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| 部門       | 部門名
| 型式       | 型式名
| リンク      | リンク
| 状態       | 状態
| 文章版     | 発行元
Sufficient Condition for Multivalently Starlikeness

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Abstract

It is the purpose of the present paper to obtain a sufficient condition for multivalently starlikeness.

1 Introduction.

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N} = 1, 2, 3, \ldots)$$

which are analytic in $U = \{z : |z| < 1\}$.

A function $f(z) \in A(p)$ is said to be $p$-valently starlike if and only if

$$\text{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in} \quad U.$$ 

In [4], R. Singh and S. Singh proved the following result.

Theorem A. If $f(z) \in A(1)$ satisfies

$$1 + \text{Re} \frac{zf''(z)}{f'(z)} < \frac{3}{2} \quad \text{in} \quad U,$$

then $f(z)$ is starlike in $U$ or

$$\text{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in} \quad U.$$

**Theorem B.** If \( f(z) \in A(p) \) satisfies

\[
1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \frac{p + \frac{1}{2}}{2} \quad \text{in} \quad U,
\]

then \( f(z) \) is \( p \)-valently starlike in \( U \) and

\[
0 < \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \frac{2p(p + 1)}{2p + 1} \quad \text{in} \quad U.
\]

**2 Main result.**

**Theorem.** Let \( f(z) \in A(p) \) and suppose that

\[
(1) \quad 1 + \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \alpha \quad \text{in} \quad U
\]

where \( p < \alpha < p + \frac{1}{2} \).

Then we have

\[
\text{Re} \left( \frac{f(z)}{zf'(z)} \right) > \frac{2\alpha}{p(2\alpha + 1)} \quad \text{in} \quad U
\]

and

\[
0 < \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \frac{p(2\alpha + 1)}{2\alpha} \quad \text{in} \quad U.
\]

**Proof.** Let us put

\[
(2) \quad p(z) = p \left( \frac{1 + \beta}{q(z) + \beta} \right)
\]

where \( p(z) \) is analytic in \( U \), \( p(0) = p \), \( \beta = 2\alpha \), and \( 2p < \beta < 2p + 1 \), then we have \( q(0) = 1 \).

Then it follows that

\[
p(z) + \frac{zp'(z)}{p(z)} = p \left( \frac{1 + \beta}{q(z) + \beta} \right) - \frac{zp'(z)}{q(z) + \beta}.
\]
If there exists a point \( z_0 \in U \) such that

\[(3) \quad \text{Re}(q(z)) > 0 \quad \text{for} \quad |z| < |z_0|, \quad \text{Re}(z_0) = 0 \quad \text{and} \quad q(z_0) = ia,\]

then from [2, p.152], we have

\[-z_0q'(z_0) \geq \frac{1}{2}(1 + a^2).\]

Therefore we have

\[
\text{Re}(p(z_0) + \frac{z_0p'(z_0)}{p(z_0)}) = \text{Re}(p \frac{\beta + 1}{ia + \beta} - \frac{z_0q'(z_0)}{ia + \beta}) \\
\geq p \frac{\beta(1 + \beta)}{\beta^2 + a^2} + \frac{1}{2}(1 + a^2) \frac{\beta}{\beta^2 + a^2} \\
= \frac{\beta}{2(\beta^2 + a^2)} \left\{ 2(1 + \beta) + (1 + a^2) \right\}.
\]

Putting

\[
g(x) = \frac{\beta}{2(\beta^2 + x^2)} \left\{ 2(1 + \beta) + (1 + x^2) \right\} \quad \text{for} \quad -\infty < x < \infty,
\]

then it follows that

\[
g'(x) = \frac{\beta x}{(\beta^2 + x^2)^2} (\beta^2 - 2\beta p - 2p - 1).
\]

This shows that \( g(x) \) takes its minimum at \( x = \infty \) and \( x = -\infty \) and so

\[(4) \quad g(x) \geq \frac{\beta}{2} = \alpha \quad \text{for} \quad -\infty < x < \infty.\]

On the other hand, putting

\[
p(z) = \frac{zf''(z)}{f(z)},
\]

then it follows that

\[(5) \quad p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}.\]

From (2), (3), (4) and (5), this contradicts (1).

Therefore we must have

\[\text{Re}(q(z)) > 0 \quad \text{in} \quad U.\]
Then we easily have

\[ \Re \frac{f(z)}{zf'(z)} > \frac{2\alpha}{p(2\alpha + 1)} \quad \text{in} \quad U \]

and

\[ 0 < \Re \frac{zf'(z)}{f(z)} < \frac{p(2\alpha + 1)}{2\alpha} \quad \text{in} \quad U \]

where \( p < \alpha < p + \frac{1}{2} \).

**Remark.** Putting \( \alpha = p + \frac{1}{2} \) in the Theorem, we have Theorem B.

**References**


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