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Some Properties for Convolutions of Generalized Hypergeometric Functions

Shigeyoshi Owa, Ji A Kim and Nak Eun Cho

Abstract

With the convolution products of generalized hypergeometric functions \( pF_q(z) \) and analytic functions \( f(z) \) in the open unit disk, the operator \( I_{b_1,b_2,...,b_q}^{a_1,a_2,...,a_p}(f) \) is introduced. The object of the present paper is to derive some interesting properties of operator \( I_{b_1,b_2,...,b_q}^{a_1,a_2,...,a_p}(f) \) associated with some classes of univalent functions.

1. Introduction

Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( U = \{ z : |z| < 1 \} \). Denote by \( S \) the class of all functions in \( A \) which are univalent in \( U \).

A function \( f(z) \in A \) is said to be in the class \( R^t(A, B) \) if

\[
\left| \frac{f'(z) - 1}{t(A - B) - B(f'(z) - 1)} \right| < 1,
\]

where \( A \) and \( B \) are arbitrary fixed numbers with \(-1 \leq B < A \leq 1\) and \( t \in C \setminus \{0\} \) (\( C \) is the set of all complex numbers). Clearly, a function \( f(z) \) belongs to \( R^t(A, B) \) if and only if there exists a function \( w(z) \) regular in \( U \) satisfying \( w(0) = 0 \) and \( |w(z)| < 1 \) (\( z \in U \)) such that

\[
1 + \frac{1}{t}(f'(z) - 1) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in U).
\]

The class \( R^t(A, B) \) was introduced by Dixit and Pal [4], recently.

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By giving specific values $t$, $A$ and $B$ in (1.2), we obtain the following subclasses studied by various researchers in earlier works:

(i) For $t = e^{-in} \cos \eta$ ($|\eta| < \frac{\pi}{2}$), $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $B = -1$, we obtain the class of functions $f$ satisfying the condition

\[
\frac{\epsilon_{n}(f'(z) - 1)}{2(1 - \alpha) \cos \eta + \epsilon_{n}(f'(z) - 1)} < 1 \quad (z \in U).
\]

In this case, the class $R_{\eta}(A, B)$ is equivalent to the class $R_{\eta}(\alpha)$ which is studied by Ponnu-samy and Rphonning [11]. Here $R_{\eta}(\alpha)$ is the class of functions $f(z) \in A$ satisfying the condition

\[
Re(e^{i\eta}(f'(z) - \alpha)) > 0 \quad (|\eta| < \frac{\pi}{2}, 0 \leq \alpha < 1, z \in U).
\]

(ii) For $t = e^{-in} \cos \eta$ ($|\eta| < \frac{\pi}{2}$), we obtain the class of functions $f(z) \in A$ satisfying the condition

\[
\frac{\epsilon_{n}(f'(z) - 1)}{Be^{i\eta}f'(z) - (A \cos \eta + iB \sin \eta)} < 1 \quad (z \in U),
\]

which was studied by Dashrath [3].

(iii) For $t = 1$, $A = \beta$ and $B = -\beta$ ($0 < \beta \leq 1$), we obtain the class of functions $f(z)$ satisfying the condition

\[
\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta \quad (z \in U),
\]

which was studied by Padmanabhan [10] and Caplinger and Cauchy [2].

Let $S^{*}(\alpha)$ and $C(\alpha)$ denote the subclasses of $S$ consisting of starlike and convex functions of order $\alpha$ ($0 \leq \alpha < 1$) in $U$, respectively. It is well-known that $S^{*}(\alpha) \subset S^{*}(0) \equiv S^{*}$, $C(\alpha) \subset C(0) \equiv C$ and $C(\alpha) \subset S^{*}(\alpha) \subset S$. For $\lambda > 0$, define the classes $S_{\lambda}^{*}$ and $C_{\lambda}$ by

\[
S_{\lambda}^{*} = \{f(z) \in A : \frac{zf'(z)}{f(z)} < \lambda, z \in U\}
\]

and

\[
C_{\lambda} = \{f(z) \in A : zf'(z) \in S_{\lambda}^{*}\},
\]

respectively. It is a known fact that a sufficient condition for $f(z) \in A$ of the form (1.1) to belong to the class $S^{*}$ is that $\sum_{n=2}^{\infty} n|a_{n}| \leq 1$. A simple extension of this result is the following [16]:

\[
\sum_{n=2}^{\infty} (n + \lambda - 1)|a_{n}| \leq \lambda \Rightarrow f(z) \in S_{\lambda}^{*}.
\]
For $\lambda = \frac{1}{2}$, this was previously proved by Schild [18]. Since $f(z) \in C_\lambda$ if and only if $zf'(z) \in S_\lambda^*$, we have a corresponding result for $C_\lambda$,

\begin{equation}
\sum_{n=2}^{\infty} n(n+\lambda-1)|a_n| \leq \lambda \Rightarrow f(z) \in C_\lambda.
\end{equation}

In this paper, we consider the generalized hypergeometric series $pF_q(z)$ defined by

\begin{equation}
pF_q(z) \equiv pFq\left( \begin{array}{c}
a_1, a_2, \cdots, a_p \\
b_1, b_2, \cdots, b_q \\
z \end{array} \right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{i=1}^{q}(b_i)_n} \frac{z^n}{(1)_n}.
\end{equation}

where $p$ and $q$ are positive integers and we assume that the variable $z$, the numerator parameters $a_1, a_2, \cdots, a_p$ and the denominator parameters $b_1, b_2, \cdots, b_q$ take on complex values, provided that $b_i \neq 0, -1, -2, \cdots; i = 1, 2, \cdots, q$. Here $(\lambda)_n$ is the Pochhammer symbol defined by

\begin{equation}
(\lambda)_n = \begin{cases} 
1 & \text{if } n = 0 \\
\lambda(\lambda+1) \cdots (\lambda+n-1) & \text{if } n \in N = \{1, 2, \cdots\}.
\end{cases}
\end{equation}

For any complex number $\lambda$, we also use the ascending factorial notation

\begin{equation}
(\lambda)_n = \lambda(\lambda+1)_{n-1}
\end{equation}

for $n \geq 1$ and $(\lambda)_0 = 1$ for $\lambda \neq 0$. If $\lambda$ is neither zero nor a negative integer, then using the definition of the Gamma function, we can write

\begin{equation}
(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}.
\end{equation}

Furthermore, if we set

\begin{equation}
\omega = \sum_{i=1}^{q} b_i - \sum_{j=1}^{p} a_j
\end{equation}

it is known that the series $pF_q(z)$, with $p = q + 1$, is

(i) absolutely convergent for $|z| = 1$ if $\Re \omega > 0$,

(ii) conditionally convergent for $|z| = 1$, $z \neq 1$ if $-1 < \Re \omega \leq 0$

and

(iii) divergent for $|z| = 1$ if $\Re \omega \leq -1$.

As in the case of the function $2F_1(z)$, we are led to the well-known Gauss summation theorem:

\begin{equation}
2F_1\left( \begin{array}{c}
a_1, a_2 \\
b_1 \\
1 \end{array} \right) = \frac{\Gamma(b_1)\Gamma(b_1 - a_1 - a_2)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)}, \quad \Re(b_1 - a_1 - a_2) > 0.
\end{equation}
We recall that the function \( _2F_1(z) \) is bounded if \( \text{Re}(b_1 - a_1 - a_2) > 0 \) and has a pole at \( z = 1 \) if \( \text{Re}(b_1 - a_1 - a_2) \leq 0 \) (cf. [1]). Univalence, starlikeness and convexity properties of \( _2F_1(a_1, a_2; z) \) have been studied extensively in [12, 15].

For \( f(z) \in A \), we define the operator \( I_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \cdots, a_p}(f) \) by

\[
[I_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \cdots, a_p}(f)](z) = z_{p}F_{q}(a_1, a_2b_1, b_2, \cdots, b_q; \ z) * f(z),
\]

where the symbol " * " denotes the usual Hadamard product or convolution of power series.

2. Properties of the operators with \( R^t(A, B) \)

Now we introduce several lemmas which are needed for the proof of our main results.

**Lemma 2.1 ([8])** Let \( w(z) \) be analytic in \( U \) with \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value on the circle \( |z| = r < 1 \) at a point \( z_1 \in U \), then we can write

\[
z_1w'(z_1) = mw(z_1),
\]

where \( m \) is real and \( m \geq 1 \).

**Lemma 2.2 ([4])** Let a function \( f(z) \) of the form (1.1) be in \( R^t(A, B) \). Then

\[
|a_n| \leq \frac{(A - B)|t|}{n}.
\]

Then result is sharp for the function

\[
f(z) = \int_0^z \left( 1 + \frac{(A - B)tz^{n-1}}{1 + Bz^{n-1}} \right) \ dz \quad (n \geq 2, z \in U).
\]

**Lemma 2.3 ([4])** Let a function \( f(z) \) of the form (1.1) be in \( A \). If

\[
\sum_{n=2}^{\infty} (1 + |B|)n|a_n| \leq (A - B)|t| \quad (-1 \leq B < A \leq 1, \ t \in C \setminus \{0\})
\]

then \( f(z) \in R^t(A, B) \). The result is sharp for function

\[
f(z) = z + \frac{(A - B)t}{(1 + |B|)n}z^n \quad (n \geq 2, z \in U).
\]

Our first result for the operators is contained in
Theorem 2.1 If \( f(z) \in A \) satisfies
\[
|I_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \ldots, a_p}(f) - 1|^{1-\beta} < \left( \frac{1}{2} \right)^{\beta}
\]
for some fixed \( \beta \geq 0 \), then \( I_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \ldots, a_p}(f) \) is univalent (close-to-convex) in \( U \).

Proof. We note that
\[
I_{b_1, b_2, \ldots, b_q}(f) = z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{n-1}}{\prod_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}} a_n z^n
\]
in \( A \). Define \( w(z) \) by
\[
w(z) = \frac{I_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \ldots, a_p}(f)}{z} - 1
\]
for \( z \in U \). Then it follows that \( w(z) \) is analytic in \( U \) with \( w(0) = 0 \). By (2.1), it is clear that
\[
|w(z)|^{1-\beta} \left| \frac{zw'(z)}{1+w(z)} \right|^{\beta} = |w(z)| \left( \frac{zw'(z)}{w(z)(1+w(z))} \right)^{\beta} < \left( \frac{1}{2} \right)^{\beta}
\]
Suppose that there exists a point \( z_1 \in U \) such that
\[
\max_{|z| \leq |z_1|} |w(z)| = |w(z_1)| = 1.
\]
Then we can put
\[
\frac{z_1w'(z_1)}{w(z_1)} = m \geq 1,
\]
by Lemma 2.1. Therefore we obtain
\[
|w(z_1)| \left| \frac{z_1w'(z_1)}{w(z_1)(1+w(z_1))} \right|^{\beta} \geq \left( \frac{m}{2} \right)^{\beta} \geq \left( \frac{1}{2} \right)^{\beta},
\]
which contradicts the condition (2.2). This shows that
\[
|w(z)| = \left| \frac{I_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \ldots, a_p}(f)}{z} - 1 \right| < 1,
\]
which implies that \( \text{Re} \left[ I_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \ldots, a_p}(f) \right]'(z) > 0 \) for \( z \in U \). Therefore, by Noshiro-Warschawski Theorem [5], \( I_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \ldots, a_p}(f) \) is univalent (close-to-convex) in \( U \).
Theorem 2.2 Let $a_j \ (j = 1, 2, \cdots, p) \in C \setminus \{0\}$, $b_i \ (i = 1, 2, \cdots, q) \in C \setminus \{0\}$, $\Re b_i > 0 \ (i = 1, 2, \cdots, q)$, and $\sum_{i=1}^{q} \Re b_i > \sum_{j=1}^{p} |a_j|$. If $f(z) \in R^t(A, B)$ satisfies

\begin{equation}
\label{eq:2.3}
p_F_q \left( \frac{|a_1|, |a_2|, \cdots, |a_p|}{\Re b_1, \Re b_2, \cdots, \Re b_q} ; 1 \right) \leq \frac{1}{1+|B|} + 1,
\end{equation}

then

\begin{equation}
\label{eq:2.2.2}
z \cdot p_F_q \left( \frac{a_1, a_2, \cdots, a_p}{b_1, b_2, \cdots, b_q} ; z^k \right) * f(z) \in R^t(A, B),
\end{equation}

where $k \in N$.

**Proof.** By Lemma 2.3, it suffices to show that

\begin{equation}
\label{eq:2.4}
T_1 := \sum_{n=2}^{\infty} (1 + |B|) (k(n - 1) + 1) \left\{ \frac{\prod_{j=1}^{p} (a_j n - 1)}{\prod_{i=1}^{q} (\Re b_i n - 1)} a_k(n-1) + 1 \right\} \leq (A - B)|t|.
\end{equation}

From Lemma 2.2 and the fact that $|(a)_n| \leq (|a|)_n$ and $(\Re b)_n \leq (b)_n$, $\Re b > 0$, we have

\begin{align*}
T_1 & \leq \sum_{n=2}^{\infty} (A - B)(1 + |B|)|t| \left\{ \frac{\prod_{j=1}^{p} (a_j n - 1)}{\prod_{i=1}^{q} (\Re b_i n - 1)} \right\} \\
& = (A - B)(1 + |B|)|t| \left\{ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j n)}{\prod_{i=1}^{q} (\Re b_i n)(1)_{n-1}} - 1 \right\} \\
& = (A - B)(1 + |B|)|t| \left\{ p_F_q \left( \frac{|a_1|, |a_2|, \cdots, |a_p|}{\Re b_1, \Re b_2, \cdots, \Re b_q} ; 1 \right) - 1 \right\} \\
& \leq (A - B)|t|
\end{align*}

by (2.3). This completes the proof of Theorem 2.2.

**Corollary 2.1** Let $a_j \ (j = 1, 2, \cdots, q + 1) \in C \setminus \{0\}$, $b_i \ (i = 1, 2, \cdots, q) \in C \setminus \{0\}$, $\Re b_m > |a_m| + 1 \ (m = 1, 2, \cdots, q - 1)$, and $\Re b_q > |a_q| + |a_{q+1}|$. If $f(z) \in R^t(A, B)$ satisfies

\begin{equation}
\label{eq:2.5}
\frac{\Gamma(\Re b_q) \Gamma(\Re b_q - |a_q| - |a_{q+1}|)}{\Gamma(\Re b_q - |a_q|) \Gamma(\Re b_q - |a_{q+1}|)} \left( \prod_{m=1}^{q-1} \frac{\Re b_m - 1}{\Re b_m - |a_m| - 1} \right) \leq \frac{1}{1 + |B|} + 1,
\end{equation}

then

\begin{equation}
\label{eq:2.2.1}
z \cdot q+1 F_q \left( \frac{a_1, a_2, \cdots, a_{q+1}}{b_1, b_2, \cdots, b_q} ; z^k \right) * f(z) \in R^t(A, B),
\end{equation}

where $k \in N$. 

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Proof. We note that

\[ q+1F_{q}\left(\left|a_{1}\right|,\left|a_{2}\right|,\ldots,\left|a_{q+1}\right|;1\right) = \sum_{n=0}^{\infty} \frac{\left(|a_{1}|\right)_{n} \cdots \left(|a_{q+1}|\right)_{n}}{(\text{Re} b_{1})_{n} \cdots (\text{Re} b_{q})_{n}(1)_{n}}. \]

\[ = \left( \prod_{m=1}^{q-1} \frac{\Gamma(\text{Re} b_{m})}{\Gamma(\text{Re} b_{m} - |a_{m}| - 1)} \right) \frac{\Gamma(\text{Re} b_{q})}{\Gamma(\text{Re} b_{q} - |a_{q}| - |a_{q+1}|)} \]

\[ = \left( \prod_{m=1}^{q-1} \frac{\text{Re} b_{m} - 1}{\text{Re} b_{m} - |a_{m}| - 1} \right) \frac{\Gamma(\text{Re} b_{q})}{\Gamma(\text{Re} b_{q} - |a_{q}| - |a_{q+1}|)}. \]

Hence we have

\[ \sum_{n=2}^{\infty} (1 + |B|)(k(n-1) + 1) \left| \frac{\prod_{j=1}^{p}(a_{j})_{n-1}}{\prod_{i=1}^{q}(b_{i})_{n-1}(1)_{n-1}} a_{k(n-1)+1} \right| \leq (A - B)(1 + |B|)|t| \left\{ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(|a_{j}|)_{n}}{\prod_{i=1}^{q}(\text{Re} b_{i})_{n}(1)_{n}} - 1 \right\} \]

\[ \leq (A - B)|t|, \]

by assumption. This completes our proof.

Theorem 2.3 Let \( a_{j} (j = 1, 2, \ldots, p) \in C \setminus \{0\} \), \( b_{i} (i = 1, 2, \ldots, q) \in C \setminus \{0\} \), \( \text{Re} b_{i} > 0 (i = 1, 2, \ldots, q) \), and \( \sum_{i=1}^{q}\text{Re} b_{i} > \sum_{j=1}^{p}|a_{j}| \). If \( f(z) \in R^{t}(A, B) \) satisfies

\[ p+2F_{q+2}\left(\left|a_{1}\right|,\left|a_{2}\right|,\ldots,\left|a_{p}\right|,\lambda + 1, 1;1\right) \leq \frac{1}{(A - B)|t|} + 1, \]

then \( I_{b_{1},b_{2},\ldots,b_{q}}^{a_{1},a_{2},\ldots,a_{p}}(f) \in S_{\lambda} \) where \( \lambda > 0 \).

Proof. Suppose that \( f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n} \in R^{t}(A, B) \). Then, by (1.5) it suffices to show that

\[ T_{2} := \sum_{n=2}^{\infty} (n + \lambda - 1) \left| \frac{\prod_{j=1}^{p}(a_{j})_{n-1}}{\prod_{i=1}^{q}(b_{i})_{n-1}(1)_{n-1}} a_{n} \right| \leq \lambda. \]

From Lemma 2.3, we observe that

\[ T_{2} \leq \sum_{n=2}^{\infty} (n + \lambda - 1) \frac{(A - B)|t|}{n} \left\{ \frac{\prod_{j=1}^{p}(|a_{j}|)_{n}}{\prod_{i=1}^{q}(\text{Re} b_{i})_{n}(1)_{n}} \right\} \]
\[
(A - B)|t| \sum_{n=1}^{\infty} \lambda(\lambda + 1)_n \frac{(1)_n}{(2)_n} \left\{ \prod_{i=1}^{p} |a_j|_n \left\{ \prod_{i=1}^{q} \left( \text{Re} b_i \right)_n (1)_n \right\} \right\}
\leq \lambda
\]

by (2.7), which completes the proof of Theorem 2.3.

Corollary 2.2 Let \( a_j \) (\( j = 1, 2, \ldots, q + 1 \)) \( \in C \setminus \{0\} \), \( b_i \) (\( i = 1, 2, \ldots, q \)) \( \in C \setminus \{0\} \), \( \text{Re} b_m > |a_m| + 1 \) (\( m = 1, 2, \ldots, q - 1 \)), \( |a_q| < 1 \) and \( \text{Re} b_q - 2 > \lambda > |a_{q+1}| + 1 \). If \( f(z) \in R^t(A, B) \) satisfies

\[
\frac{\lambda(\lambda - 1) \left( \text{Re} b_q - 1 \right)}{\left( 1 - |a_q| \right) \left( \lambda - |a_{q+1}| - 1 \right) \left( \text{Re} b_q - \lambda - 2 \right)} \times \left( \prod_{m=1}^{q-1} \frac{\text{Re} b_m - 1}{\text{Re} b_m - |a_m| - 1} \right) \lambda \leq \frac{1}{(A - B)|t| + 1},
\]

then \( f_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \ldots, a_{q+1}}(f) \in S_\lambda^* \) where \( \lambda > 0 \).

Proof. We note that

\[
q+3 F_{q+2} \left( \begin{array}{c}
|a_1|, |a_2|, \ldots, |a_{q+1}|, \lambda + 1, 1 \\
\text{Re} b_1, \text{Re} b_2, \ldots, \text{Re} b_q, \lambda, 2
\end{array} \right)
\]

\[
= \left( \prod_{m=1}^{q} \frac{\Gamma(\text{Re} b_m) \Gamma(\text{Re} b_m - |a_m| - 1)}{\Gamma(\text{Re} b_m - |a_m|) \Gamma(\text{Re} b_m - 1)} \right) \Gamma(2) \Gamma(1 - |a_q|) \Gamma(\lambda) \Gamma(\lambda - |a_{q+1}| - 1) \Gamma(2 - |a_q|) \Gamma(1) \Gamma(\lambda - |a_{q+1}|) \Gamma(\lambda - 1) \times \frac{\Gamma(\text{Re} b_q) \Gamma(\text{Re} b_q - \lambda - 2)}{\Gamma(\text{Re} b_q - 1) \Gamma(\text{Re} b_q - \lambda - 1)}
\]

\[
= \left( \prod_{m=1}^{q-1} \frac{\text{Re} b_m - 1}{\text{Re} b_m - |a_m| - 1} \right) \frac{1}{1 - |a_q|} \frac{\lambda - 1}{\lambda - |a_{q+1}| - 1} \frac{\text{Re} b_q - 1}{\text{Re} b_q - \lambda - 2}.
\]

Hence we have

\[
\sum_{n=2}^{\infty} \frac{n + \lambda - 1}{\prod_{j=1}^{q+1} (a_j)_n - a_n}
= (A - B)|t| \sum_{n=1}^{\infty} \lambda(\lambda + 1)_n \frac{(1)_n}{(2)_n} \left\{ \prod_{i=1}^{q} |a_j|_n \left\{ \prod_{i=1}^{q} \left( \text{Re} b_i \right)_n (1)_n \right\} \right\}
\]
\[ \lambda(A-B)|t| \left\{ \begin{array}{c} q+3F_{q+2} \left( |a_1|, |a_2|, \ldots, |a_{q+1}|, \lambda + 1, 1 \\
Re b_1, Re b_2, \ldots, Re b_q, \lambda, 2 \end{array} \right) - 1 \right\} \leq \lambda, \]

by assumption, which completes the proof of Corollary 2.2.

**Theorem 2.4** Let \( a_j \ (j = 1, 2, \cdots, p) \in C \setminus \{0\}, \) \( b_i \ (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \) \( \text{Re} b_i > 0 \ (i = 1, 2, \cdots, q), \) and \( \sum_{i=1}^{q} \text{Re} b_i > \sum_{j=1}^{p} |a_j| + 1. \) If \( f(z) \in R^t(A, B) \) satisfies

\[ p+1F_{q+1} \left( \begin{array}{c} |a_1|, |a_2|, \ldots, |a_p|, \lambda + 1 \\
Re b_1, Re b_2, \ldots, Re b_q, \lambda \end{array} ; 1 \right) \leq \frac{1}{(A-B)|t|} + 1, \]

then \( I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_{q+1}} (f) \in C_{\lambda} \) where \( \lambda > 0. \)

**Proof.** Since the proof follows from Lemma 2.3 and by using the method of the proof of Theorem 2.3, we omit the details.

**Corollary 2.3** Let \( a_j \ (j = 1, 2, \cdots, q+1) \in C \setminus \{0\}, \) \( b_i \ (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \) \( \text{Re} b_m > |a_m| + 1 \ (m = 1, 2, \cdots, q-1), \) and \( \text{Re} b_q - 2 > \lambda > |a_q| + |a_{q+1}|. \) If \( f(z) \in R^t(A, B) \) satisfies

\[ \frac{(\text{Re} b_q - 1)\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{(\text{Re} b_q - \lambda - 2)\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)} \times \left( \prod_{m=1}^{q} \frac{\text{Re} b_m - 1}{\text{Re} b_m - |a_m| - 1} \right) \leq \frac{1}{(A-B)|t|} + 1, \]

then \( I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_{q+1}} (f) \in C_{\lambda} \) where \( \lambda > 0. \)

**Proof.** We note that

\[ \begin{align*}
q+2F_{q+1} & \left( \begin{array}{c} |a_1|, |a_2|, \ldots, |a_{q+1}|, \lambda + 1 \\
\text{Re} b_1, \text{Re} b_2, \ldots, \text{Re} b_q, \lambda \end{array} ; 1 \right) \\
& = \left( \prod_{m=1}^{q} \frac{\Gamma(\text{Re} b_m)\Gamma(\text{Re} b_m - |a_m| - 1)}{\Gamma(\text{Re} b_m - |a_m|)\Gamma(\text{Re} b_m - 1)} \right) \frac{\Gamma(\text{Re} b_q)\Gamma(\text{Re} b_q - \lambda - 2)}{\Gamma(\text{Re} b_q - 1)\Gamma(\text{Re} b_q - \lambda - 1)} \\
& \times \frac{\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)} \\
& = \left( \prod_{m=1}^{q} \frac{\text{Re} b_m - 1}{\text{Re} b_m - |a_m| - 1} \right) \frac{\text{Re} b_q - 1}{\text{Re} b_q - \lambda - 2} \frac{\Gamma(\lambda)\Gamma(\lambda - |a_q| - |a_{q+1}|)}{\Gamma(\lambda - |a_q|)\Gamma(\lambda - |a_{q+1}|)}. 
\end{align*} \]
Hence we observe that
\[
\sum_{n=2}^\infty n(n + \lambda - 1) |\frac{\Pi_{j=1}^{p}(a_j)_{n-1}}{\Pi_{i=1}^{q}(b_i)_{n-1}(1)_{n-1}}a_n| = (A - B)|t| \sum_{n=1}^\infty \frac{\lambda(\lambda + 1)_{n}}{(\lambda)_{n}} \left\{ \frac{\Pi_{j=1}^{p}(|a_j|)_{n}}{\Pi_{i=1}^{q}(\text{Re} b_i)_{n-1}(1)_{n-1}} \right\} \leq \lambda,
\]
by assumption. This completes our proof.

**Theorem 2.5** Let \( a_j \ (j = 1, 2, \cdots, p) \in C \setminus \{0\}, \ b_i \ (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \ \text{Re} b_i > 0 \ (i = 1, 2, \cdots, q), \ \text{and} \ \Sigma_{i=1}^{q} \text{Re} b_i > \Sigma_{j=1}^{p} |a_j| + 1. \ If

\[
(2.12) \quad k \frac{\Pi_{j=1}^{p} |a_j|}{\Pi_{i=1}^{q} \text{Re} b_i} pF_q\left( \frac{|a_1| + 1, |a_2| + 1, \cdots, |a_p| + 1}{\text{Re} b_1 + 1, \text{Re} b_2 + 1, \cdots, \text{Re} b_q + 1} ; 1 \right) \leq (A - B)|t| + 1,
\]

then

\[
z_{pF_q}\left( a_1, a_2, \cdots, a_p \mid b_1, b_2, \cdots, b_q ; z^k \right) \in R^t(A, B),
\]

where \( k \in N. \)

**Proof.** By Lemma 2.3, it suffices to show that

\[
(2.13) \quad T_3 := \sum_{n=2}^\infty (1 + |B|)(k(n - 1) + 1) \left| \frac{\Pi_{j=1}^{p} (a_j)_{n-1}}{\Pi_{i=1}^{q} (b_i)_{n-1}(1)_{n-1}} \right| \leq (A - B)|t|.
\]

Then we have,

\[
T_3 \leq (1 + |B|) \sum_{n=2}^\infty (kn - (k - 1)) \frac{\Pi_{j=1}^{p} (|a_j|)_{n-1}}{\Pi_{i=1}^{q} (\text{Re} b_i)_{n-1}(1)_{n-1}}
= (1 + |B|)k_{p+1}pF_{q+1}\left( |a_1|, |a_2|, \cdots, |a_p|, 2 \mid \text{Re} b_1, \text{Re} b_2, \cdots, \text{Re} b_q, 1 \right)
\]
\begin{align*}
-(1 + |B|)(k - 1)_{pq}F_q & \left( \begin{array}{c} |a_1|, |a_2|, \ldots, |a_p| \\ Reb_1, Reb_2, \ldots, Reb_q 
\end{array} ; 1 \right) - (1 + |B|) \\
= & \ (1 + |B|) k \frac{\prod_{j=1}^{p} |a_j|}{\prod_{i=1}^{q} \text{Re} b_i} \ p F_q \left( \begin{array}{c} |a_1| + 1, |a_2| + 1, \ldots, |a_p| + 1 \\ Reb_1 + 1, Reb_2 + 1, \ldots, Reb_q + 1 
\end{array} ; 1 \right) \\
+ & \ (1 + |B|)_{pq}F_q \left( \begin{array}{c} |a_1|, |a_2|, \ldots, |a_p| \\ Reb_1, Reb_2, \ldots, Reb_q 
\end{array} ; 1 \right) - (1 + |B|) \\
\leq & \ (A - B) |t| 
\end{align*}

by (2.12), which completes the proof of Theorem 2.5.

**Corollary 2.4** Let \( a_j \ (j = 1, 2, \ldots, q + 1) \in C \setminus \{0\}, \ b_i \ (i = 1, 2, \ldots, q) \in C \setminus \{0\}, \ Reb_m > |a_m| + 1 \ (m = 1, 2, \ldots, q - 1), \) and \( Reb_q > |a_q| + |a_{q+1}| + 1. \) If

\begin{equation}
(2.14)
\frac{\Gamma(Reb_q)\Gamma(Reb_q - |a_q| - |a_{q+1}| - 1)}{\Gamma(Reb_q - |a_q|)\Gamma(Reb_q - |a_{q+1}| - 1)} \left( \prod_{m=1}^{q-1} \frac{1}{Reb_m - |a_m| - 1} \right)
\times \left\{ k \prod_{j=1}^{q+1} |a_j| + \left( \prod_{m=1}^{q} \text{Re} b_m - 1 \right) (Reb_q - |a_q| - |a_{q+1}| - 1) \right\}
\leq \frac{(A - B) |t|}{1 + |B|} + 1,
\end{equation}

then

\[ z_{q+1} F_q \left( \begin{array}{c} a_1, a_2, \ldots, a_{q+1} \\ b_1, b_2, \ldots, b_q 
\end{array} ; z^k \right) \in R^4(A, B), \]

where \( k \in \mathbb{N}. \)

**Proof.** We note that

\[
\begin{align*}
q+1_{F}_q & \left( \begin{array}{c} |a_1| + 1, |a_2| + 1, \ldots, |a_{q+1}| + 1 \\ Reb_1 + 1, Reb_2 + 1, \ldots, Reb_q + 1 
\end{array} ; 1 \right) \\
= & \left( \prod_{m=1}^{q-1} \frac{\Gamma(Reb_m + 1)\Gamma(Reb_m - |a_m| - 1)}{\Gamma(Reb_m)\Gamma(Reb_m - |a_m|)} \right) \Gamma(Reb_q + 1)\Gamma(Reb_q - |a_q| - |a_{q+1}| - 1) \\
& \Gamma(Reb_q - |a_q|)\Gamma(Reb_q - |a_{q+1}|) \\
= & \left( \prod_{m=1}^{q-1} \frac{Reb_m}{Reb_m - |a_m| - 1} \right) \Gamma(Reb_q + 1)\Gamma(Reb_q - |a_q| - |a_{q+1}| - 1) \\
& \Gamma(Reb_q - |a_q|)\Gamma(Reb_q - |a_{q+1}|).
\end{align*}
\]

From above equality and (2.6), we have the result of Corollary 2.4.
3. Uniformly starlikeness and convexity

A function $f(z) \in \mathcal{A}$ is said to be uniformly starlike in $U$ if it satisfies

\[(3.1) \quad \Re \left\{ \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} \right\} \geq 0 \]

for all $(z, \zeta) \in U \times U$. We denote by $UST$ the subclass of $\mathcal{A}$ consisting of all uniformly starlike functions in $U$. Further, a function $f(z) \in \mathcal{A}$ is said to be uniformly convex in $U$ if and only if

\[(3.2) \quad \Re \left\{ 1 + (z - \zeta)\frac{f''(z)}{f'(z)} \right\} \geq 0 \]

for all $(z, \zeta) \in U \times U$. We also denote by $UCV$ the class of all such functions.

The classes $UST$ and $UCV$ were defined by Goodman [6,7] and studied recently by Rønning [13]. By the result of Rønning [13], we see that $f(z) \in UCV$ if and only if

\[(3.3) \quad \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in U). \]

In view of definitions of $UST$ and $UCV$, we define the following classes:

**Definition 3.1** A function $f(z)$ in $\mathcal{A}$ is said to be a member of the class $UST(\alpha)$ if it satisfies

\[(3.4) \quad \Re \left\{ \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} \right\} \geq \alpha \quad ((z, \zeta) \in U \times U) \]

for some real $\alpha$ ($0 \leq \alpha < 1$).

**Definition 3.2** A function $f(z)$ belonging to $\mathcal{A}$ is called as a member of the class $UCV(\alpha)$ if and only if

\[(3.5) \quad \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \alpha \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in U) \]

for some real $\alpha$ ($\alpha \geq 0$).

Note that $UST(\alpha) \subset UST$ ($0 \leq \alpha < 1$), $UCV(\alpha) \subset UCV$ ($\alpha \geq 1$) and $UCV \subset UCV(\alpha)$ ($0 \leq \alpha < 1$). Now, we derive the following lemmas for functions $f(z) \in \mathcal{A}$ to be in the classes $UST(\alpha)$ and $UCV(\alpha)$.

**Lemma 3.1** If $f(z) \in \mathcal{A}$ satisfies $\sum_{n=2}^{\infty} n(n(\alpha+1)-\alpha)|a_n| \leq 1$, then $f(z)$ is in $UCV(\alpha)$.
Proof. It suffices to show that

\begin{equation}
\alpha \left| \frac{zf''(z)}{f'(z)} \right| - \Re \left( \frac{zf''(z)}{f'(z)} \right) \leq 1.
\end{equation}

We have

\begin{align*}
\alpha \left| \frac{zf''(z)}{f'(z)} \right| &- \Re \left( \frac{zf''(z)}{f'(z)} \right) \\
&\leq (\alpha + 1) \left| \frac{zf''(z)}{f'(z)} \right| \\
&= \left| \frac{(\alpha + 1) \sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}} \right| \\
&\leq \frac{\sum_{n=2}^{\infty} (\alpha + 1)n(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|}.
\end{align*}

Now this last expression is bounded above by 1 provided that \( \sum_{n=2}^{\infty} n(n+1)(1-\alpha)|a_n| \leq 1. \)

Lemma 3.2 If \( f(z) \in A \) satisfies \( \sum_{n=2}^{\infty} (3-\alpha)n-2|a_n| \leq 1-\alpha, \) then \( f(z) \) is in \( UST(\alpha). \)

Proof. It suffices to show that

\begin{equation}
\left| \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} - 1 \right| \leq 1 - \alpha.
\end{equation}

We have

\begin{align*}
\left| \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} - 1 \right| &= \left| \frac{\sum_{n=2}^{\infty} a_n (z^{n-1} + z^{n-2} \zeta + \cdots + \zeta^{n-1}) - \sum_{n=2}^{\infty} n a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}} \right| \\
&\leq \frac{\sum_{n=2}^{\infty} 2(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|},
\end{align*}

which is bounded above by \( 1 - \alpha \) if \( \sum_{n=2}^{\infty} ((3-\alpha)n-2)|a_n| \leq 1 - \alpha. \)

Theorem 3.1 Let \( a_j (j = 1, 2, \cdots, p) \in C \setminus \{0\}, b_i (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \) \( \Re b_i > 0 (i = 1, 2, \cdots, q), \) and \( \sum_{i=1}^{q} \Re b_i > \sum_{j=1}^{p} |a_j| + 1. \) If \( f(z) \in R^t(A, B) \) satisfies

\begin{equation}
(1 + \alpha) \frac{\prod_{j=1}^{p} |a_j|}{\prod_{i=1}^{q} \Re b_i} P_F q \left( |a_1| + 1, |a_2| + 1, \cdots, |a_p| + 1 \right) \\
+ p F_q \left( |a_1|, |a_2|, \cdots, |a_p| \right) \leq \frac{1}{(A-B)|t|} + 1,
\end{equation}

then \( f_{b_1, b_2, \cdots, b_q}(f) \in UCV(\alpha). \)
Proof. By Lemma 3.1, we need only to show that

\[(3.9) \quad S_1 := \sum_{n=2}^{\infty} n(n(\alpha + 1) - \alpha) \left| \frac{\prod_{j=1}^{p} (a_j)_{n-1}}{\prod_{i=1}^{q} (b_i)_{n-1}} a_n \right| \leq 1.\]

From Lemma 2.2, we have,

\[
S_1 \leq (A - B)|t| \sum_{n=2}^{\infty} (n(\alpha + 1) - \alpha) \frac{\prod_{j=1}^{p} (|a_j|)_{n-1}}{\prod_{i=1}^{q} (\Re b_i)_{n-1}} \infty \\
\leq (A - B)|t| (\alpha + 1)_{\mathrm{P} + 1} F_{q+1} + (A - B)|t| \leq 1 \text{ by (3.8), which completes the proof of Theorem 3.1.}
\]

Corollary 3.1 Let \(a_j (j = 1, 2, \ldots, q + 1) \in C \setminus \{0\}, b_i (i = 1, 2, \ldots, q) \in C \setminus \{0\}, \Re b_m > |a_m| + 1 (m = 1, 2, \ldots, q - 1), \text{ and } \Re b_q > |a_q| + |a_{q+1}| + 1. \) If \(f(z) \in R_1(A, B)\) satisfies

\[(3.10) \quad \frac{\Gamma(\Re b_q) \Gamma(\Re b_q - |a_q| - |a_{q+1}| - 1)}{\Gamma(\Re b_q - |a_q|) \Gamma(\Re b_q - |a_{q+1}|)} \left( \prod_{m=1}^{q-1} \frac{\Re b_m - |a_m| - 1}{\Re b_m - |a_m| - 1} \right) \times \left\{(\alpha + 1) \prod_{j=1}^{q+1} |a_j| + \left( \prod_{m=1}^{q-1} \Re b_m - 1 \right)(\Re b_q - |a_q| - |a_{q+1}| - 1) \right\} \leq \frac{1}{(A - B)|t|} + 1,
\]

then \(f_{b_1, b_2, \ldots, b_q}^{a_1, a_2, \ldots, a_{q+1}}(f) \in UCV(\alpha).\)

Proof. Since the proof is similarly the proof of Corollary 2.4, we omit the details.
Theorem 3.2 Let \(a_j (j = 1, 2, \cdots, p) \in C \setminus \{0\}, \) \(b_i (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \) \(\text{Re} b_i > 0 (i = 1, 2, \cdots, q), \) and \(\Sigma_{j=1}^{p} \text{Re} b_j > \Sigma_{j=1}^{p} |a_j|. \) If \(f(z) \in R^t(A, B)\) satisfies

\[
(3.11) \quad (3 - \alpha) \mathcal{F}_q^{p} \left( \begin{array}{c} |a_1|, |a_2|, \cdots, |a_p| \\ \text{Re} b_1, \text{Re} b_2, \cdots, \text{Re} b_q & ; 1 \end{array} \right) - 2 (3 - \alpha) \mathcal{F}_q^{p+1} \left( \begin{array}{c} |a_1|, |a_2|, \cdots, |a_p|, 1 \\ \text{Re} b_1, \text{Re} b_2, \cdots, \text{Re} b_{q+1} & ; 1 \end{array} \right) \leq (1 - \alpha) \left( \frac{1}{(A - B)|t|} + 1 \right),
\]

then \(I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_p}(f) \in UST(\alpha), \) for some \(\alpha (0 \leq \alpha < 1). \)

Proof. By Lemma 3.2, we need only to show that

\[
S_2 := \sum_{n=2}^{\infty} \left( (3 - \alpha)n - 2 \right) \frac{\prod_{j=1}^{p} |a_j|^{n-1}}{\prod_{i=1}^{q} \text{Re} b_i^{n-1}(1)_{n-1}} a_n \leq 1 - \alpha.
\]

From Lemma 2.2, we have,

\[
S_2 \leq (A - B)|t|(3 - \alpha) \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p} |a_j|^{n-1}}{\prod_{i=1}^{q} \text{Re} b_i^{n-1}(1)_{n-1}} a_n \leq 1 - \alpha
\]

by (3.11), which completes the proof of Theorem 3.2.

Corollary 3.2 Let \(a_j (j = 1, 2, \cdots, q + 1) \in C \setminus \{0\}, \) \(b_i (i = 1, 2, \cdots, q) \in C \setminus \{0\}, \) \(\text{Re} b_m > |a_m| + 1 (m = 1, 2, \cdots, q - 1), \) and \(\text{Re} b_q > |a_q| + 1, |a_{q+1}| < 1. \) If \(f(z) \in R^t(A, B)\) satisfies

\[
(3.13) \quad \left\{ (3 - \alpha) \frac{\Gamma(\text{Re} b_q - 1)\Gamma(\text{Re} b_q - |a_q| - |a_{q+1}|)}{\Gamma(\text{Re} b_q - |a_q| - 1)\Gamma(\text{Re} b_q - |a_{q+1}|)} \right\} \left( \prod_{m=1}^{q} \frac{\text{Re} b_m - 1}{\text{Re} b_m - |a_m| - 1} \right) \leq (1 - \alpha) \left( \frac{1}{(A - B)|t|} + 1 \right),
\]

then \(I_{b_1, b_2, \cdots, b_q}^{a_1, a_2, \cdots, a_p}(f) \in UST(\alpha), \) for some \(\alpha (0 \leq \alpha < 1). \)
Proof. We note that

\[
q+2F_{q+1}
\begin{bmatrix}
|a_1|, |a_2|, \ldots, |a_{q+1}|, 1 \\
\Re b_1, \Re b_2, \ldots, \Re b_q, 2
\end{bmatrix}
= \left( \prod_{m=1}^{q-1} \frac{\Re b_m - 1}{\Re b_m - |a_m| - 1} \right) \frac{\Re b_q - 1}{(\Re b_q - |a_q| - 1)(1 - |a_{q+1}|)}.
\]

From above equality and (2.6), we have the result of Corollary 3.2.

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