数理解析研究所講究録
1012 巻 1997 年 164-177

New Problems of Coefficient Inequalities

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Abstract

H.Silverman determines some coefficient inequalities and distortion theorems for univalent functions with negative coefficients that are starlike of order α or convex of order α. The same coefficient inequalities and distortion theorems are obtained for univalent functions with coefficients other than negative coefficients. We give some examples of univalent functions with nagative coefficients or with coefficients other than negative coefficients by using elementary functions. Further we investigate some properties of the convolutions of such univalent functions.

1 Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \quad (a_n\in C,\ n\in N) \tag{1.1}$$

that are analytic in the unit disk $U=\{z:|z|<1\}$.

Let S be the subclass of A consisting of functions which are univalent in the unit disk U. And we denote by S^* the subclass of S consisting of starlike functions. Further we denote by K the subclass of S consisting of convex functions. The following theorems for S^* and K are well known.

Theorem A(Kobori [1]). A function $f(z)$ in A is in S^* if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0. \tag{1.2}$$

Theorem B(Kobori [1]). A function $f(z)$ in A is in K if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0. \tag{1.3}$$

M.S.Robertson[2] defined the subclasses $S^*(\alpha)$ and $K(\alpha)$ of A as follows.

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A function $f(z)$ in A is said to be starlike of order α if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.4}$$

for some $\alpha(0 \leq \alpha < 1)$, and for all $z \in U$. The subclass of A consisting of all starlike functions of order α is denoted by $S^*(\alpha)$.

A function $f(z)$ in A is said to be convex of order α if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.5}$$

for some $\alpha(0 \leq \alpha < 1)$, and for all $z \in U$. The subclass of A consisting of all such functions is denoted by $K(\alpha)$.

Let $A(1)$ denote the subclass of A consisting of functions whose nonzero coefficients, from the second on, are negative, that is,

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0, n \in N). \tag{1.6}$$

A function $f(z)$ in $A(1)$ is called analytic function with negative coefficients. We denote by $T^*(\alpha)$ and $C(\alpha)$ the subclass of $A(1)$ that are, starlike of order α and convex of order α, respectively.

In [3], H.Silverman determines the coefficient inequalities for the functions belonging to $T^*(\alpha)$ and $C(\alpha)$, respectively as follows.

Theorem C(Silverman [3]). *A function $f(z)$ in $A(1)$ is in $T^*(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n \leq 1-\alpha. \tag{1.7}$$

Theorem D(Silverman [3]). *A function $f(z)$ in $A(1)$ is in $C(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1-\alpha. \tag{1.8}$$

We note that $f(z) \in C(\alpha)$ if and only if $zf'(z) \in T^*(\alpha)$ for $0 \leq \alpha < 1$.

Let $A(n,\theta)$ denote the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k z^k \quad (a_k \geq 0,\ n \in N). \tag{1.9}$$

Then we have that $A(1,0) = A(1)$. That is, $A(1,0)$ is a class of analytic functions with negative coefficients.

We denote by $B(n,\theta,\alpha)$ and $C(n,\theta,\alpha)$ the subclasses of $A(n,\theta)$ that are, respectively starlike of order α and convex of order α. That is,

$$\begin{aligned} B(n,\theta,\alpha) &= A(n,\theta)\cap S^*(\alpha), \\ C(n,\theta,\alpha) &= A(n,\theta)\cap K(\alpha). \end{aligned}$$

Then $B(1,0,\alpha)$ and $C(1,0,\alpha)$ are equivalent to $T^*(\alpha)$ and $C(\alpha)$, respectively. We note that $f(z)\in C(n,\theta,\alpha)$ if and only if $zf'(z)\in B(n,\theta,\alpha)$ for $0\leq\alpha<1$.

2 Coefficients inequalities

Theorem 2. 1 *A function $f(z)$ in $A(n,\theta)$ is in $B(n,\theta,\alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty}(k-\alpha)a_k\leq 1-\alpha. \tag{2.1}$$

Proof. If $f(z)$ is in $A(n,\theta)$ and coefficient inequality $\sum_{k=n+1}^{\infty}(k-\alpha)a_k\leq 1-\alpha$ hold true, then

$$\begin{aligned} \left|\frac{zf'(z)}{f(z)}-1\right| &= \left|\frac{\sum_{k=n+1}^{\infty}(k-1)e^{i(k-1)\theta}a_kz^{k-1}}{1-\sum_{k=n+1}^{\infty}e^{i(k-1)\theta}a_kz^{k-1}}\right| \\ &\leq \frac{\sum_{k=n+1}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=n+1}^{\infty}a_k|z|^{k-1}} \\ &< \frac{\sum_{k=n+1}^{\infty}(k-1)a_k}{1-\sum_{k=n+1}^{\infty}a_k} \\ &\leq 1-\alpha \end{aligned}$$

Therefore since the values for $\dfrac{zf'(z)}{f(z)}$ lie in a circle centered at $w=1$ whose radius is $1-\alpha$, we have

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha.$$

The sufficiency of the condition is proved.

We shall prove the necessity of the condition.

If $f(z)$ is in $B(n,\theta,\alpha)$, then we have

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}=\mathrm{Re}\left\{\frac{z-\sum_{k=n+1}^{\infty}e^{i(k-1)\theta}ka_kz^k}{z-\sum_{k=n+1}^{\infty}e^{i(k-1)\theta}a_kz^k}\right\}>\alpha$$

for all $z \in U$. Choose the values of z on half line $z = re^{-i\theta}(0 \leq r < 1)$, then

(2.2)
$$\mathrm{Re}\left\{\frac{z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} k a_k z^k}{z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k z^k}\right\} = \frac{1 - \sum_{k=n+1}^{\infty} k a_k r^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k r^{k-1}} > \alpha.$$

Since $1 - \sum_{k=n+1}^{\infty} k a_k r^{k-1} > 0, 1 - \sum_{k=n+1}^{\infty} a_k r^{k-1} > 0$, we have by the inequality (2.2) that

(2.3)
$$1 - \sum_{k=n+1}^{\infty} k a_k r^{k-1} > \alpha \left(1 - \sum_{k=n+1}^{\infty} a_k r^{k-1}\right).$$

By letting $r \longrightarrow 1$ through half line $z = re^{-i\theta}(0 \leq r < 1)$ in (2.3), we have

(2.4)
$$1 - \sum_{k=n+1}^{\infty} k a_k \geq \alpha \left(1 - \sum_{k=n+1}^{\infty} a_k\right).$$

That is,

$$\sum_{k=n+1}^{\infty} (k - \alpha) a_k \leq 1 - \alpha$$

and the proof of the theorem is completed.

Theorem 2. 2 *A function $f(z)$ in $A(n, \theta)$ is in $C(n, \theta, \alpha)$ if and only if*

(2.5)
$$\sum_{k=n+1}^{\infty} k(k - \alpha) a_k \leq 1 - \alpha.$$

Proof. As we noted in Introduction,

$$f(z) \in C(n, \theta, \alpha) \text{ if and only if } zf'(z) \in B(n, \theta, \alpha)$$

for $0 \leq \alpha < 1$.

Since

$$zf'(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} k a_k z^k,$$

if we put ka_k insted of a_k in Theorem 2.1, we obtain Theorem 2.2.

Special cases of Theorem 2.1 and Theorem 2.2 are Theorem C and Theorem D by H.Silverman [3], $n = 1, \theta = 0$, respectively.

Now, we shall show some examples of function $f(z) \in B(2, \theta, 0)$ and $C(2, \theta, 0)$, respectively. For the proof we refer to T. Sekine and T. Yamanaka [5].

Theorem 2. 3 *The function*

$$f(z) = \frac{1}{2e^{i\theta}}(1-e^{i\theta}z)^2\log(1-e^{i\theta}z) + \frac{3}{2}z - \frac{3e^{i\theta}}{4}z^2 \tag{2.6}$$

belongs to $B(2,\theta,0)$.

Proof. We note that

$$1 + e^{i\theta}z + (e^{i\theta}z)^2 + (e^{i\theta}z)^3 + (e^{i\theta}z)^4 + \cdots = \frac{1}{1-e^{i\theta}z} \quad (|z|<1). \tag{2.7}$$

Hence we have the following equation by (2.13) above.

$$\begin{aligned} z + e^{i\theta}\frac{z^2}{2} + e^{i2\theta}\frac{z^3}{3} + e^{i3\theta}\frac{z^4}{4} + \cdots &= \int_0^z \frac{1}{1-e^{i\theta}\xi}d\xi \\ &= -\frac{1}{e^{i\theta}}\log(1-e^{i\theta}z). \end{aligned}$$

Further we get the following equation.

$$\begin{aligned} \frac{z^2}{2} + \frac{e^{i\theta}z^3}{2\cdot 3} + \frac{e^{i2\theta}z^4}{3\cdot 4} + \cdots &= -\frac{1}{e^{i\theta}}\int_0^z \log(1-e^{i\theta}\xi)d\xi \\ &= -\frac{1}{e^{i\theta}}\left\{(z-\frac{1}{e^{i\theta}})\log(1-e^{i\theta}z) - z\right\}. \end{aligned} \tag{2.8}$$

Let furtheremore integrate both sides of the equation (2.8) from 0 to z, similarly. Then we get

$$\frac{z^3}{2\cdot 3} + \frac{e^{i\theta}z^4}{2\cdot 3\cdot 4} + \frac{e^{i2\theta}z^5}{3\cdot 4\cdot 5}\cdots = -\frac{1}{2e^{i3\theta}}(1-e^{i\theta}z)^2\log(1-e^{i\theta}z) - \frac{1}{2e^{i2\theta}}z + \frac{3}{4e^{i\theta}}z^2. \tag{2.9}$$

By multiplying $e^{i2\theta}$ to both sides of (2.9), we have

$$\frac{e^{i2\theta}z^3}{2\cdot 3} + \frac{e^{i3\theta}z^4}{2\cdot 3\cdot 4} + \frac{e^{i4\theta}z^5}{3\cdot 4\cdot 5}\cdots = -\frac{1}{2e^{i\theta}}(1-e^{i\theta}z)^2\log(1-e^{i\theta}z) - \frac{1}{2}z + \frac{3e^{i\theta}}{4}z^2. \tag{2.10}$$

We therefore define a function $f(z)$ as follows.

$$\begin{aligned} f(z) &= \frac{1}{2e^{i\theta}}(1-e^{i\theta}z)^2\log(1-e^{i\theta}z) + \frac{3}{2}z - \frac{3e^{i\theta}}{4}z^2 \\ &= z - \frac{e^{i2\theta}}{1\cdot 2\cdot 3}z^3 - \frac{e^{i3\theta}}{2\cdot 3\cdot 4}z^4 - \frac{e^{i4\theta}}{3\cdot 4\cdot 5}z^5 - \cdots. \end{aligned} \tag{2.11}$$

That is,

$$f(z) = z - \sum_{k=3}^{\infty} e^{i(k-1)\theta}a_k z^k \quad (|z|<1) \tag{2.12}$$

where $a_k = \dfrac{1}{(k-2)(k-1)k}$.

In this case, we have

$$\begin{aligned}\sum_{k=3}^{\infty} ka_k &= \frac{3}{1\cdot 2\cdot 3}+\frac{4}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+\cdots\\ &= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\\ &= 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots\\ &= 1.\end{aligned}$$

Hence we have that $f(z)$ belongs to $B(2,\theta,0)$ by Theorem 2.1.

By virtue of (2.12) of Theorem 2.1, and Theorem 2.2 we have the following corollary.

Corollary 2. 1 *The function*

$$f(z) = z - \sum_{k=3}^{\infty} e^{i(k-1)\theta}\frac{1}{(k-2)(k-1)k^2}z^k \tag{2.13}$$

belongs to $C(2,\theta,0)$.

Further putting $\theta=0$ or $\theta=\pi$ in the theorem 2. 3 , we have the following examples.

Example 2. 1 (T.Sekine and T.Yamanaka [5]) *The function*

$$\begin{aligned} f(z) &= \frac{1}{2}(1-z)^2\log(1-z)+\frac{3}{2}z-\frac{3}{4}z^2\\ &= z-\frac{z^3}{2\cdot 3}-\frac{z^4}{2\cdot 3\cdot 4}-\frac{z^5}{3\cdot 4\cdot 5}-\cdots\end{aligned} \tag{2.14}$$

belongs to $B(2,0,0)=T^*(0)$.

Example 2. 2 *The function*

$$\begin{aligned} f(z) &= -\frac{1}{2}(1+z)^2\log(1+z)+\frac{3}{2}z+\frac{3}{4}z^2\\ &= z-\frac{z^3}{2\cdot 3}+\frac{z^4}{2\cdot 3\cdot 4}-\frac{z^5}{3\cdot 4\cdot 5}+\cdots\end{aligned} \tag{2.15}$$

belongs to $B(2,\pi,0)$.

3 Distortion theorems for $B(n,\theta,\alpha)$ and $C(n,\theta,\alpha)$

Theorem 3. 1 *If* $f(z)$ *is in* $B(n,\theta,\alpha)$, *then*

$$|z|-\frac{1-\alpha}{n+1-\alpha}|z|^{n+1}\le |f(z)|\le |z|+\frac{1-\alpha}{n+1-\alpha}|z|^{n+1}. \tag{3.1}$$

Right-hand equality holds for the function

$$f(z)=z-e^{in(\theta+\frac{\pi}{n})}\frac{1-\alpha}{n+1-\alpha}z^{n+1}\quad (z=re^{-i\theta}) \tag{3.2}$$

and Left-hand equality holds for the function

$$f(z) = z - e^{in\theta}\frac{1-\alpha}{n+1-\alpha}z^{n+1} \quad (z = re^{-i\theta}). \tag{3.3}$$

Proof. From the assumption of the theorem, note that

$$(n+1-\alpha)\sum_{k=n+1}^{\infty} a_k \leq \sum_{k=n+1}^{\infty}(k-\alpha)a_k \leq 1-\alpha.$$

Hence we have

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{1-\alpha}{n+1-\alpha}. \tag{3.4}$$

Using the coefficient inequality (3.4), we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{k=n+1}^{\infty} a_k|z|^k \\ &\leq |z| + |z|^{n+1}\sum_{k=n+1}^{\infty} a_k \\ &\leq |z| + \frac{1-\alpha}{n+1-\alpha}|z|^{n+1}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{k=n+1}^{\infty} a_k|z|^k \\ &\geq |z| - |z|^{n+1}\sum_{k=n+1}^{\infty} a_k \\ &\geq |z| - \frac{1-\alpha}{n+1-\alpha}|z|^{n+1}. \end{aligned}$$

Theorem 3. 2 *If $f(z)$ is in $C(n,\theta,\alpha)$, then*

$$|z| - \frac{1-\alpha}{(n+1)(n+1-\alpha)}|z|^{n+1} \leq |f(z)| \leq |z| + \frac{1-\alpha}{(n+1)(n+1-\alpha)}|z|^{n+1}. \tag{3.5}$$

Right-hand equality holds for the function

$$f(z) = z - e^{in(\theta+\frac{\pi}{n})}\frac{1-\alpha}{(n+1)(n+1-\alpha)}z^{n+1} \quad (z = re^{-i\theta}) \tag{3.6}$$

and Left-hand equality holds for the function

$$f(z) = z - e^{in\theta}\frac{1-\alpha}{(n+1)(n+1-\alpha)}z^{n+1} \quad (z = re^{-i\theta}). \tag{3.7}$$

Proof. Using Theorem 2.2, we shall prove Theorem 3.2 similarly as in the proof of the theorem 3.1.

Theorem 3. 3 *If $f(z)$ is in $B(n,\theta,\alpha)$, then*

$$1-\frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n \le |f'(z)| \le 1+\frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n. \tag{3.8}$$

Right-hand equality holds for the function

$$f(z)=z-e^{in(\theta+\frac{\pi}{n})}\frac{1-\alpha}{n+1-\alpha}z^{n+1} \quad (z=re^{-i\theta}) \tag{3.9}$$

and left-hand equality holds for the function

$$f(z)=z-e^{in\theta}\frac{1-\alpha}{n+1-\alpha}z^{n+1} \quad (z=re^{-i\theta}). \tag{3.10}$$

Proof. By the assumption of the theorem and the coefficient inequality (3.4) in the theorem 3.1, we have

$$\begin{aligned}\sum_{k=n+1}^{\infty} ka_k &\le \alpha \sum_{k=n+1}^{\infty} a_k + (1-\alpha) \\ &\le \alpha\cdot\frac{1-\alpha}{n+1-\alpha}+(1-\alpha) \\ &= \frac{(n+1)(1-\alpha)}{n+1-\alpha}.\end{aligned}$$

Hence

$$\begin{aligned}|f'(z)| &\le 1+\sum_{k=n+1}^{\infty} ka_k|z|^{k-1} \\ &\le 1+|z|^n\sum_{k=n+1}^{\infty} ka_k \\ &\le 1+\frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n.\end{aligned}$$

Similarly, we have

$$\begin{aligned}|f'(z)| &\ge 1-\sum_{k=n+1}^{\infty} ka_k|z|^{k-1} \\ &\ge 1-|z|^n\sum_{k=n+1}^{\infty} ka_k \\ &\ge 1-\frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n.\end{aligned}$$

Theorem 3. 4 *If $f(z)$ is in $C(n,\theta,\alpha)$, then*

$$1-\frac{1-\alpha}{n+1-\alpha}|z|^n \le |f'(z)| \le 1+\frac{1-\alpha}{n+1-\alpha}|z|^n. \tag{3.11}$$

Right-hand equality holds for the function

$$f(z) = z - e^{in(\theta+\frac{\pi}{n})} \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1} \quad (z = re^{-i\theta}) \tag{3.12}$$

and left-hand equality holds for the function

$$f(z) = z - e^{in\theta} \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1} \quad (z = re^{-i\theta}). \tag{3.13}$$

4 Convolutions of functions from subclasses of $A(n, \theta)$

Let $f(z)$ and $g(z)$ be in $A(n, \theta_1)$ and $A(n, \theta_2)$, respectively. That is,

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta_1} a_k z^k, \tag{4.1}$$

$$g(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta_2} b_k z^k. \tag{4.2}$$

Then we define by $f(z) * g(z)$ the convolution of $f(z)$ and $g(z)$, that is,

$$f(z) * g(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)(\theta_1+\theta_2)} a_k b_k z^k. \tag{4.3}$$

By using the convolution technique, which Schild and Silverman[4] introduced, we shall prove the following theorems in the same way.

Theorem 4. 1 *If $f(z)$ is in $B(n, \theta_1, \alpha)$ and $g(z)$ is in $B(n, \theta_2, \beta)$, then $f(z) * g(z)$ is an element of $B(n, \theta_1 + \theta_2, \gamma)$, where $\gamma = \dfrac{2-\alpha\beta}{3-\alpha-\beta}$.*
The result is sharp for the functions

$$f(z) = z - e^{i\theta_1} \frac{1-\alpha}{2-\alpha} z^2 \in B(1, \theta_1, \alpha) \text{ and } g(z) = z - e^{i\theta_2} \frac{1-\beta}{2-\beta} z^2 \in B(1, \theta_2, \beta).$$

Proof. By virtue of Theorem 2.1 we wish to find the largest $\gamma = \gamma(\alpha, \beta)$ such that

$$\sum_{k=n+1}^{\infty} (k-\gamma) a_k b_k \leq 1 - \gamma.$$

Hence it is equivalent to show that

$$\sum_{k=n+1}^{\infty} (k-\alpha) a_k \leq 1 - \alpha \tag{4.4}$$

and

$$\sum_{k=n+1}^{\infty} (k-\beta) b_k \leq 1 - \beta \tag{4.5}$$

imply that

$$\sum_{k=n+1}^{\infty}(k-\gamma)a_kb_k \le 1-\gamma \tag{4.6}$$

for all $\gamma=\gamma(\alpha,\beta)\le \dfrac{2-\alpha\beta}{3-\alpha-\beta}$.

Using Cauchy-Schwarz inequality for (4.4) and (4.5) we have the following inequality.

$$\sum_{k=n+1}^{\infty}\frac{\sqrt{k-\alpha}\sqrt{k-\beta}}{\sqrt{1-\alpha}\sqrt{1-\beta}}\sqrt{a_kb_k}\le 1. \tag{4.7}$$

Therefore it suffices to show that

$$\sqrt{a_kb_k}\le\left(\frac{\sqrt{k-\alpha}}{\sqrt{1-\alpha}}\right)\left(\frac{\sqrt{k-\beta}}{\sqrt{1-\beta}}\right)\left(\frac{1-\gamma}{k-\gamma}\right)\quad (k=2,3,\cdots).$$

Since we have

$$\sqrt{a_kb_k}\le\left(\frac{\sqrt{1-\alpha}}{\sqrt{k-\alpha}}\right)\left(\frac{\sqrt{1-\beta}}{\sqrt{k-\beta}}\right)$$

for each k by (4), it will be suffice to show that

$$\left(\frac{\sqrt{1-\alpha}}{\sqrt{k-\alpha}}\right)\left(\frac{\sqrt{1-\beta}}{\sqrt{k-\beta}}\right)\le\left(\frac{\sqrt{k-\alpha}}{\sqrt{k-\alpha}}\right)\left(\frac{\sqrt{1-\beta}}{\sqrt{1-\beta}}\right)\left(\frac{1-\gamma}{k-\gamma}\right) \tag{4.8}$$

for all k.

Inequality (4.8) is equivalent to

$$\gamma\le\gamma(\alpha,\beta)=\frac{1-k\left(\dfrac{1-\alpha}{k-\alpha}\right)\left(\dfrac{1-\beta}{k-\beta}\right)}{1-\left(\dfrac{1-\alpha}{k-\alpha}\right)\left(\dfrac{1-\alpha}{k-\alpha}\right)}. \tag{4.9}$$

Because the right hand side of (4.9) is an increasing functions of $k(k=2,3,\cdots)$, we have

$$\gamma\le\frac{1-2\left(\dfrac{1-\alpha}{2-\alpha}\right)\left(\dfrac{1-\beta}{2-\beta}\right)}{1-\left(\dfrac{1-\alpha}{2-\alpha}\right)\left(\dfrac{1-\beta}{2-\beta}\right)}=\frac{2-\alpha\beta}{3-\alpha-\beta}.$$

Corollary 4. 1 *If $f(z)$ and $g(z)$ are in $B(1,0,\alpha)$, then $f(z)*g(z)$ is an element of $B(1,0,\gamma)$, where $\gamma=\dfrac{2-\alpha^2}{3-2\alpha}$. The result is sharp for the functions*

$$f(z)=g(z)=z-\frac{1-\alpha}{2-\alpha}z^2\in B(1,0,\alpha).$$

Remark 4. 1

$$B\left(1,0,\frac{2-\alpha^2}{3-2\alpha}\right)=T^*\left(\frac{2-\alpha^2}{3-2\alpha}\right).$$

Corollary 4. 2 *If $f(z)$ is in $B(1,0,\alpha)$ and $g(z)$ is in $B(1,\pi,\alpha)$, then $f(z)*g(z)$ belongs to $B(1,\pi,\gamma)$, where $\gamma=\dfrac{2-\alpha^2}{3-2\alpha}$. The result is sharp for the functions*

$$f(z)=z-\frac{1-\alpha}{2-\alpha}z^2\in B(1,0,\alpha) \text{ and } g(z)=z+\frac{1-\beta}{2-\beta}z^2\in B(1,\pi,\alpha).$$

Corollary 4. 3 *If $f(z)$ and $g(z)$ are in $B(1,\pi,\alpha)$, then $f(z)*g(z)$ belongs to $B(1,0,\gamma)$, where $\gamma=\dfrac{2-\alpha^2}{3-2\alpha}$. The result is sharp for the functions*

$$f(z)=g(z)=z+\frac{1-\alpha}{2-\alpha}z^2\in B(1,\pi,\alpha).$$

Theorem 4. 2 *If $f(z)$ and $g(z)$ are in $C(n,\theta_1,\alpha)$ and $C(n,\theta_2,\beta)$, respectively. Then $f(z)*g(z)$ is an element of $C(n,\theta_1+\theta_2,\gamma)$, where $\gamma=\dfrac{2(3-\alpha-\beta)}{7-3\alpha-3\beta+\alpha\beta}$. The result is sharp for the functions*

$$f(z)=z-e^{i\theta_1}\frac{1-\alpha}{2(2-\alpha)}z^2\in C(1,\theta_1,\alpha) \text{ and } g(z)=z-e^{i\theta_2}\frac{1-\beta}{2(2-\beta)}z^2\in C(1,\theta_2,\beta).$$

Proof. From Theorem 2.1 we want to get the largest $\gamma=\gamma(\alpha,\beta)$ such that

$$\sum_{k=n+1}^{\infty}k(k-\gamma)a_kb_k\le 1-\gamma.$$

It is equivalent to show that

$$\sum_{k=n+1}^{\infty}k(k-\alpha)a_k\le 1-\alpha \tag{4.10}$$

and

$$\sum_{k=n+1}^{\infty}k(k-\beta)b_k\le 1-\beta \tag{4.11}$$

imply that

$$\sum_{k=n+1}^{\infty}k(k-\gamma)a_kb_k\le 1-\gamma \tag{4.12}$$

for all $\gamma=\gamma(\alpha,\beta)\le\dfrac{2(3-\alpha-\beta)}{7-3\alpha-3\beta+\alpha\beta}$.

Proceeding similarly as in the proof of the preceding theorem, we have

$$\gamma(\alpha,\beta)\le\frac{1-\dfrac{(1-\alpha)(1-\beta)}{(k-\alpha)(k-\beta)}}{1-\dfrac{(1-\alpha)(1-\beta)}{k(k-\alpha)(k-\beta)}}. \tag{4.13}$$

Since the right hand side of (4.13) is an increasing functions of $k(k = 2, 3, \cdots)$ as in Theorem 4.1, setting $n = 2$ in (4.13) we have

$$\gamma \leq \frac{1 - \dfrac{(1-\alpha)(1-\beta)}{(2-\alpha)(2-\beta)}}{1 - \dfrac{(1-\alpha)(1-\beta)}{2(2-\alpha)(2-\beta)}} = \frac{2(3-\alpha-\beta)}{7-3\alpha-3\beta+\alpha\beta}. \tag{4.14}$$

Corollary 4. 4 *If $f(z)$ and $g(z)$ are in $C(1,0,\alpha)$, then $f(z) * g(z)$ is an element of $C(1,0,\gamma)$, where $\gamma = \dfrac{2(3-2\alpha)}{7-6\alpha+\alpha^2}$. The result is sharp for the functions*

$$f(z) = g(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2 \in C(1,0,\alpha).$$

Corollary 4. 5 *If $f(z)$ is in $C(1,0,\alpha)$ and $g(z)$ is in $C(1,\pi,\alpha)$, then $f(z) * g(z)$ belongs to $C(1,\pi,\gamma)$, where $\gamma = \dfrac{2(3-2\alpha)}{7-6\alpha+\alpha^2}$. The result is sharp for the functions*

$$f(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2 \in C(1,0,\alpha) \text{ and } g(z) = z + \frac{1-\alpha}{2(2-\alpha)} z^2 \in C(1,\pi,\alpha).$$

Corollary 4. 6 *If $f(z)$ and $g(z)$ are in $C(1,\pi,\alpha)$, then $f(z) * g(z)$ belongs to $C(1,0,\gamma)$, where $\gamma = \dfrac{2(3-2\alpha)}{7-6\alpha+\alpha^2}$. The result is sharp for the functions*

$$f(z) = g(z) = z + \frac{1-\alpha}{2(2-\alpha)} z^2 \in C(1,\pi,\alpha).$$

Theorem 4. 3 *If $f(z)$ and $g(z)$ are in $B(n,\theta_1,\alpha)$ and $B(n,\theta_2,\beta)$, respectively. Then $f(z) * g(z)$ is an element of $C(n,\theta_1+\theta_2,\gamma)$, where $\gamma = \dfrac{2\alpha+2\beta-3\alpha\beta}{2-\alpha\beta}$. The result is sharp for the functions*

$$f(z) = z - e^{i\theta_1}\frac{1-\alpha}{2-\alpha} z^2 \in B(1,\theta_1,\alpha) \text{ and } g(z) = z - e^{i\theta_2}\frac{1-\beta}{2-\beta} z^2 \in B(1,\theta_2,\beta).$$

Proof. We shall prove that

$$\sum_{k=n+1}^{\infty} (k-\alpha)a_k \leq 1-\alpha \tag{4.15}$$

and

$$\sum_{k=n+1}^{\infty} (k-\beta)b_k \leq 1-\beta \tag{4.16}$$

imply that

$$\sum_{k=n+1}^{\infty} k(k-\gamma)a_k b_k \leq 1-\gamma \tag{4.17}$$

for all $\gamma = \gamma(\alpha,\beta) \leq \dfrac{2\alpha+2\beta-3\alpha\beta}{2-\alpha\beta}$.

Proceeding similarly as in the proof of the theorem 4.1 or the theorem 4.2, we have

$$\gamma = \gamma(\alpha,\beta) \leq \frac{1-\dfrac{k^2(1-\alpha)(1-\beta)}{(k-\alpha)(k-\beta)}}{1-\dfrac{k(1-\alpha)(1-\beta)}{(k-\alpha)(k-\beta)}}. \tag{4.18}$$

Because the right hand side of (4.18) is an increasing functions of $k(k=2,3,\cdots)$, we have

$$\gamma \leq \frac{1-\dfrac{4(1-\alpha)(1-\beta)}{(2-\alpha)(2-\beta)}}{1-\dfrac{2(1-\alpha)(1-\beta)}{(2-\alpha)(2-\beta)}} = \frac{2\alpha+2\beta-3\alpha\beta}{2-\alpha\beta}. \tag{4.19}$$

Corollary 4. 7 *If $f(z)$ and $g(z)$ are in $B(1,0,\alpha)$, then $f(z)*g(z)$ is an element of $C(1,0,\gamma)$, where $\gamma = \dfrac{4\alpha-3\alpha^2}{2-\alpha^2}$. The result is sharp for the functions*

$$f(z) = g(z) = z - \frac{1-\alpha}{2-\alpha}z^2 \in B(1,0,\alpha).$$

Corollary 4. 8 *If $f(z)$ is in $B(1,0,\alpha)$ and $g(z)$ is in $B(1,\pi,\alpha)$, then $f(z)*g(z)$ belongs to $C(1,\pi,\gamma)$, where $\gamma = \dfrac{4\alpha-3\alpha^2}{2-\alpha^2}$. The result is sharp for the functions*

$$f(z) = z - \frac{1-\alpha}{2-\alpha}z^2 \in B(1,0,\alpha) \text{ and } g(z) = z + \frac{1-\alpha}{2-\alpha}z^2 \in B(1,\pi,\alpha).$$

Corollary 4. 9 *If $f(z)$ and $g(z)$ are in $B(1,\pi,\alpha)$, then $f(z)*g(z)$ belongs to $C(1,0,\gamma)$, where $\gamma = \dfrac{4\alpha-3\alpha^2}{2-\alpha^2}$. The result is sharp for the functions*

$$f(z) = g(z) = z + \frac{1-\alpha}{2-\alpha}z^2 \in B(1,\pi,\alpha).$$

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