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ON MEROMORPHIC CONVEX AND STARLIKE FUNCTIONS

By

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Abstract

Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in \(|z| < 1\) and

\[
1 + \frac{z f''(z)}{f'(z)} > 0 \quad \text{in} \quad |z| < 1.
\]

Then it is well known that [1, 3]

\[
\frac{zf'(z)}{f(z)} > \frac{1}{2} \quad \text{in} \quad |z| < 1.
\]

Corresponding the above theorem, if \( f(z) = 1/z + \sum_{n=2}^{\infty} a_n z^n \) is analytic in the punctured disk \( 0 < |z| < 1 \) and

\[
\text{Re}[-(1 + \frac{zf''(z)}{f'(z)})] > 0 \quad \text{in} \quad |z| < 1,
\]

then there exists no positive constant \( \alpha > 0 \) for which

\[
\text{Re}[-\frac{zf'(z)}{f(z)}] > \alpha \quad \text{in} \quad |z| < 1.
\]

1. Introduction.

Let \( \Sigma \) denote the class of function of the form

\[
f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n
\]

which are regular in the punctured disk \( \mathcal{B} = \{z : 0 < |z| < 1\} \).

A function \( f(z) \in \Sigma \) is called meromorphic starlike of order \( \alpha \) \((0 \leq \alpha < 1)\) if

\[
\text{Re}[-\frac{zf'(z)}{f(z)}] > \alpha
\]

for all \( z \in \mathcal{U} = \{z : |z| < 1\} \).

We denote by \( \Sigma^*(\alpha) \) the subclass of \( \Sigma \) consisting of functions which are meromorphic starlike of order \( \alpha \) in \( \mathcal{U} \).

Further, a function \( f \in \Sigma \) is called meromorphic convex of order \( \alpha \) \((0 \leq \alpha < 1)\) if

\[
\text{Re}[-(1 + \frac{zf''(z)}{f'(z)})] > \alpha
\]

for all \( z \in \mathcal{U} \).
We denote by $\Sigma_{c}(\alpha)$ the subclass of $\Sigma$ consisting of functions which are meromorphic convex of order $\alpha$ in $U$.

2. Preliminaries.

**Lemma 1.** [1, Theorem 1] Let $p(z)$ be regular in $U$, $p(0) = 1$ and suppose that there exists a point $z_0 \in U$ such that

$$\Re p(z) > 0 \quad \text{for} \quad |z| < |z_0|,$$

$$\Re p(z_0) = 0 \quad \text{and} \quad p(z_0) \neq 0.$$  

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i k$$

where $k$ is real and $|k| \geq 1$.

**Lemma 2.** Let $p(z)$ be regular in $U$, $p(0) = 1$ and

\[(1) \quad \Re (p(z) - \frac{zp'(z)}{p(z)}) > 0 \quad (z \in U),\]

then

$$\Re p(z) > 0 \quad (z \in U).$$

Then this result is sharp for the function $p(z) = \frac{1+z}{1-z}$.

**Proof.** First, we want to prove $p(z) \neq 0 \ (z \in U)$.

If $p(z)$ has a zero of order $n (n \geq 1)$ at a point $z_0 (z_0 \neq 0)$, then $p(z)$ can be written as $p(z) = (z - z_0)^n g(z)$ ($g(z_0) \neq 0$, $g(z)$ is regular in $U$), and it follows that

$$p(z) - \frac{zp'(z)}{p(z)} = (z - z_0)^n g(z) - \frac{n z}{z - z_0} - \frac{z g'(z)}{g(z)}.$$

When $z$ approaches to $z_0$ on the line segment satisfying the conditions $\arg z = \arg z_0 = \theta$ and $|z_0| < |z| < 1$, we have

\[
\lim_{z \to z_0 \atop \arg z = \arg z_0, |z| < 1} \Re (p(z) - \frac{zp'(z)}{p(z)}) = \lim_{z \to z_0 \atop \arg z = \arg z_0, |z| < 1} \Re ((z - z_0)^n g(z) - \frac{n z}{z - z_0} - \frac{z g'(z)}{g(z)}) = \text{negative infinite real value},
\]
because we have

\[
\lim_{z \to z_0} \arg(-\frac{nz}{z-z_0}) = \lim_{z \to z_0} \arg(-1 + \arg nz - \arg(z - z_0)) = \pi + \theta - \theta = \pi.
\]

This result contradicts (1).

Therefore we have

\[ p(z) \neq 0 \quad (z \in U). \]

If there exists a point \( z_0 \in U \) such that

\[
\Re p(z_0) > 0 \quad \text{for} \quad |z| < |z_0|,
\]

\[
\Re p(z_0) = 0 \quad \text{and} \quad p(z_0) \neq 0,
\]

then from Lemma 1, we have

\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik
\]

where \( k \) is real and \( |k| \geq 1 \).

For the case \( p(z_0) = ia \) (\( a > 0 \)), we have

\[
\Re \left( \frac{z_0 p'(z_0)}{p(z_0)} - p(z_0) \right) = \Re (ik - ia) = 0.
\]

This contradicts our assumption.

For the case \( p(z_0) = -ia \) (\( a > 0 \)), applying the same method as the above, we have

\[
\Re \left( \frac{z_0 p'(z_0)}{p(z_0)} - p(z_0) \right) = 0.
\]

This contradicts our assumption.

Therefore we complete our proof.

The result is sharp for the function \( p(z) = \frac{1+z}{1-z} \).

3. Main result.

**Theorem.** If \( f(z) \in \Sigma_c(0) \), then \( f(z) \in \Sigma^*(0) \), and there exists no positive constant \( \alpha > 0 \) such that \( \Sigma_c(0) \subset \Sigma^*(\alpha) \).

**Proof.** Setting

\[
p(z) = -\frac{zf''(z)}{f'(z)},
\]

then we have \( p(0) = 1 \) and

\[
-(1 + \frac{zf''(z)}{f'(z)}) = p(z) - \frac{zp'(z)}{p(z)}.
\]
From the assumption of theorem, we have
\[ \text{Re}[-(1 + \frac{zf''(z)}{f'(z)}))] = \text{Re}[p(z) - \frac{zp'(z)}{p(z)}] > 0 \quad \text{in } U, \]
then from Lemma 2, we have
\[ \text{Re}(\frac{zf'(z)}{f(z)}) = \text{Re}(p(z)) > 0 \quad \text{in } U. \]

Next, we prove that there exists no positive constant \( \alpha > 0 \) such that \( \Sigma_c(0) \subset \Sigma^*(\alpha) \). Because the extremal function of Lemma 2 is
\[ p(z) = \frac{1+z}{1-z}, \]
so we put
\[ -\frac{zf'(z)}{f(z)} = \frac{1+z}{1-z}. \]
Then by a brief calculation, we have
\[ \frac{f'(z)}{f(z)} = -\frac{1}{z} - \frac{2}{1-z}. \]
Adding \( 1/z \) to both sides and integrating from zero to \( z \) (\( 0 < |z| < 1 \)), we have
\[ \int_0^z \left( \frac{1}{z} + \frac{f'(z)}{f(z)} \right) dz = -\int_0^z \frac{2}{1-z} dz, \]
and it follows that
\[ f(z) = \frac{(1-z)^2}{z}. \]
This function belong to \( \Sigma_c(0) \) and \( \Sigma^*(0) \) but there exists no positive constant \( \alpha > 0 \) for which \( f(z) \in \Sigma^*(\alpha) \).

References

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