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ON BLOCH FUNCTIONS AND THE CONTRACTION OF TEICHMÜLLER METRICS

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ABSTRACT. In this note, we consider the properties of Bloch functions determined by Beltrami coefficient. A sufficient condition for extremal quasiconformal mapping with nonexistence of degenerating sequence is obtained. As a result, we consider the contraction or preserved of Teichmüller metrics for the related Beltrami lines under the projection mapping π.

1. INTRODUCTION

Let $Q_{f}$ be the class of quasiconformal mappings $f$ of the unit disk $D = \{z||z| \leq 1\}$ onto itself with $f(0) = f(1) - 1 = 0$, $\mu_f$ be the complex dilatation of $f$, $k_f = \|\mu_f\|_{\infty} = \text{esssup}_{z \in D} |\mu_f|$, $k_0(f) = \inf_{g} k_g$, where $g \in Q_{f}$ with $g|_{\partial D} = f|_{\partial D}$. We say that $f(z)$ is extremal if $k_f = k_0(f)$, and the corresponding $\mu_f$ is called extremal.

We know that the universal Teichmüller space $T(1)$ can be represented as a quotient space of $QS$ by the Möbius group $PSL(2, R)$, where $QS$ is the group of all quasi-symmetric homeomorphisms of a circle, and the Teichmüller distance $d([f], [g])$, from a point $[g]$ to another point $[f]$ in $T(1)$, is equal to

\begin{equation}
(1.1) \quad d([f], [g]) = \frac{1}{2} \log \frac{1 + k_0(g \circ f^{-1})}{1 - k_0(g \circ f^{-1})}.
\end{equation}

$QS$ contains another topological subgroup, which is much larger than $PSL(2, R)$, the subgroup $S$ of symmetric homeomorphisms. Gardiner-Sullivan [1] showed that $QS mod S$ also has a natural complex Banach manifold structure and a natural quotient metric $\overline{d}$, called the Teichmüller metric in $QS mod S$. Let $\overline{k}_f = \inf_{U} \text{esssup}_{x \in U} |\mu_f(x)|$, where $U$ moves all neighborhoods of $\partial D$ in $D$, $\overline{k}_f$ is called the boundary dilatation of $f$. Set $\overline{k}_0(f) = \inf_{g} \overline{k}_g$, where $g$ moves all quasiconformal mappings of $D$ with the same boundary values as $f$. If $\overline{k}_0(f) = \overline{k}_f$, then $f(z)$ is called extremal in $QS mod S$. The distance between two points $\pi[f]$ and $\pi[g]$ in $QS mod S$ is equal to

\begin{equation}
(1.2) \quad \overline{d}(\pi[f], \pi[g]) = \frac{1}{2} \log \frac{1 + \overline{k}_0(g \circ f^{-1})}{1 - \overline{k}_0(g \circ f^{-1})}.
\end{equation}

Suppose $\mu(z)$ is a given Beltrami coefficient, we consider the Beltrami line $C_{\mu} = \{[f^t]| -1 \leq t \leq 1\}$ or $\pi C_{\mu} = \{\pi[f^t]| -1 \leq t \leq 1\}$, where $f^t = t\frac{\mu}{||\mu||_\infty}$. If $\mu$ is

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extremal in $T(1)$ or in $Q S m o d S$, then the natural mapping $t \mapsto t\|\mu\|_\infty$ from the
open interval $(-1, 1)$ with the Poincaré metric onto $C_\mu$ or $\pi C_\mu$ with the Teichmüller
metric is an isometry. Whether $\mu$ is extremal or not, such mapping is weakly
contracting. The following problem is very interesting and considered by many
authors(cf. [2],[3]):

For which points $[f] \in T(1)$, does the Teichmüller distance from 0 to $[f]$ in $QS$
strictly greater than the distance from 0 to $\pi[f]$ in $Q S m o d S$?

In this note, we will investigate some properties for Bloch functions determined
by $\mu$ and partially solve the above problem.

2. MAIN RESULTS AND THEIR PROOFS

Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be analytic in $D$, $f(z)$ is called a Bloch function if

$$(2.1) \quad \|f\|_B = \sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty.$$ 

The Bloch functions will be denoted by $B$. $B_0$ will be the subset of $B$ with

$$(2.2) \quad \|f\|_{B_0} = \lim_{|z| \to 1} \sup_{D} (1 - |z|^2)|f'(z)| = 0.$$ 

$A(D) = \{f(z) | f(z) \text{ is analytic in } D, \|f(z)\|_1 = \frac{1}{\pi} \iint_D |f(z)| \, dxdy < \infty\}$. The quasi-
conformal mapping $f$ from $D$ onto itself is called a Teichmüller mapping of finite
type, if $\mu_f = \|\mu(z)\|_\infty \frac{\varphi^\kappa}{|\varphi|}, \varphi_0 \in A(D)$. From Reich's example(cf.[4]), we know that
even the point $[f]$ corresponds to a Teichmüller mapping of finite type, the distance
from 0 to $[f]$ under the projection $\pi$ may not contract. However, if $[f] \in T(1)$, and
$d(0, \pi[f]) < d(0, [f])$, then $[f]$ contains a Teichmüller mapping of finite type. This
makes the above problem more complicated.

Suppose $\kappa(z) \in L^\infty(D)$, the space of complex-valued bounded measurable functions in $D$ with $\|\kappa\|_\infty = \text{esssup}_{z \in D}|\kappa(z)|$, we consider a linear functional $L_\kappa$ on
$A(D)$

$$(2.3) \quad L_\kappa(f) = \frac{1}{\pi} \iint_D \kappa(z)f(z) \, dxdy, \quad f(z) \in A(D),$$

then

$$(2.4) \quad \|L_\kappa\| \leq \|\kappa\|_\infty.$$ 

Hamilton, Reich and Streble [5, 6] showed that

Theorem A. A Beltrami coefficient $\mu$ is extremal if and only if one of the following
statements holds:

1) There exist $\varphi \in A(D)$ and $k \in [0, 1)$ such that $\mu = k\varphi/|\varphi|$ for almost every-
where on $D$.

2) There is a degeneration sequence $\{\varphi_n\} \in A(D)$, $\|\varphi_n\|_1 = 1$, converging to 0
locally uniformly in $D$, such that
For a given Beltrami coefficient $\mu(z)$, let

$$b_n = \frac{n+2}{\pi} \int_D \mu(z) z^n \, dx \, dy,$$

it is clearly that $|b_n| \leq 2\|\mu(z)\|_{\infty}$ and $g(\zeta)$ is analytic in $D$. We call that the analytic function $g(\zeta)$ is determined by $\mu(z)$.

Let $G(\zeta) = (g(\zeta)$, Anderson proved in [7] the following

**Theorem B.** For a given $\mu(z) \in L^\infty(D)$, then

$$\|L_\mu\| \leq \|G(\zeta)\|_B \leq 4\|L_\mu\|,$$

where $G'(\zeta) = \frac{2}{\pi} \iint_D \frac{\mu(z)}{(1-(\zeta)^2)^3} \, dx \, dy$.

**Theorem C.** If $\mu(z)$ possesses a degenerating sequence, then

$$\|L_\mu\| \leq \lim_{|z| \to 1} \sup (1-|z|^2)|G'(z)|,$$

then $\mu(z) = \|\mu\|_{\infty} \frac{\varphi_0(z)}{|\varphi_0(z)|}, \varphi_0 \in A(D)$, for almost all $z \in D$.

Theorem C means that if $\mu(z)$ is extremal and $\lim_{|z| \to 1} \sup (1-|z|^2)|G'(z)| = 0$, then

$$\mu(z) = \|\mu\|_{\infty} \varphi_0 / |\varphi_0|, \quad \varphi_0 \in A(D),$$

for almost everywhere $z \in D$. For an extremal quasiconformal mapping $f^{\mu(z)} \in Q_I$, in what case, is it a finite type Teichmüller mapping or even has it no degenerating sequence? This problem is very interesting itself(cf. [8, 9] and the references cited there). First, we will prove the following

**Theorem 1.** Suppose $\mu(z)$ is extremal, let $g(z)$ be defined in (2.6), if there exists a $\rho_0$, $0 < \rho_0 < 1$, such that

$$\sup_{\rho_0 < |z| < 1} (1-|z|^2)|g'(z)| < 1,$$

then there exists a $\varphi_0 \in A(D)$ with $\mu(z) = \|\mu(z)\|_{\infty} \frac{\varphi_0(z)}{|\varphi_0(z)|}$ for almost all $z \in D$. In particular, $\mu(z)$ possesses no degenerating sequence.

The proof of Theorem 1. If $\mu(z)$ is an extremal Beltrami coefficient, let $g(\zeta)$ be defined in (2.6), if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A(D)$, $0 < \rho < 1$, we have

$$L_\mu(f(\rho z)) = \sum_{n=0}^{\infty} a_n \rho^n L_\mu(z^n) = \sum_{n=0}^{\infty} \frac{a_n b_n}{n+2 \rho^n}.$$
Since \( \|f(\rho z) - f(z)\|_1 \to 0 \), when \( \rho \to 1^- \), then we have

\[
L_{\mu}(f) = \lim_{\rho \to 1^-} \sum_{n=0}^{\infty} \frac{a_n b_n}{n + 2} \rho^n.
\]

On the other hand, if \( G(\zeta) = \zeta g(\zeta) \), then

\[
\frac{1}{2\pi} \int_{0}^{2\pi} f(re^{i\theta})G'(\zeta re^{-i\theta}) d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \right) \left( \sum_{n=0}^{\infty} (n+1) b_n \zeta^n r^n e^{-in\theta} \right) d\theta
\]

\[
= \sum_{n=0}^{\infty} (n+1) a_n b_n \zeta^n r^{2n}.
\]

Thus, we have

\[
(2.11) \quad \lim_{\rho \to 1^-} \sum_{n=0}^{\infty} \frac{a_n b_n}{n + 2} \rho^n = \frac{1}{\pi} \int_{0}^{1} \int_{0}^{2\pi} f(re^{i\theta})G'(\zeta re^{-i\theta})(1 - r^2) r, dr d\theta,
\]

for any \( f(z) \in A(D) \). Since

\[
g(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n = \sum_{n=0}^{\infty} \left( \frac{n+2}{\pi} \right) \int_{D} z^n \mu(z) dx dy \zeta^n
\]

\[
= \frac{1}{\pi} \int_{D} \left( \sum_{n=0}^{\infty} (n+2) z^n \mu(z) \right) dx dy
\]

\[
= \frac{1}{\pi} \int_{D} \left[ \frac{2 - z\zeta}{(1 - z\zeta)^2} \right] \mu(z) dx dy,
\]

then,

\[
(2.11) \quad |g(\zeta)| \leq \frac{3\|\mu\|_{\infty}}{\pi|\zeta|} \log \frac{1 + |\zeta|}{1 - |\zeta|} = o((1 - |\zeta|^2)^{-1}), \quad |\zeta| \to 1^-.
\]

If \( \{f_n(z)\} \) is a degenerating sequence for \( \mu(z) \) with \( \|f_n\|_1 = 1 \), by Theorem B and (2.11), we can choose a \( \rho' \) with \( \rho_0 < \rho' < 1 \) such that

\[
|L_{\mu}(f_n)| \leq \frac{4\|\mu\|_{\infty}}{\pi} \int_{|z| \leq \rho'} |f_n(re^{i\theta})| r dr d\theta + \sup_{\rho' < |z| < 1} (1 - |z|^2) |g(z)|
\]

\[
+ \sup_{\rho' < |z| < 1} (1 - |z|^2) |g'(z)| < 1, \quad \text{for} \ n \to \infty,
\]

which contradicts that \( \{f_n(z)\} \) is a degenerating sequence. By Theorem A, Theorem 1 is proved.

The following example 1 shows that there is non-extremal Beltrami coefficient \( \mu(z) \) with the bound \( \sup_{\rho_0 < |z| < 1} (1 - |z|^2) |g'(z)| = \frac{2}{\pi} \).
Example 1. Set Beltrami coefficient
\[
\mu(z) = \begin{cases} 
1, & \text{for } \Re z \geq 0, |z| < 1 \\
0, & \text{for } \Re z < 0, |z| < 1.
\end{cases}
\]

Then by [8, Theorem 1], we see that \(\mu(z)\) is not extremal. In this case, by calculation, we have
\[
g'(z) = 2 + \frac{2i}{\pi} \left[ z + \frac{1}{3} z^3 + \cdots + \frac{1}{2n-1} z^{2n-1} + \cdots \right]
\]
and \(\lim_{|z| \to 1}(1 - |z|^2)|g'(z)| = \frac{2}{\pi}\).

Next we will investigate the relationship between extremal Beltrami coefficient \(\mu\) and the coefficients of \(g(z)\) defined in (2.6).

From [11] and Theorem 1, we know that if \(\mu(z)\) is extremal and the determined analytic function \(g(z) \in B_0\), then \(\lim_{n \to \infty} |b_n| = 0\). However, we also know that even if \(f(z) \in B\) and \(\lim_{n \to \infty} |b_n| = 0\), one cannot derive that \(f(z) \in B_0\). From this we will prove the following

Corollary 1. Suppose \(\mu(z)\) is extremal, and let \(g(z) = \sum_{n=0}^{\infty} b_n z^n\) be defined in (2.6), if there exist a positive number \(N_0\) and \(l, 0 < l < \frac{1}{2}\), such that
\[
|b_n| < \frac{l}{n}, \quad \text{holds for } n > N_0,
\]
then there exists a \(\varphi_0(z) \in A(D)\) with
\[
\mu(z) = ||\mu||_{\infty} \varphi_0 / |\varphi_0|, \quad \text{for almost all } z \in D.
\]

The proof of Collary 1. If \(\mu(z)\) is extremal, and let \(g(z) = \sum_{n=0}^{\infty} b_n z^n\) be defined in (2.6), we have
\[
|g'(z)| \leq |\sum_{n=0}^{N_0} n b_n z^n| + \sum_{n=N_0+1}^{\infty} l |z^n|
\]
\[
= |\sum_{n=0}^{N_0} n b_n z^n| + \frac{l |z|^{N_0+1}}{1 - |z|},
\]
thus there exists a \(\rho_0 > 0\), such that \(\sup_{\rho_0 < |z| < 1}(1 - |z|^2)|g'(z)| < 1\), by Theorem 1, we obtain the assertion.

Let \(\Pi\) denote the subset of \(T(1)\) consisting of elements of \([f]\) which correspond to Teichmüller mappings of finite type whose complex dilatations \(\mu = \mu_f\) satisfy the following condition: There exists a \(\rho_0, 0 < \rho_0 < 1\), such that \(\sup_{\rho_0 < |\zeta| < 1}(1 - |\zeta|^2)|g'(\zeta)| < 1\), where \(g(\zeta)\) is defined in (2.6). We will prove the following

Theorem 2. For \([f] \in \Pi\), then \(d(0, \pi([f])) < d(0, [f])\).

In order to prove Theorem 2, we need the following Theorem D due to Gardiner [2].
Theorem D. For every \([f] \in T(1)\), then \(\bar{k}_f = \bar{k}_0(f)\) if and only if

\[
\sup \limsup_{\{\varphi_n\} \to \infty} \left| \operatorname{Re} \int_D \varphi_n \mu_f \, dx \, dy \right| = \bar{k}_f,
\]

where the supremum is taken over all degenerating sequences \(\{\varphi_n\}\) for \(\mu_f\) with \(\|\varphi_n\|_1 = 1\) in \(A(D)\).

The proof of Theorem 2. We use the same way as in [3] to prove Theorem 2. If \([f] \in \Pi\), then we conclude that \(\bar{k}_0(f) = k_0(f)\). On the contrary, by Theorem D, we can find a degenerating sequence \(\{\varphi_n\}\) with \(\|\varphi_n\|_1 = 1\) such that

\[
\lim_{n \to \infty} \operatorname{Re} \int_D \varphi_n \mu_f \, dx \, dy = \|\mu_f\|_\infty = k_0(f) = \bar{k}_0(f),
\]

which is impossible by Theorem 1.

Thus we have \(\bar{k}_0(f) < k_0(f)\), which is equivalent to \(d(0, \pi([f])) < d(0, [f])\).

On the other hand, comparing with Theorem 2, we will prove the following

Theorem 3. Suppose \([f] \in T(1)\), and \(b_n = \frac{n+2}{\pi} \int_D \mu_f \, x^n \, dx \, dy\), if \(\lim_{n \to \infty} b_n = 2\|\mu_f\|_\infty\), then \(d(0, \pi([f])) = d(0, [f])\). The constant 2 is the best.

The proof of Theorem 3. First, from Fehlmann and Sakan’s paper in [10], we know that the subset of \(T(1)\) satisfying the conditions in Theorem 3 is not empty, and by the example of Fehlmann and Sakan made in [10], there exists an extremal Beltrami coefficient \(\mu\) such that the coefficients of \(g(z)\) satisfy \(\lim_{n \to \infty} b_n = 2\|\mu\|_\infty\), thus the constant 2 is the best. Now, if \(\lim_{n \to \infty} b_n = 2\|\mu_f\|_\infty\), then we have \(\lim_{n \to \infty} \frac{b_n}{2} = 2\|\mu_f\|_\infty\), and the sequence \(\{\varphi_n(z) = \frac{n+2}{2} z^n\}\) is a degenerating sequence for the Beltrami coefficient \(\mu_f\), with \(\|\varphi_n\|_1 = 1\), by Theorem D, we conclude that \(\bar{k}_0(f) = k_0(f)\), thus \(d(0, \pi([f])) = d(0, [f])\).

To consider the contraction of Teichmüller metrics, we need the following Principle of Teichmüller contraction due to Gardiner [2].

Principle of Teichmüller contraction. Assume \(\|\mu\| = 1\), \(0 < k_1 < k_2 < 1\), and \(d(0, [f^{k_1}]) \leq \lambda_1 d_f(0, k_1)\) or \(d(0, \pi([f^{k_1}])) \leq \lambda_1 d_f(0, k_1)\) with some \(\lambda_1 < 1\), where and in the sequel, \(f^k\) is the quasiconformal mapping of \(D\) on to itself such that \(\mu_f = k_\mu\) for every positive \(k < 1\). Then there exists a \(\lambda_2 < 1\) depending only on \(k_1, k_2\), and \(\lambda_1\) such that

\[
d(0, [f^k]) \leq \lambda_2 d_f(0, k) \quad \text{or} \quad d(0, \pi([f^k])) \leq \lambda_2 d_f(0, k)
\]

respectively, for all \(k\) with \(0 \leq k \leq k_2\).

Using Theorem 2 and the Principle of Teichmüller contraction, we can obtain the following
Corollary 2. Under the same circumstance as in Theorem 2, let $k = \|\mu_f\|_\infty$ and $\lambda = \frac{\bar{d}(0, \pi([f]))}{d(0, [f])}$. Fix $k' < 1$ and let $f^t$ be the quasiconformal mapping of $D$ onto itself such that $\mu_{f^t} = \frac{t}{k}\mu_f$ for every $t \in [0, k')$. Then there exists $\lambda' < 1$ depending only on $k$, $k'$, and $\lambda$ such that

$$\bar{d}(0, \pi([f^t])) \leq \lambda' d_p(0, f),$$

for every $t$ with $0 \leq t \leq k'$, where $d_p$ denotes the Poincaré metric on $D$.

The proof of Corollary 2. By Theorem 2, we have $d(0, [f]) = d_p(0, k)$ and $\lambda = \frac{\bar{d}(0, \pi([f]))}{d(0, [f])} < 1$, using the principle of Teichmüller contraction, the Corollary 2 is obtained.

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