequence \(\{\varphi_n\}\) with \(\|\varphi_n\|_1 = 1\) such that

\[
\lim_{n \to \infty} \left| \int_D \varphi_n \mu_f \, dxdy \right| = \|\mu_f\|_{\infty} = k_0(f) = k_0(f),
\]

which is impossible by Theorem 1.

Thus we have \(k_0(f) < k_0(f)\), which is equivalent to \(d(0, \pi([f])) < d(0, [f])\).

On the other hand, comparing with Theorem 2, we will prove the following

Theorem 3. Suppose \([f] \in T(1)\), and \(b_n = \frac{n+2}{\pi} \int_D \mu_f x^n \, dxdy\), if \(\lim_{n \to \infty} b_n = 2\|\mu_f\|_{\infty}\), then \(d(0, \pi([f])) = d(0, [f])\). The constant 2 is the best.

The proof of Theorem 3. First, from Fehlmann and Sakam's paper in [10], we know that the subset of \(T(1)\) satisfying the conditions in Theorem 3 is not empty, and by the example of Fehlmann and Sakam made in [10], there exists an extremal Beltrami coefficient \(\mu\) such that the coefficients of \(g(z)\) satisfy \(\lim_{n \to \infty} b_n = 2\|\mu\|_{\infty}\), thus the constant 2 is the best. Now, if \(\lim_{n \to \infty} b_n = 2\|\mu_f\|_{\infty}\), then we have \(\lim_{j \to \infty} b_{n_j} = 2\|\mu_f\|_{\infty}\), and the sequence \(\{\varphi_{n_j}(z) = \frac{n_j+2}{2} z^{n_j}\}\) is a degenerating sequence for the Beltrami coefficient \(\mu_f\), with \(\|\varphi_{n_j}\|_1 = 1\), by Theorem D, we conclude that \(k_0(f) = k_0(f)\), thus \(d(0, \pi([f])) = d(0, [f])\).

To consider the contraction of Teichmüller metrics, we need the following Principle of Teichmüller contraction due to Gardiner [2].

Principle of Teichmüller contraction. Assume \(\|\mu\| = 1\), \(0 < k_1 < k_2 < 1\), and \(d(0, [f^{k_1}]) \leq \lambda_1 d_p(0, k_1)\) or \(d(0, \pi([f^{k_1}])) \leq \lambda_1 d_p(0, k_1)\) with some \(\lambda_1 < 1\), while and in the sequel, \(f^k\) is the quasiconformal mapping of \(D\) on to itself such that \(\mu_f = k \mu\) for every positive \(k < 1\). Then there exists a \(\lambda_2 < 1\) depending only on \(k_1, k_2\), and \(\lambda_1\) such that

\[
d(0, [f^k]) \leq \lambda_2 d_p(0, k) \quad \text{or} \quad d(0, \pi([f^k])) \leq \lambda_2 d_p(0, k)
\]

respectively, for all \(k\) with \(0 \leq k \leq k_2\).

Using Theorem 2 and the Principle of Teichmüller contraction, we can obtain the following
Corollary 2. Under the same circumstance as in Theorem 2, let \( k = \| \mu_f \|_\infty \) and 
\[ \lambda = \frac{d(0, \pi([f]))}{d(0, [f])}. \]
Fix \( k' < 1 \) and let \( f^t \) be the quasiconformal mapping of \( D \) onto itself such that \( \mu_{f^t} = (t/k) \mu_f \) for every \( t \in [0, k') \). Then there exists \( \lambda' < 1 \) depending only on \( k, k' \), and \( \lambda \) such that

\[ \bar{d}(0, \pi([f^t])) \leq \lambda' d_p(0, f), \]

for every \( t \) with \( 0 \leq t \leq k' \), where \( d_p \) denotes the Poincaré metric on \( D \).

The proof of Corollary 2. By Theorem 2, we have \( d(0, [f]) = d_p(0, k) \) and \( \lambda = \frac{d(0, \pi([f]))}{d(0, [f])} < 1 \), using the principle of Teichmüller contraction, the Corollary 2 is obtained.

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