On meromorphic $\alpha$-starlike functions

by

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Abstract

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E = \{z : |z| < 1\}$, let for a real number $\alpha$

\[
\Re \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f(z)} \right) \right] > 0 \quad \text{in } E.
\]

Then it is well known that [1, 2]

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in } E.
\]

Corresponding to this, we take the analytic function $f(z) = 1/z + \sum_{n=0}^{\infty} a_n z^n$ in the punctured disk $U = \{z : 0 < |z| < 1\}$ satisfying

\[
\Re \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f(z)} \right) \right] < 0 \quad \text{in } E.
\]

Then we prove

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} < 0 \quad \text{in } E.
\]

1. Introduction.

Let $\Sigma$ denote the class of function of the form

\[ f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \]

which are analytic in the punctured disk $U = \{z : 0 < |z| < 1\}$.

A function $f(z)$ belonging to the class is said to be meromorphic starlike of order $\alpha$ ($0 \leq \alpha < 1$) in $E = \{z : |z| < 1\}$ if and only if

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} < -\alpha
\]

for all $z \in E$. We denote by $\Sigma^*(\alpha)$ the class of all functions in $\Sigma$ which are meromorphic starlike of order $\alpha$ in $U$. We note also that

\[ \Sigma^*(\alpha) \subseteq \Sigma^*(0) \equiv \Sigma^* \quad (0 \leq \alpha < 1), \]

where $\Sigma^*$ denote the subclass of $A$ consisting of functions which are meromorphic starlike in $U$. The meromorphic starlike is meant that the complement of $f(E)$ is starlike with respect to the origin.
Definition 1. Let $\alpha$ be a real number and suppose that $f(z) \in \Sigma$ with $f(z)f'(z) \neq 0$ in $U$. If $f(z)$ satisfies the condition
\[ \text{Re} \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < 0 \quad \text{in} \quad E, \]
then $f(z)$ is said to be a meromorphic $\alpha$-starlike function.

2. Preliminary Results.

Lemma 1. Let $p(z)$ be analytic in $E$, $p(0) = 1$ and suppose that there exists a point $z_0 \in E$ such that
\[ \text{Re} \{p(z)\} > 0 \quad \text{for} \quad |z| < |z_0|, \]
\[ \text{Re} \{p(z_0)\} = 0 \quad \text{and} \quad p(z_0) = ia \quad (a \neq 0). \]
Then we have
\[ \frac{z_0p'(z_0)}{p(z_0)} = ik, \]
where
\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \geq 1 \quad \text{when} \quad a > 0 \]
and
\[ k \leq \frac{1}{2} \left( a + \frac{1}{a} \right) \leq -1 \quad \text{when} \quad a < 0. \]

We owe this lemma to [3, Theorem 1].

Lemma 2. Let $\alpha, \beta$ be positive real number ($\alpha > 1, 0 < \beta < 1$) and $p(z)$ be analytic in $E$, $p(0) = 1$, $p(z) \neq \beta$ in $E$, and suppose that

(i) for the case $0 < \beta \leq 1/2$
\[ \text{Re} \left( \alpha \frac{zp'(z)}{p(z)} - p(z) \right) > -\frac{\alpha \beta}{2(1 - \beta)} - \beta \quad \text{in} \quad E, \]
where $\alpha > 2(1 - \beta)^2/\beta$;

(ii) for the case $1/2 < \beta < 1$
\[ \text{Re} \left( \alpha \frac{zp'(z)}{p(z)} - p(z) \right) > -\frac{\alpha(1 - \beta)}{2\beta} - \beta \quad \text{in} \quad E, \]
where $\alpha > 2\beta$.

Then we have
\[ \text{Re} \{p(z)\} > \beta \quad \text{in} \quad E. \]
Proof. If we put

$$q(z) = \frac{1 - \beta}{p(z) - \beta},$$

then $q(z)$ is analytic in $E$, $q(0) = 1$ and $q(z) \neq 0$ in $E$.

At first, we want to prove $\Re \{p(z)\} > \beta$ in $E$, i.e., $\Re \{q(z)\} > 0$ in $E$. If there exists a point $z_0 \in E$ such that

$$\Re \{q(z)\} > 0 \quad \text{for} \quad |z| < |z_0| < 1,$$

$$\Re \{q(z_0)\} = 0 \quad \text{and} \quad q(z_0) = ia \ (a \neq 0),$$

then from Lemma 1, we have

$$\Re \left( \alpha \frac{z_0 p'(z_0)}{p(z_0)} - p(z_0) \right) = \Re \left( -\alpha \frac{1 - \beta - ik}{1 - \beta + \beta ia} \right)$$

$$= -\frac{\alpha \beta ka (1 - \beta)}{(1 - \beta)^2 + \beta^2 a^2} - \beta$$

$$\leq -\frac{\alpha \beta (1 - \beta)}{2} \frac{1 + a^2}{(1 - \beta)^2 + a^2 \beta^2} - \beta$$

by virtue of (1), (2). Let us put

$$\varphi(x) = \frac{1 + x^2}{(1 - \beta)^2 + x^2 \beta^2}$$

and simple calculation leads to

$$\varphi'(x) = \frac{2x(1 - 2\beta)}{((1 - \beta)^2 + x^2 \beta^2)^2}.$$

For the case $0 < \beta \leq 1/2$, $\varphi(x)$ takes its minimum value at $x = 0$

$$\varphi(0) = \frac{1}{(1 - \beta)^2}.$$

Therefore we have

$$\Re \left( \alpha \frac{z_0 p'(z_0)}{p(z_0)} - p(z_0) \right) \leq -\frac{\alpha \beta}{2(1 - \beta)} - \beta.$$

Next, if $1/2 < \beta < 1$, $\varphi(x)$ takes its minimum at $x = \infty$

$$\lim_{x \to \infty} \varphi(x) = \lim_{x \to \infty} \frac{1 + x^2}{(1 - \beta)^2 + x^2 \beta^2} = \frac{1}{\beta^2},$$

and we have

$$\Re \left( \alpha \frac{z_0 p'(z_0)}{p(z_0)} - p(z_0) \right) \leq -\frac{\alpha (1 - \beta)}{2 \beta} - \beta.$$

This contradicts the assumption of Lemma 2. Therefore we have $\Re \{q(z)\} > 0$ in $E$ and then

$$\Re \{p(z)\} > \beta \quad \text{in} \quad E.$$
This completes our proof.

3. Main Results.

Theorem 1. Let $f(z)$ be a meromorphic $\alpha$-starlike function, and suppose that

\[
\text{Re} \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < 0 \quad \text{in} \ E,
\]

where $\alpha$ is a real number. Then we have

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < 0 \quad \text{in} \ E.
\]

Proof. Let us put

\[
p(z) = -\frac{zf'(z)}{f(z)}.
\]

By simple calculation, we obtain

\[
\frac{zp'(z)}{p(z)} - p(z) = 1 + \frac{zf''(z)}{f'(z)},
\]

or

\[
\text{Re} \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] = \text{Re} \left[ \alpha \frac{zp'(z)}{p(z)} - p(z) \right].
\]

At first, we want to prove $\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < 0$ in $E$, which means $\text{Re} \{ p(z) \} > 0$ in $E$. If there exists a point $z_0 \in E$ such that

\[
\text{Re} \{ p(z) \} > 0 \quad \text{for} \quad |z| < |z_0|,
\]

\[
\text{Re} \{ p(z_0) \} = 0 \quad \text{and} \quad p(z_0) = ia \quad (a \neq 0),
\]

then from Lemma 1 we have

\[
\frac{z_0p'(z_0)}{p(z_0)} = ik,
\]

where $k$ is real and $|k| \geq 1$. Thus

\[
\text{Re} \left[ \alpha \frac{z_0p'(z_0)}{p(z_0)} - p(z_0) \right] = \text{Re} [aik - ia] = 0.
\]

This contradicts the assumption of the theorem. Therefore we have

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < 0 \quad \text{in} \ E.
\]

This completes our proof.
Theorem 2. Let \( \alpha, \beta \) be positive real number \((\alpha > 1, 0 < \beta < 1)\), \( f(z) \) be a meromorphic \( \alpha \)-starlike function and suppose that

(i) for the case \( 0 < \beta \leq 1/2 \)

\[
\text{Re} \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] > -\frac{\alpha \beta}{2(1 - \beta)} - \beta \quad \text{in } E,
\]
where \( \alpha > 2(\beta - 1)^2/\beta \);

(ii) for the case \( 1/2 < \beta < 1 \)

\[
\text{Re} \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] > -\frac{\alpha(1 - \beta)}{2\beta} - \beta \quad \text{in } E,
\]
where \( \alpha > 2\beta \).

Then we have

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < -\beta \quad \text{in } E.
\]

Proof. Applying (4), (5) and (6), we can easily prove the theorem. Therefore from the assumption of the theorem and Lemma 2, we have

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} = \text{Re} \{ -p(z) \} < -\beta \quad \text{in } E.
\]

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References


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