数理解析研究所講究録
1012 巻 1997 年 178-185



# ON THE NORM OF PRE-SCHWARZIAN DERIVATIVES OF STRONGLY STARLIKE FUNCTIONS

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August 5, 1997

ABSTRACT. For a constant $\alpha \in (0,1]$, a normalized analytic function $f(z) = z + a_2z^2 + \cdots$ on the unit disk is said to be strongly starlike of order $\alpha$ if $|\arg zf'(z)/f(z)| < \alpha\pi/2$ for any point $z$ in the unit disk. In this note, we shall present an optimal but not explicit esitimate of the norm of $f''/f'$, for such a function $f$. And we provide a sufficiently good esitimate for the optimal constants. We also refer to the related topics.

## 1. INTRODUCTION

Let $A$ denote the set of analytic functions $f$ on the unit disk $\Delta$ normalized so that $f(0) = 0$ and $f'(0) = 1$. For a constant $\alpha \in (0,1]$, a function $f \in A$ is called *strongly starlike of order* $\alpha$ if $|\arg(zf'(z)/f(z))| < \pi\alpha/2$ in $\Delta$. We denote by $S^*(\alpha)$ the set of strongly starlike functions of order $\alpha$. Note that a function in $A$ is strongly starlike of order 1 if and only if it is starlike, i.e., $\mathrm{Re}(zf'(z)/f(z)) > 0$, in particular univalent in $\Delta$. These classes of the functions have been considered by several authors, for example, Stankiewicz [6], Brannan-Kirwan [1] and Chiang [2].

For a locally univalent holomorphic function $f$, we define

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = (T_f)' - \frac{1}{2}(T_f)^2,$$

these are called the pre-Schwarzian derivative (or nonlinearity) and the Schwarzian derivative of $f$, respectively. For a locally univalent holomorphic function $f$ in the unit disk, we define norms of $T_f$ and $S_f$ by

$$\|T_f\|_1 = \sup_{z\in\Delta} |T_f(z)|(1-|z|^2), \quad \text{and} \quad \|S_f\|_2 = \sup_{z\in\Delta} |S_f(z)|(1-|z|^2)^2,$$

respectively. These norms have a significant meaning in the theory of Teichmüller spaces.

1991 *Mathematics Subject Classification.* 30C45.
*Key words and phrases.* pre-Schwarzian derivative, Schwarzian derivative, starlike functions.

For a univalent function $f$, it is well-known that $\|T_f\|_1 \leq 6, \|S_f\|_2 \leq 6$ and that these estimates are best possible. Moreover, if $f$ can be extended to a $k$-quasiconformal automorphism of the Riemann sphere $\widehat{\mathbb{C}}$ then we have $\|T_f\|_1 \leq 6k$ and $\|S_f\|_2 \leq 6k$.

On the other hand, for any $f \in A$, it is also known that if $\|T_f\|_1 \leq 1$ or $\|S_f\|_2 \leq 2$ the functions $f$ is univalent in $\Delta$.

In [5] Fait, Krzyż and Zygmunt showed that any function $f \in S^*(\alpha)$ can be extended to a $\sin(\pi\alpha/2)$-quasiconformal automorphism of $\widehat{\mathbb{C}}$, therefore we have $\|T_f\|_1 \leq 6\sin(\pi\alpha/2)$ and $\|S_f\|_2 \leq 6\sin(\pi\alpha/2)$ (cf. [3]). Moreover, Chiang [3](Theorem 2.4.3) proved that $\|T_f\|_1 \leq 6\alpha$ however the formulation is slightly different from here. He also states this estimate is best possible, but his argument has a gap. In fact, we shall give the following best possible estimate. The proof will be given in Section 2.

**Theorem 1.1.** *For any $f \in S^*(\alpha)$, where $0 < \alpha < 1$, we have*

$$\|T_f\|_1 = \sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq M(\alpha) + 2\alpha, \tag{1.1}$$

*where, $M(\alpha)$ is given by*

$$M(\alpha) = \frac{4\alpha c(\alpha)}{(1-\alpha)c(\alpha)^2 + 1 + \alpha} = \frac{4\alpha c(\alpha)^{\alpha+1}}{c(\alpha)^2+1}, \tag{1.2}$$

*and $c(\alpha)$ is the unique solution of the following equation with respect to $x$ in the interval $(1,\infty)$ :*

$$(1-\alpha)x^{\alpha+2} + (1+\alpha)x^{\alpha} - x^2 - 1 = 0. \tag{1.3}$$

*Moreover, the equality in (1.1) holds precisely if $T_f(z) = (\frac{1+\varepsilon z}{1-\varepsilon z})^\alpha$ for a constant $\varepsilon$ with $|\varepsilon| = 1$.*

In case $\alpha = 1$, as is well-known, the Koebe function $K(z) = z(1-z)^{-2}$ belongs to $S^*(1)$ and satisfies that $\|T_K\|_1 = 6$.

By the expression of $c(\alpha)$ above, we can immediately see that $M(\alpha) < 4\alpha$. Moreover, modifying the method in [3], we can show that

$$M(\alpha) < 2\alpha(1+\alpha). \tag{1.4}$$

The method of estimation of this might be interesting in itself, so we include it in the proof of Theorem 1.1.

It seems difficult to determine the exact value of $c(\alpha)$ in terms of $\alpha$, although we have a sufficiently good estimate of it.

**Theorem 1.2.** *Under the same hypothesis of Theorem* 1.1, *we can estimate* $c(\alpha)$ *and* $M(\alpha)$ *as*

$$\frac{(1-\alpha/2)(1+\alpha)}{1-\alpha} < c(\alpha) < \frac{1+\alpha}{1-\alpha}, \tag{1.5}$$

*and*

$$\frac{2\alpha(1+\alpha)^{1+\alpha}(1-\alpha)^{1-\alpha}}{1+\alpha^2} < M(\alpha) < 2\alpha(1-\alpha/2)^{\alpha}(1+\alpha)^{1+\alpha}(1-\alpha)^{1-\alpha}. \tag{1.6}$$

*Remark.* The ratio $r(\alpha) = (1-\alpha/2)^{\alpha}(1+\alpha^2)$ of the right-most term and the left-most one in (1.6) is very near to 1. In fact, for $0 < \alpha < 1$ it holds that $1 < r(\alpha) \leq 1.10244\cdots$, where the maximum is attained by $\alpha = 0.679508\cdots$. Therefore, we are convinced that the esimates in the above are nearly sharp. And, one can also deduce (1.4) from the estimate (1.6).

The proof of Theorem 1.2 will be given in Section 3.

On the other hand, it is unknown if the estimate $\|S_f\|_2 \leq 6\sin(\pi\alpha/2)$ is sharp. In this direction, we will show that this bound cannot be replaced by a smaller number than $6\alpha$ in Section 4.

## 2. Proof of Thereom 1.1

Now we shall prove the Theorem 1.1 by following the method developed by Chiang [3]. First we set $p(z) = P_f(z) = zf'(z)/f(z)$. Then, by assumption, $p$ is a holomorphic function on $\Delta$ satisfying $p(0) = 1$ and $p(\Delta) \subset \{w \in \mathbb{C}^*; |\arg w| < \pi\alpha/2\}$. Since $p$ is subordinate to the univalent map $q(z) = (\frac{1+z}{1-z})^{\alpha}$, there exists a holomorphic function $\omega : \Delta \to \Delta$ with $\omega(0) = 0$ such that

$$p = q \circ \omega = \left(\frac{1+\omega}{1-\omega}\right)^{\alpha}. \tag{2.1}$$

Let $F = F_\alpha \in A$ be the function with $P_F = q$, i.e. $F(z) = z\exp(\int_0^z \frac{(q(t)-1)dt}{t})$. Here, for later use, we note an elementary fact that $|q(z) - 1| \leq q(|z|) - 1$ because

$$|q(z)-1| = \left|\int_0^z \frac{2\alpha}{1-t^2}\left(\frac{1+t}{1-t}\right)^{\alpha} dt\right| \leq \int_0^z \frac{2\alpha}{1-|t|^2}\left(\frac{1+|t|}{1-|t|}\right)^{\alpha} |dt| = q(|z|) - 1.$$

By the logarithmic differentiation of $p$, we have

$$\frac{1}{z} + \frac{f''}{f'} - \frac{f'}{f} = \alpha\left(\frac{\omega'}{1+\omega} + \frac{\omega'}{1-\omega}\right), \text{ thus}$$

$$zT_f(z) = \frac{zf''(z)}{f'(z)} = \frac{2\alpha z\omega'}{1-\omega^2} + p - 1. \tag{2.2}$$

By the Schwarz-Pick lemma: $|\omega'(z)|/(1-|\omega(z)|^2) \le 1/(1-|z|^2)$ and the fact that $|\omega(z)| \le |z|$ and $p = q \circ \omega$, we can estimate as

$$\begin{aligned}|T_f(z)| &\le \frac{2\alpha|\omega'|}{1-|\omega|^2} + \frac{|q(\omega)-1|}{|z|} \le \frac{2\alpha}{1-|z|^2} + \frac{q(|\omega|)-1}{|z|} \\ &\le \frac{2\alpha}{1-|z|^2} + \frac{q(|z|)-1}{|z|} = T_F(|z|),\end{aligned}$$

where the inequality is strict unless $\omega/z$ is a constant with absolute value 1. In particular, we immediately see that $\|T_f\|_1 \le \|T_F\|_1$.

Since $(1-t^2)T_F(t) = \frac{1-t^2}{t}(q(t)-1) + 2\alpha$ tends to $2\alpha$ as $t \to 1-0$, if the equality $\|T_f\|_1 = \|T_F\|_1 (> 2\alpha)$ holds then $|T_f(z_0)| = T_F(|z_0|)$ for some $z_0 \in \Delta$, hence we conclude that $T_f(z) = q(\epsilon z)$ for some constant $\varepsilon$ with $|\varepsilon| = 1$.

From now on, we may restrict our attension on the norm of $T_F$. What we need is to evaluate $M(\alpha) = \sup_{0<t<1} \frac{1-t^2}{t}(q(t)-1)$. Changing the variable by $x = \frac{1+t}{1-t}$, we have $M(\alpha) = \sup_{1<x} g(x)$, where $g(x) = \frac{4(x^{1+\alpha}-x)}{x^2-1}$.

By the logarithmic differentiation, we have

$$\frac{g'(x)}{g(x)} = -\frac{(1-\alpha)x^{2+\alpha} + (1+\alpha)x^{\alpha} - x^2 - 1}{(x^{1+\alpha}-x)(x^2-1)}.$$

We set $h(x) = (1-\alpha)x^{2+\alpha} + (1+\alpha)x^{\alpha} - x^2 - 1$, then we obtain $h(1) = 0, h(+\infty) = +\infty$ and

$$\begin{aligned}h'(x) &= (1-\alpha)(2+\alpha)x^{1+\alpha} + \alpha(1+\alpha)x^{\alpha-1} - 2x, \\ h''(x) &= (1-\alpha^2)(2+\alpha)x^{\alpha} - \alpha(1-\alpha^2)x^{\alpha-2} - 2, \\ h'''(x) &= \alpha(1-\alpha^2)x^{\alpha-3}((2+\alpha)x^2 + 2 - \alpha) > 0.\end{aligned}$$

In particular, $h''$ is increasing, thus $h''$ has a unique zero in $(1,+\infty)$, say $x = x_1$, because $h''(1) = -2\alpha^2 < 0$ and $h''(x) \to +\infty \quad (x \to +\infty)$. Since $h'(1) = 0$ and $h'(x) \to +\infty \quad (x \to +\infty)$, $h'$ has a unique zero $x_2 > x_1$ in $(1,+\infty)$. By the same reasoning, $h$ has a unique zero (nothing but the solution of (1.3)!) $c(\alpha) > x_2$ in $(1,+\infty)$, since $h(0) = 0$ and $h(x) \to +\infty \quad (x \to +\infty)$. By these observations, we can see that $g' > 0$ in $(1, c(\alpha))$ and $g' < 0$ in $(c(\alpha), +\infty)$. Thus $g$ assumes its maximum at $x = c(\alpha)$, therefore we obtain $M(\alpha) = g(c(\alpha))$. Since $c = c(\alpha)$ satisfies $c^{\alpha} = \frac{c^2+1}{(1-\alpha)c^2+1+\alpha}$, we have also

$$M(\alpha) = g(c) = \frac{4c\left(\frac{c^2+1}{(1-\alpha)c^2+1+\alpha} - 1\right)}{c^2-1} = \frac{4\alpha c}{(1-\alpha)c^2+1+\alpha} = \frac{4\alpha c^{1+\alpha}}{c^2+1}.$$

Now the proof is completed.

Here we should note that $M(\alpha) = g(c(\alpha)) > \lim_{x \to 1+0} g(x) = 2\alpha$, thus $M(\alpha) - 2\alpha = \frac{4\alpha c}{(1-\alpha)c^2+1+\alpha} - 2\alpha = \frac{2\alpha(c-1)(\alpha(c+1)-(c-1))}{(1-\alpha)c^2+1+\alpha} > 0$, which proves that $\alpha > \frac{c-1}{c+1}$, i.e. $c = c(\alpha) > \frac{1+\alpha}{1-\alpha}$.

As we promised in Introduction, we shall show (1.4). The proof below is rather geometric and does not use the knowledge about $c(\alpha)$. In order to estimate $M(\alpha)$, we shall investigate the function $q(z) = 1 + c_1 z + c_2 z^2 + \cdots$, here we can compute that $c_1 = 2\alpha$ and $c_2 = 2\alpha^2$.

First, by the following lemma due to Loewner, we know that

$$|c_n| \leq |c_1| = 2\alpha \tag{2.3}$$

for all $n$.

**Lemma 2.1 (cf. Duren [4]).** *If a function $f(z) = z + a_2 z^2 + \cdots \in A$ is convex, then $|a_n| \leq 1$ for all $n \geq 2$. And all these inequalities are strict unless $f$ is a rotation of the function $\frac{z}{1-z}$.*

Here we devide $q$ into the even part $q_e$ and the odd part $q_o$, i.e.,

$$q_e(z) = \frac{1}{2}(q(z) + q(-z)) = \frac{1}{2}\left(q(z) + \frac{1}{q(z)}\right) = 1 + c_2 z^2 + c_4 z^4 + \cdots,$$

and

$$q_o(z) = \frac{1}{2}(q(z) - q(-z)) = c_1 z + c_3 z^3 + \cdots.$$

Noting that $q_e(\sqrt{z})$ is a univalent function whose image is a component of the exterior of the hyperbola:

$$\{z = x + iy; \left(\frac{x}{\cos(\pi\alpha/2)}\right)^2 - \left(\frac{y}{\sin(\pi\alpha/2)}\right)^2 > 1 \quad \text{and} \quad x > 0\},$$

thus convex, by the above lemma again, we have

$$|c_{2n}| \leq |c_2| = 2\alpha^2 \tag{2.4}$$

for all $n = 1, 2, \cdots$.

We remark that $q_o : \Delta \to q_o'(\Delta)$ is a conformal mapping onto the interior of the hyperbola:

$$\{z = x + iy; \left(\frac{y}{\sin(\pi\alpha/2)}\right)^2 - \left(\frac{x}{\cos(\pi\alpha/2)}\right)^2 < 1\}.$$

By virtue of (2.3) and (2.4), we have

$$\begin{aligned}|q(z)-1| &\le \sum_{n=0}^{\infty}|c_{2n+1}||z|^{2n+1}+\sum_{n=1}^{\infty}|c_{2n}||z|^{2n}\\ &\le \sum_{n=0}^{\infty}2\alpha|z|^{2n+1}+\sum_{n=1}^{\infty}2\alpha^2|z|^{2n}\\ &= \frac{2\alpha|z|}{1-|z|^2}+\frac{2\alpha^2|z|^2}{1-|z|^2}=\frac{2\alpha|z|(|z|+\alpha)}{1-|z|^2}.\end{aligned}$$

Thus we have proved that $\frac{1-|z|^2}{|z|}|q(z)-1|\le 2\alpha(|z|+\alpha)$, from which we can deduce (1.4).

## 3. More about $c(\alpha)$ and $M(\alpha)$

In this section, we shall prove Theorem 1.2 by investigating $c(\alpha)$ and $M(\alpha)$ more.

Already, we have proved that $c(\alpha)<\frac{1+\alpha}{1-\alpha}$ below the proof of Theorem 1.1 in the previous section. Our next task is to show that $\tau_0 := \frac{(1-\alpha/2)(1+\alpha)}{1-\alpha} < c(\alpha)$. In order to prove this, it suffices to show that $h(\tau_0)<0$, where $h$ is the function defined in Section 2. By substituting $\tau_0$ to the expression of $h$, we calculate as

$$h(\tau_0)=(1-\alpha)^{-2}$$
$$\times\left[\left(2-\alpha-\frac{3\alpha^2}{4}+\frac{\alpha^3}{4}\right)(1-\frac{\alpha}{2})^{\alpha}(1+\alpha)^{1+\alpha}(1-\alpha)^{1-\alpha}-\left(2-\alpha+\frac{\alpha^2}{4}-\frac{\alpha^3}{2}+\frac{\alpha^4}{4}\right)\right].$$

Set

$$k(\alpha)=\log\frac{(2-\alpha-\frac{3\alpha^2}{4}+\frac{\alpha^3}{4})(1-\frac{\alpha}{2})^{\alpha}(1+\alpha)^{1+\alpha}(1-\alpha)^{1-\alpha}}{2-\alpha+\frac{\alpha^2}{4}-\frac{\alpha^3}{2}+\frac{\alpha^4}{4}},$$

then we have

$$k''(\alpha)=\frac{\alpha^3 l(\alpha)}{(8-4\alpha-3\alpha^2+\alpha^3)^2(8-4\alpha+\alpha^2-2\alpha^3+\alpha^4)^2(2-\alpha)^2(1-\alpha^2)},$$

where

$$\begin{aligned}l(\alpha)=&-30720-109056\alpha+139072\alpha^2-58176\alpha^3-29136\alpha^4+34392\alpha^5-1293\alpha^6\\ &-12397\alpha^7+7478\alpha^8-2053\alpha^9+468\alpha^{10}-215\alpha^{11}+82\alpha^{12}-15\alpha^{13}+\alpha^{14}.\end{aligned}$$

Since

$$\begin{aligned}l'(\alpha)=&109056-278144\alpha+174528\alpha^2+116544\alpha^3-171960\alpha^4+7758\alpha^5+86779\alpha^6\\ &-59824\alpha^7+18477\alpha^8-4680\alpha^9+2365\alpha^{10}-984\alpha^{11}+195\alpha^{12}-14\alpha^{13}\\ =&96+5920\beta+28320\beta^2+50048\beta^3+36830\beta^4+5046\beta^5-9555\beta^6\\ &-9008\beta^7-1071\beta^8+2260\beta^9+407\beta^{10}-264\beta^{11}+13\beta^{12}+14\beta^{13},\end{aligned}$$

where $\beta = 1-\alpha$, we see that $l'(\alpha) \geq 96 > 0$, hence $l$ is increasing. On the other hand, $l(0) = -30720 < 0$ and $l(1) = 128 > 0$, so we conclude that $l$ has a unique zero $\alpha_1$ in $(0,1)$. Taking account of that $k'(0) = 0$ and $\lim_{\alpha\to 1-0} k'(\alpha) = +\infty$, we see that $k'$ has a unique zero $\alpha_2 > \alpha_1$ in $(0,1)$ and that $k' < 0$ in $(0,\alpha_2)$ and $k' > 0$ in $(\alpha_2, 1)$. Therefore, we have $k(\alpha) < \max\{k(0), k(1)\} = 0$ for all $0 < \alpha < 1$, hence $h(\tau_0) < 0$.

Now, we shall estimate $M(\alpha)$. First, we note that the function $u(x) = \frac{4\alpha x^{1+\alpha}}{x^2+1}$ is decreasing in $[\tau_0, +\infty)$, where $\tau_0 = \frac{(1-\alpha/2)(1+\alpha)}{1-\alpha}$, in fact, $u'(x)/u(x) = \frac{1+\alpha-(1-\alpha)x^2}{x(x^2+1)}$, and $(1-\alpha)x^2 \geq (1-\alpha)(1-\alpha/2)^2(\frac{1+\alpha}{1-\alpha})^2 \geq (1+\alpha)^2\frac{1-\alpha+\alpha^2/4}{1-\alpha} > 1+\alpha$, thus $u'(x) < 0$. In particular, in view of (1.5), we have $u(\tau_1) < M(\alpha) = u(c(\alpha)) < u(\tau_0)$, where $\tau_1 = \frac{1+\alpha}{1-\alpha}$. Noting that $u(\tau_1) = \frac{2\alpha(1+\alpha)^{1+\alpha}(1-\alpha)^{1-\alpha}}{1+\alpha^2}$ and

$$u(\tau_0) = \frac{4\alpha(1-\alpha/2)^{1+\alpha}(1+\alpha)^{1+\alpha}(1-\alpha)^{1-\alpha}}{2-\alpha+\alpha^2(1-\alpha)^2/4} \leq 2\alpha(1-\alpha/2)^{\alpha}(1+\alpha)^{1+\alpha}(1-\alpha)^{1-\alpha},$$

we have the inequality (1.6).

## 4. Schwarzian derivatives

In this section, we remark a corresponding result on the Schwarzian derivatives of strongly starlike functions. As was seen previously, if $f \in S^*(\alpha)$ then $p = zf'/f$ can be represented as $p = q\circ\omega$, where $q(z) = (\frac{1+z}{1-z})^{\alpha}$ and $\omega : \Delta \to \Delta$ is holomorphic function with $\omega(0) = 0$.

Differentiating the both sides of $zf''/f' + 1 - p = zp'/p (= 2\alpha\omega'/(1-\omega^2))$, we have $z(f''/f')' + f''/f' - p' = (zp'/p)'$, thus

$$z^2(f''/f')' = zp' - zf''/f' + z(zp'/p)' = zp' - p + 1 - zp'/p + z(zp'/p)'.$$

Noting these equalities, we can calculate as

$$\begin{aligned} z^2 S_f &= z^2(f''/f')' - \frac{1}{2}(zf''/f')^2 \\ &= z(zp'/p)' - zp'/p + zp' - p + 1 - \frac{1}{2}\left\{(zp'/p)^2 + p^2 + 1 + 2zp' - 2zp'/p - 2p\right\} \\ &= z(zp'/p)' - \frac{1}{2}(zp'/p)^2 + \frac{1}{2}(1-p^2) \\ &= \frac{2\alpha z\left\{z(z\omega'' + \omega')(1-\omega^2) + (2z\omega - \alpha z)(\omega')^2\right\}}{(1-\omega^2)^2} + \frac{1}{2}(1-p^2). \end{aligned}$$

Though we know that $\|S_f\|_2 \leq 6\sin(\pi\alpha/2)$, we feel much difficulty to estimate $\|S_f\|_2$ directly by the above expression of $S_f$.

Analogously as in the case of pre-Schwarzian derivatives, one might expect that the function $F = F_\alpha$ determined by $P_F = q$ plays an extremal role, but this is not the case. By the above calculations, we can see that $z^2 S_F(z) = \frac{2\alpha z(1-\alpha z+z^2)}{(1-z^2)^2} + \frac{1}{2}(1-q^2)$

in this case. A direct but tedious calculation yields that $\|S_F\|_2 = 2\alpha(2+\alpha)$. But, the same estimate does not hold for general $f \in S^*(\alpha)$. In fact, if $P_f = (\frac{1+z^2}{1-z^2})^\alpha$, then $z^2 S_f(z) = \frac{2\alpha z(4z-4(1+\alpha)z^3+8z^5)}{(1-z^4)^2} + \frac{1}{2}(1-q(z^2))$, in particular, $S_f(0) = 6\alpha$, hence $\|S_f\|_2 \geq 6\alpha$. (In fact, in this case $\|S_f\|_2 = 6\alpha$.)

The author does not know if there exists a function $f \in S^*(\alpha)$ such that $\|S_f\|_2 > 6\alpha$.

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