ON AN OPEN PROBLEM OF S. OWA

by

Shigeyoshi OWA and Grigore Stefan SALAGEAN

Let $U$ denote the unit disc, $U = \{ z \in \mathbb{C}; |z| < 1 \}$, let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N} = \{ 1, 2, 3, \ldots \}$ and let $H(U)$ denote the set of functions which are holomorphic in $U$.

For $n \in \mathbb{N}$ let

$$ T_n = \left\{ f \in H(U); \frac{f(z)}{z} \neq 0, (z \in \mathbb{C} - \{ 0 \}), f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, a_k \geq 0, (k \in \mathbb{N}, k > n) \right\}. $$

For $n \in \mathbb{N}$ and $b \in \mathbb{C} - \{ 0 \}$ we define the next subclasses of $T_n$

$$ T_n^*(b) = \left\{ f \in T_n : \text{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf''(z)}{f(z)} - 1 \right) \right\} > 0, (z \in U) \right\}, $$

$$ O_n^*(b) = \left\{ f \in T_n : \sum_{k=n+1}^{\infty} (k-1+|b|)a_k \leq |b| \right\} $$

and

$$ P_n^*(b) = \left\{ f \in T_n : \sum_{k=n+1}^{\infty} \left[ (k-1)\frac{\text{Re} b}{|b|} + |b| \right] a_k \leq |b| \right\}. $$

The functions in $T_n^*(b)$ are the functions with negative coefficients starlike of the complex order $b$ (see [1, 2]).

The class $T_1^*(1-\alpha)$, $\alpha \in [0,1)$ is the class of functions with negative coefficients starlike of order $\alpha$ introduced and studied by H. Silverman [4].

The class $O_n^*(b)$ was introduced by S. Owa in [3, p.163-164], where he conjectured that $T_n^*(b) = O_n^*(b)$. In this paper we give an answer to this conjecture.

THEOREM. Let $n \in \mathbb{N}$ and let $b \in \mathbb{C} - \{ 0 \}$; then

1) $O_n^*(b) \subset T_n^*(b)$;
2) $T_n^*(b) \subset P_n^*(b)$;
3) If \( b \in (0, \infty) \) (\( b \) is a positive real number), then
\[
O_n^*(b) = T_n^*(b) = P_n^*(b);
\]

4) If \( -n/2 < \Re b \leq 0 \), then \( P_n^*(b) \not\subset T_n^*(b) \);

5) If \( b \in (-\infty, -n) \cup (-n/2, 0) \), then \( T_n^*(b) \not\subset O_n^*(b) \).

Proof. 1). Let \( f \in O_n^*(b) \). We prove that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < |b|, \quad z \in U.
\]

We suppose that \( f \) has the series expansion
\[
f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad a_k \geq 0.
\]

We have
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| - |b| = \left| \sum_{k=n+1}^{\infty} (k-1) a_k z^{k-1} \right| - |b| \leq \sum_{k=n+1}^{\infty} (k-1) a_k |z|^{k-1} - |b|.
\]

We use the fact that \( f(z) \neq 0 \) when \( z \in U \setminus \{0\} \) and \( \lim_{z \to 0} \frac{f(z)}{z} = 1 \); these imply
\[
1 - \sum_{k=n+1}^{\infty} a_k |z|^k > 0,
\]
when \( |z| = r \in [0, 1] \).

From (3) and (4) we deduce
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| - |b| < \frac{\sum_{k=n+1}^{\infty} (k-1) a_k}{1 - \sum_{k=n+1}^{\infty} a_k} - |b|
\]
\[
= \frac{\sum_{k=n+1}^{\infty} (k-1) a_k |z|^{k-1} - |b|}{1 - \sum_{k=n+1}^{\infty} a_k}.
\]

By using the definition of \( O_n^*(b) \) we obtain (1) and this implies
\[
\Re \left\{ \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > -1, \quad z \in U,
\]
hence \( f \in T_n^*(b) \).
2). Let \( f \) be in \( T_n^*(b) \). Then
\[
\text{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \quad (z \in U)
\]
and, by using (2), this is equivalent to
\[
\text{Re} \left\{ \frac{1}{b} \sum_{k=n+1}^{\infty} \frac{(1-k)a_k z^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k z^{k-1}} \right\} > -1 \quad (z \in U).
\]
If \( z = r \in [0,1) \) and for \( r \to 1^- \), from (5) we obtain
\[
\frac{\sum_{k=n+1}^{\infty} (1-k)a_k}{1 - \sum_{k=n+1}^{\infty} a_k} \text{Re} \frac{1}{b} > -1
\]
which is equivalent to
\[
\sum_{k=n+1}^{\infty} \text{Re} b (1-k)a_k > -|b|^2 \left( 1 - \sum_{k=n+1}^{\infty} a_k \right)
\]
or
\[
\sum_{k=n+1}^{\infty} \left[ (k-1)\text{Re} b/|b| + |b| \right] a_k < |b|
\]
hence \( f \in P_n^*(b) \).

3). If \( b \) is a real positive number, then the definition of \( O_n^* \) and \( P_n^* \) are equivalent, hence \( O_n^*(b) = P_n^*(b) \). By using 1) and 2) from this theorem we obtain the conclusion of 3).

4). Let
\[
f_n(z) = z - z^{n+1};
\]
then \( f_n \in P_n^*(b) \) when \( b \in \mathbb{C} - \{0\} \) and \( \text{Re} b < 0 \), because
\[
\sum_{k=n+1}^{\infty} \left[ |b| + ((k-1)\text{Re} b)/|b| \right] a_k
\]
\[
= \left[ |b| + [(n+1) - 1]\text{Re} b/|b| \right] \cdot 1 = |b| + n\text{Re} b/|b| \leq |b|.
\]
Now let \( \rho = \text{Re} b < 0 \) and let \( s \) be a negative real number such that
\[
n + 2\rho(1-s) > 0
\]
for \( n \in \mathbb{N} \) fixed. If we choose \( z_0 \) one of the roots of the equation

\[
    z^n = \frac{b(1-s)}{n+b(1-s)},
\]

then \( z_0 \in U \) and for \( f_n \) given by (6) we have

\[
    1 + \frac{1}{b} \left( \frac{z_0 f_n'(z_0)}{f_n(z_0)} - 1 \right) = s < 0,
\]

hence \( f_n \notin T_n^*(b) \).

5). Let \( b \in (-\infty, -n) \); we verify that the functions

\[
    f_{n,\lambda}(z) = z - \lambda z^{n+1}
\]

belong to \( T_n^*(b) \) for \( \lambda > b/(n+b) \) and that \( f_{n,\lambda} \notin O_n^*(b) \).

Indeed we have

\[
    \sum_{k=n+1}^{\infty} (k-1+|b|)a_k = (n+|b|)\lambda > |b|,
\]

because \( \lambda > b/(n+b) > 1 \).

We also have

\[
    \text{Re} \left\{ 1 + \frac{1}{b} \left( \frac{z f_{n,\lambda}'(z)}{f_{n,\lambda}(z)} - 1 \right) \right\} = \text{Re} \left\{ 1 + \frac{n\lambda z^n}{b(\lambda z^n - 1)} \right\} > 0, \ z \in U,
\]

for \( \lambda > b/(n+b) \) and \( b < -n \), hence \( f_{n,\lambda} \in T_n^*(b) \).

Let now \( b \in (-n/2, 0) \), and let \( f_{n,\lambda} \) be defined by (7), where

\[
    -b/(n-b) < \lambda < -b/(n+b).
\]

Then \( \lambda > -b/(n-b) \) implies \( f_{n,\lambda} \notin O_n^*(b) \) and for \( \lambda < -b/(n+b) \), \(-n/2 < b < 0 \) the inequality (8) also is verified, hence \( f_{n,\lambda} \in T_n^*(b) \).
References


Shigeyoshi OWA
Kinki University
Department of Mathematics
Higashi-Osaka,
Osaka 577
JAPAN

Grigore Stefan SALAGEAN
Babes-Bolyai University
Faculty of Mathematics and Computer Science
str. M. Kogalniceanu nr. 1
3400 Cluj-Napoca
ROMANIA