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STARLIKE AND CONVEX FUNCTION OF COMPLEX ORDER INVOLVING A CERTAIN FRACTIONAL INTEGRAL OPERATOR

Abstract

Let the classes $S_0^*(b)$, $K_0(b)$ and $C_0(b)$ consist of functions which are starlike, convex and close-to-convex of complex order $b$ introduced by Nasr and Aouf [2], [3]. The main object of the present paper is to investigate the starlike and convex functions of complex order involving a certain fractional integral operator. Further relevant connections are also pointed out with various earlier results involving the Hadamard product.

Key words: fractional integral, Hadamard product, starlike and convex functions of complex order

AMS Subject Classification: 30C45
1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form:

\[(1.1)\quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n\]

which are analytic in the unit disk $\mathcal{U} = \{ z : |z| < 1 \}$. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be starlike of complex order $b$ ($b \in \mathbb{C} \setminus \{0\}$) if and only if $z^{-1} f(z) \neq 0 (z \in \mathcal{U})$ and

\[(1.2)\quad \text{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \quad (z \in \mathcal{U}).\]

We denote by $\mathcal{S}_0^*(b)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of complex order $b$. Further, let $\mathcal{S}_1^*(b)$ denote the class of functions $f(z) \in \mathcal{A}$ satisfying

\[(1.3)\quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < |b| \quad (b \in \mathbb{C} \setminus \{0\}).\]

Here the inequality (1.2) is equivalent to

\[(1.4)\quad \text{Re} \left\{ \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > -1.\]

If $f(z) \in S_1^*(b)$, then $f(z)$ satisfies (1.4) which implies that

\[(1.7)\quad \text{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0.\]

Thus $\mathcal{S}_1^*(b)$ is a subclass of $\mathcal{S}_0^*(b)$.

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be convex of complex order $b$ ($b \in \mathbb{C} \setminus \{0\}$) if and only if $f'(z) \neq 0 (z \in \mathcal{U})$ and

\[(1.5)\quad \text{Re} \left\{ 1 + \frac{1}{b} \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathcal{U}).\]

We denote by $\mathcal{K}_0(b)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of complex order $b$. Furthermore, let $\mathcal{K}_1(b)$ denote the class of functions $f(z) \in \mathcal{A}$ satisfying

\[(1.6)\quad \left| \frac{zf''(z)}{f'(z)} \right| < |b| \quad (b \in \mathbb{C} \setminus \{0\}).\]

We note that

\[(1.7)\quad f(z) \in \mathcal{K}_0(b) \iff zf'(z) \in \mathcal{S}_0^*(b)\]
and

\[(1.8) \quad f(z) \in \mathcal{K}_1(b) \iff zf'(z) \in S_1^*(b) \]

for \( b \in \mathbb{C} \setminus \{0\} \).

A function \( f(z) \) belonging to the class \( \mathcal{A} \) is said to be close-to-convex of complex order \( b \) (\( b \in \mathbb{C} \setminus \{0\} \)) if and only if there exists a function \( g(z) \in \mathcal{K}_0(c) \) (\( c \in \mathbb{C} \setminus \{0\} \)) satisfying the condition

\[(1.9) \quad \Re \left\{ 1 + \frac{1}{b} \left( \frac{f'(z)}{g'(z)} - 1 \right) \right\} > 0 \quad (z \in \mathcal{U}).\]

We denote by \( \mathcal{C}_0(b) \) the subclass of \( \mathcal{A} \) consisting of functions which are close-to-convex of complex order \( b \). Also let \( \mathcal{C}_1(b) \) denote the class of functions \( f(z) \in \mathcal{A} \) satisfying

\[(1.10) \quad \left| \frac{f'(z)}{g'(z)} - 1 \right| < |b| \]

for some \( g \in \mathcal{K}_0(c) \) (\( c \in \mathbb{C} \setminus \{0\} \)).

We also have \( \mathcal{K}_1(b) \subset \mathcal{K}_0(b) \) and \( \mathcal{C}_1(b) \subset \mathcal{C}_0(b) \).

**Remark.** Setting \( b = 1 - \alpha \) (\( 0 \leq \alpha < 1 \)), we observe that \( S_0^*(1 - \alpha) = S^*(\alpha) \), \( \mathcal{K}_0(1 - \alpha) = \mathcal{K}(\alpha) \) and \( \mathcal{C}_0(1 - \alpha) = \mathcal{C}(\alpha) \), where \( S^*(\alpha) \), \( \mathcal{K}(\alpha) \) and \( \mathcal{C}(\alpha) \) denote the usual classes of starlike, convex and close-to-convex of real order \( \alpha \), respectively. Indeed, letting \( b = i\alpha \) (\( \alpha \in \mathbb{R} \)), we obtain that \( f \in S_0^*(i\alpha) \) implies that \( \Im(zf'(z)/f(z)) > -\alpha \).

For the functions \( f_j(z) \) (\( j = 1, 2 \)) defined by

\[(1.11) \quad f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1} z^{n+1} \quad (a_{1,1} = a_{2,1} = 1),\]

let \((f_1 * f_2)(z)\) denote the Hadamard product or convolution of \( f_1(z) \) and \( f_2(z) \), defined by

\[(1.12) \quad (f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{1,n+1} a_{2,n+1} z^{n+1}.\]

Let \( a, b, c \) be complex numbers with \( c \neq 0, -1, -2, \cdots \). The Gaussian hypergeometric function \( _2F_1(z) \) is defined by

\[(1.13) \quad _2F_1(z) \equiv _2F_1(a, b; c; z) \equiv \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},\]
where \((\lambda)_n\) denotes the Pochhammer symbol defined, in terms of \(\Gamma\)-function, by
\[
(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 
1 & (n = 0) \\
\lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\}) 
\end{cases}.
\]

Many essentially equivalent definitions of fractional calculus have been given in the literature (cf., e.g, [9], [10,p.45]). For convenience, we recall here the following definitions due to Owa [4] and Saigo [8] which have been used rather frequently in the theory of analytic functions:

**Definition 1.** The fractional integral of order \(\lambda\) (\(\lambda \in \mathbb{C}\)) is defined, for a function \(f(z)\), by
\[
D_{z}^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)} \int_{0}^{z} \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} \, d\zeta \quad (\text{Re}(\lambda) > 0),
\]
where \(f(z)\) is an analytic function in a simply-connected region of the \(z\)-plane containing the origin, and the multiplicity of \((z - \zeta)^{\lambda-1}\) is removed by requiring \(\log(z - \zeta)\) to be real for \(z - \zeta > 0\).

**Definition 2.** For \(\alpha, \beta, \eta \in \mathbb{C}\) and \(\text{Re}(\alpha) > 0\), the fractional integral operator \(I_{0,z}^{\alpha,\beta,\eta}\) is defined by
\[
I_{0,z}^{\alpha,\beta,\eta}f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{0}^{z} (z-\zeta)^{\alpha-1} _{2}F_{1}(\alpha + \beta, -\eta; \alpha; 1 - \frac{\zeta}{z}) f(\zeta) \, d\zeta,
\]
where the function \(_{2}F_{1}\) is Gauss's hypergeometric function defined by (1.13).

The definition (1.15) is an interesting extension of both the Riemann-Liouville and Erdélyi-Kober fractional operators in terms of Gauss's hypergeometric functions. Indeed, in its special case, it is treated alike the definition (1.14).

It is easy to observe that
\[
I_{0,z}^{\alpha,-\alpha,\eta}f(z) = D_{z}^{-\alpha}f(z) \quad (\text{Re}(\alpha) > 0).
\]

By using the fractional integral, we now introduce the linear operator \(\Omega^{\lambda}\) given by
\[
\Omega^{\lambda}f(z) = \Gamma(2 - \lambda)z^{\lambda}D_{z}^{\lambda}f(z) \quad (\text{Re}(\lambda) < 0)
\]
for \(f(z) \in A\).
The operator $T_{0,\alpha,\beta,\eta}^{\alpha,\beta,\eta}$ is also modified by defining $J_{0,\alpha,\beta,\eta}^{\alpha,\beta,\eta}$ in the form

$$J_{0,\alpha,\beta,\eta}^{\alpha,\beta,\eta} f(z) = \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} z^\beta T_{0,\alpha,\beta,\eta}^{\alpha,\beta,\eta} f(z)$$

for $f(z) \in A$ and $\min\{\text{Re}(\alpha+\eta), \text{Re}(-\beta+\eta), \text{Re}(-\beta)\} > -2$.

2. Main results

In order to prove our main results, we shall require the following lemmas to be used in the sequel.

**Lemma 1.** (Jack [1]) Let $\omega(z)$ be analytic in $\mathcal{U}$ with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r (r < 1)$ at a point $z_0$, we can write

$$z_0 \omega'(z_0) = k \omega(z_0),$$

where $k$ is real and $k \geq 1$.

**Lemma 2.** (Ruscheweyh and Sheil-Small [7]) Let $\phi(z)$ and $g(z)$ be analytic in $\mathcal{U}$ and satisfy

$$\phi(0) = g(0) = 0, \quad \phi'(0) \neq 0, \quad \text{and} \quad g'(0) \neq 0.$$ 

Suppose that for each $\sigma (|\sigma| = 1)$ and $\rho (|\rho| = 1)$

$$\phi(z) * \left(1 + \frac{\rho \sigma z}{1 - \sigma z}\right) g(z) \neq 0 \quad (z \in \mathcal{U} \setminus \{0\}).$$

Then, for each function $F(z)$ analytic in the unit disk $\mathcal{U}$ and satisfying the inequality $\text{Re}\{F(z)\} > 0 (z \in \mathcal{U})$, we have

$$\text{Re}\left(\frac{(\phi * G)(z)}{(\phi * g)(z)}\right) > 0 \quad (z \in \mathcal{U}),$$

where $G(z) = F(z)g(z)$.

**Lemma 3.** (Jack [7]) Let $\phi(z)$ be convex and $g(z)$ starlike in $\mathcal{U}$. Then, for each function $F(z)$ analytic in the unit disk $\mathcal{U}$ and satisfying $\text{Re}\{F(z)\} > 0 (z \in \mathcal{U})$, we have

$$\text{Re}\left(\frac{(\phi * Fg)(z)}{(\phi * g)(z)}\right) > 0 \quad (z \in \mathcal{U}),$$
Lemma 4. (c.f., Owa, Saigo and Srivastava [5]) Let $\alpha, \beta, \eta \in \mathbb{C}$ and $\operatorname{Re}(\alpha) > 0$, and let $k > \operatorname{Re}(\beta - \eta) - 1$. Then

$$I_{0, z}^{\alpha, \beta, \eta} k = \frac{\Gamma(k + 1)\Gamma(k - \beta + \eta + 1)}{\Gamma(k - \beta + \eta)\Gamma(k + \alpha + \eta + 1)} z^{k - \beta}.$$  

Applying the above lemmas, we derive

Theorem 1. Let the function $f(z)$ defined by (1.1) be in the class $S_0^*(b)$ and satisfy

$$h(z) \left( \frac{1 + \rho \sigma z}{1 - \sigma z} \right) b f(z) \neq 0 \quad (z \in \mathcal{U} \setminus \{0\})$$

for each $\rho (|\rho| = 1)$ and $\sigma (|\sigma| = 1)$, where

$$h(z) = z + \sum_{n=2}^{\infty} \frac{(2 - \beta + \eta)_{n-1}(1)n}{(2 - \beta)_{n-1}(2 + \alpha + \eta)_{n-1}} z^n,$$

and for $\alpha, \beta, \eta \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$ and $\min\{\operatorname{Re}(\alpha + \eta), \operatorname{Re}(-\beta + \eta), \operatorname{Re}(-\beta)\} > -2$. Then $J_{0, z}^{\alpha, \beta, \eta} f(z)$ belongs to the class $S_0^*(b)$.

Proof. Note from (1.18), (2.4) and (2.6) that

$$J_{0, z}^{\alpha, \beta, \eta} f(z) = z + \sum_{n=2}^{\infty} \frac{(2 - \beta + \eta)_{n-1}(1)n}{(2 - \beta)_{n-1}(2 + \alpha + \eta)_{n-1}} a_n z^n = (h \ast f)(z),$$

which readily yields

$$1 + \frac{1}{b} \left( \frac{z(J_{0, z}^{\alpha, \beta, \eta} f(z))'}{J_{0, z}^{\alpha, \beta, \eta} f(z)} - 1 \right) = \frac{h(z) \ast \left( \sum_{n=0}^{\infty} (n + b) a_{n+1} z^{n+1} \right)}{b(h \ast f)(z)}$$

$$= \frac{(h \ast [(b-1)f + zf'])(z)}{(h \ast bf)(z)}.$$

as $a_1 = 1$.

Therefore, putting $\phi(z) = h(z), g(z) = bf(z)$ and $F(z) = 1 + 1/b[(zf'(z))/f(z) - 1]$ in Lemma 2, we conclude from (2.7) that

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left( \frac{z(J_{0, z}^{\alpha, \beta, \eta} f(z))'}{J_{0, z}^{\alpha, \beta, \eta} f(z)} - 1 \right) \right\} > 0,$$

which completes the proof of Theorem 1.
Corollary 1. Let the function $f(z)$ defined by (1.1) be in the class $S_0^*(b)$ and satisfy
\[ u(z) \ast \left( \frac{1 + \rho \sigma z}{1 - \sigma z} \right) b f(z) \neq 0 \quad (z \in \mathbb{U} \setminus \{0\}) \]
for each $\rho (|\rho| = 1)$ and $\sigma (|\sigma| = 1)$, where
\[ u(z) = z + \sum_{n=2}^{\infty} \frac{(1)_n}{(2 - \lambda)_{n-1}} z^n \quad (\text{Re}(\lambda) < 0). \]
Then $\Omega^\lambda f(z)$ belongs to the class $S_0^*(b)$.

Proof. Setting $\alpha = -\beta = -\lambda$ in Theorem 1 and taking Remark 2 into account, we have Corollary 1.

Corollary 2. Let $h(z)$ be convex and let $f(z) \in S_1^*(b) (|b| \leq 1)$, where $h(z)$ is given by (2.6) with the same assumptions of $\alpha, \beta$ and $\eta$ in Theorem 1. Then $J_{0,z}^{\alpha,\beta,\eta} f(z) = (h \prec f)(z)$ belongs to the class $S_0^*(b)$.

Proof. From the hypothesis, we obtain
\[ f(z) \in S_1^*(b) \subseteq S^*(0) = S^* \quad (|b| \leq 1). \]
By applying Lemma 3 in view of Theorem 1, we have the desirous result immediately.

Theorem 2. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{K}_0(b)$ and satisfy
\[ h(z) \ast \left( \frac{1 + \rho \sigma z}{1 - \sigma z} \right) b z f'(z) \neq 0 \quad (z \in \mathbb{U} \setminus \{0\}) \]
for each $\rho (|\rho| = 1)$ and $\sigma (|\sigma| = 1)$, where $h(z)$ is given by (2.6) and for $\alpha, \beta, \eta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$ and $\min\{\text{Re}(\alpha + \eta), \text{Re}(-\beta + \eta), \text{Re}(-\beta)\} > -2$. Then $J_{0,z}^{\alpha,\beta,\eta} f(z)$ belongs to the class $\mathcal{K}_0(b)$.

Proof. Applying (1.7) and Theorem 1, we observe that
\[ f(z) \in \mathcal{K}_0(b) \iff zf'(z) \in S_0^*(b) \Rightarrow J_{0,z}^{\alpha,\beta,\eta} z f'(z) \in S_0^*(b) \]
\[ \iff (h \ast zf')(z) \in S_0^*(b) \iff z(h \ast f)'(z) \in S_0^*(b) \]
\[ \iff (h \ast f)(z) \in \mathcal{K}_0(b) \iff J_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{K}_0(b), \]
which evidently proves Theorem 2.

Taking $\alpha = -\beta = -\lambda$ in Theorem 2, we get
Corollary 3. Let the function $f(z)$ defined by (1.1) be in the class $K_0(b)$ and satisfy

$$u(z) * \left( \frac{1 + \rho \sigma z}{1 - \sigma z} \right) b z f'(z) \neq 0 \quad (z \in \mathcal{U} \setminus \{0\})$$

for each $\rho$ ($|\rho| = 1$) and $\sigma$ ($|\sigma| = 1$), where $u(z)$ is given by (2.8). Then $\Omega^\lambda f(z)$ belongs to the class $K_0(b)$.

Corollary 4. Let $h(z)$ be convex and let $f(z) \in K_1(b)$ ($|b| \leq 1$), where $h(z)$ is given by (2.6) with the same assumption of $\alpha, \beta$ and $\eta$ there. Then $J_{0_z}^{\alpha,\beta,\eta} f(z) = (h \prec f)(z)$ belongs to the class $K_0(b)$.

Theorem 3. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and satisfy

$$\left| \frac{(J_{0_z}^{\alpha,\beta,\eta} f(z))'}{g(z)} - 1 \right|^\sigma \left| \frac{z(J_{0_z}^{\alpha,\beta,\eta} f(z))''}{g'(z)} - \frac{z(J_{0_z}^{\alpha,\beta,\eta} f(z))''}{\{g'(z)\}^2} \right| ^\delta < |b|^{\sigma + \delta} \quad (z \in \mathcal{U})$$

for some $\sigma \geq 0, \delta \geq 0$ and $g(z) \in K_0(c)$. Suppose also that $\alpha, \beta, \eta \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $\min\{\Re(\alpha + \eta), \Re(-\beta + \eta), \Re(-\beta)\} > -2$. Then $J_{0_z}^{\alpha,\beta,\eta} f(z)$ belongs to the class $C_1(b)$.

Proof. If we define

$$\omega(z) = \frac{1}{b} \left( \frac{(J_{0_z}^{\alpha,\beta,\eta} f(z))'}{g'(z)} - 1 \right)$$

for $f(z) \in \mathcal{A}$ and $g(z) \in K_0(c)$, then it is an elementary matter to show that $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$. Noting that

$$b z \omega'(z) = \frac{z(J_{0_z}^{\alpha,\beta,\eta} f(z))''}{g'(z)} - \frac{z(J_{0_z}^{\alpha,\beta,\eta} f(z))''}{\{g'(z)\}^2} ,$$

we know that the condition (2.11) leads us to

$$|b \omega(z)|^\sigma |b z \omega'(z)|^\delta < |b|^{\sigma + \delta} .$$

Suppose that there exists $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1 \quad (\omega(z_0) \neq 1) .$$

Then, using Lemma 1, we see

$$|b \omega(z_0)|^\sigma |b z_0 \omega'(z_0)|^\delta = |b|^{\sigma + \delta} |k|^\delta \geq |b|^{\sigma + \delta} ,$$

which contradicts (2.11). Therefore we conclude $|\omega(z)| < 1$ for all $z \in \mathcal{U}$. This implies that

$$\left| \frac{(J_{0_z}^{\alpha,\beta,\eta} f(z))'}{g'(z)} - 1 \right| < |b| \quad (z \in \mathcal{U}) ,$$

which completes the proof of Theorem 3.

Letting $\alpha = -\beta = -\lambda$ in Theorem 3, we have
Corollary 5. Let the function $f(z)$ defined by (1.1) be in the class $A$ and satisfy
\begin{equation}
\left| \frac{(\Omega^\lambda f(z))^\prime}{g'(z)} - 1 \right|^\sigma \left| \frac{z(\Omega^\lambda f(z))''}{g'(z)} - \frac{z(\Omega^\lambda f(z))^\prime}{g'(z)} \right|^\delta < |b|^{\sigma+\delta} \quad (z \in \mathcal{U})
\end{equation}
for some $\sigma \geq 0$, $\delta \geq 0$, and $g(z) \in \mathcal{K}_0(c)$. Then $\Omega^\lambda f(z)$ belongs to the class $C_1(b)$.

Putting $g(z) = z \in \mathcal{K}_0(1)$, Theorem 3 gives

Corollary 6. Let the function $f(z)$ defined by (1.1) be in the class $A$ and satisfy
\begin{equation}
\left| (J_{0,z}^{\alpha,\beta,\eta} f(z))' - 1 \right|^\sigma \left| z(J_{0,z}^{\alpha,\beta,\eta} f(z))'' \right|^\delta < |b|^{\sigma+\delta} \quad (z \in \mathcal{U})
\end{equation}
for some $\sigma \geq 0$ and $\delta \geq 0$. Then $J_{0,z}^{\alpha,\beta,\eta} f(z)$ belongs to the class $C_1(b)$.

Theorem 4. Let the function $f(z)$ defined by (1.1) be in the class $A$ and satisfy
\begin{equation}
\left| a \left( \frac{z(J_{0,z}^{\alpha,\beta,\eta} f(z))'}{J_{0,z}^{\alpha,\beta,\eta} f(z)} - 1 \right) + (1-a) \frac{z^2(J_{0,z}^{\alpha,\beta,\eta} f(z))''}{J_{0,z}^{\alpha,\beta,\eta} f(z)} \right| < |b| \left( 1 + (1-a)(1-|b|) \right) \quad (z \in \mathcal{U})
\end{equation}
for some $a \leq 1$ and $|b| \leq 1$. Suppose also that $\alpha, \beta, \eta \in \mathbb{C}$ with Re($\alpha$) > 0 and min{Re($\alpha + \eta$), Re($-\beta + \eta$), Re($-\beta$)} > -2. Then $J_{0,z}^{\alpha,\beta,\eta} f(z)$ belongs to the class $S^*_1(b)$.

Proof. If we set
\begin{equation}
\omega(z) = \frac{1}{b} \left( \frac{z(J_{0,z}^{\alpha,\beta,\eta} f(z))'}{J_{0,z}^{\alpha,\beta,\eta} f(z)} - 1 \right) \quad (f \in A),
\end{equation}
then the function $\omega(z)$ is regular in $\mathcal{U}$ and $\omega(0) = 0$. By using the logarithmic differentiation on both sides of (2.17), we have
\[ \frac{z(J_{0,z}^{\alpha,\beta,\eta} f(z))''}{(J_{0,z}^{\alpha,\beta,\eta} f(z))'} = b\omega(z) + \frac{bz\omega'(z)}{1 + b\omega(z)}. \]
This yields
\[ a \left( \frac{z(J_{0,z}^{\alpha,\beta,\eta} f(z))'}{J_{0,z}^{\alpha,\beta,\eta} f(z)} - 1 \right) + (1-a) \frac{z^2(J_{0,z}^{\alpha,\beta,\eta} f(z))''}{J_{0,z}^{\alpha,\beta,\eta} f(z)} \]
\[ = b\omega(z) \left\{ 1 + (1-a) \left( b\omega(z) + \frac{z\omega'(z)}{\omega(z)} \right) \right\} . \]
Assume that there exists \(z_0 \in \mathcal{U}\) such that (2.13) holds true for the function \(\omega(z)\) in (2.17). Then, writing \(\omega(z_0) = e^{i\theta}\), and using Lemma 1, we deduce

\[
|b\omega(z_0)| \left\{ 1 + (1 - a) \left( b\omega(z_0) + \frac{z_0\omega'(z_0)}{\omega(z_0)} \right) \right\} = |b||1 + (1 - a)(k + be^{i\theta})| 
\geq |b||1 + (1 - a)(1 - |b|)|,
\]

which contradicts (2.16). Thus we obtain

\[
|\omega(z)| = \left| \frac{1}{b}\left( \frac{z(J_{0,z}^\alpha\beta\eta f(z))'}{(J_{0,z}^\alpha\beta\eta f(z))} - 1 \right) \right| < 1 \quad (z \in \mathcal{U}),
\]

which completes the proof of Theorem 4.

Taking \(\alpha = -\beta = -\lambda\) in Theorem 4, we have

**Corollary 7.** Let the function \(f(z)\) defined by (1.1) be in the class \(A\) and satisfy

\[
|a\left( \frac{z(\Omega^\lambda f(z))'}{\Omega^\lambda f(z)} - 1 \right) + (1 - a)\frac{z^2(\Omega^\lambda f(z))''}{\Omega^\lambda f(z)}| < |b|[1 + (1 - a)(1 - |b|)] \quad (z \in \mathcal{U})
\]

for some \(a \leq 1\) and \(|b| \leq 1\). Then \(\Omega^\lambda f(z)\) belongs to the class \(S_1^*(b)\).

**Theorem 5.** Let the function \(f(z)\) defined by (1.1) be in the class \(A\) and satisfy

\[
|a\left( \frac{z(J_{0,z}^\alpha\beta\eta f(z))'}{(J_{0,z}^\alpha\beta\eta f(z))} - 1 \right) + (1 - a)\frac{z(J_{0,z}^\alpha\beta\eta f(z))''}{(J_{0,z}^\alpha\beta\eta f(z))'}| < |b|^2 \left( 1 + \frac{1 - a}{1 + |b|} \right) \quad (z \in \mathcal{U})
\]

for some \(a \leq 1\). Suppose also that \(\alpha, \beta, \eta \in \mathbb{C}\) with \(\text{Re}(\alpha) > 0\) and \(\text{min}\{\text{Re}(\alpha + \eta), \text{Re}(\beta + \eta), \text{Re}(\beta)\} > -2\). Then \(J_{0,z}^\alpha\beta\eta f(z)\) belongs to the class \(S_1^*(b)\).

The proof of Theorem 5 is much akin to that of Theorem 4, and we omit the details involved.

**Theorem 6.** Let the function \(f(z)\) defined by (1.1) be in the class \(A\) and satisfy

\[
|a\left( \frac{z(J_{0,z}^\alpha\beta\eta f(z))'}{(J_{0,z}^\alpha\beta\eta f(z))} - 1 \right) + \frac{z(J_{0,z}^\alpha\beta\eta f(z))''}{(J_{0,z}^\alpha\beta\eta f(z))'}| < |b|^\sigma \left( 1 + \frac{2|b|}{1 + |b|} \right) \quad (z \in \mathcal{U})
\]

for some \(\sigma \geq 0\) and \(\delta \geq 0\). Suppose also that \(\alpha, \beta, \eta \in \mathbb{C}\) with \(\text{Re}(\alpha) > 0\) and \(\text{min}\{\text{Re}(\alpha + \eta), \text{Re}(\beta + \eta), \text{Re}(\beta)\} > -2\). Then \(J_{0,z}^\alpha\beta\eta f(z)\) belongs to the class \(C_1(b)\).

**Proof.** Define the function \(\omega(z)\) by

\[
\omega(z) = \frac{1}{b}\left\{ (J_{0,z}^\alpha\beta\eta f(z))' - 1 \right\}.
\]
Then it follows that $\omega(z)$ is analytic in $\mathcal{U}$ with $\omega(0) = 0$. Substituting for $J_{0,z}^{\alpha,\beta,\eta}f(z)$ into the left-hand side of (2.20) from (2.21), we get

$$\left|\left(J_{0,z}^{\alpha,\beta,\eta}f(z)\right)' - 1\right|^\sigma \left|1 + \frac{z\left(J_{0,z}^{\alpha,\beta,\eta}f(z)\right)''}{\left(J_{0,z}^{\alpha,\beta,\eta}f(z)\right)'}\right|^\delta = \left|b\omega(z)\right|^\sigma \left|\frac{1 + b\omega(z) + z\omega'(z)}{1 + b\omega(z)}\right|^\delta.$$ 

Assume that there exist a point $z_0 \in \mathcal{U}$ satisfying (2.13) for the function $\omega(z)$ in (2.21). Then, applying Lemma 1, we obtain

$$|b\omega(z_0)|^\sigma \left|\frac{1 + b(\omega(z_0) + z_0\omega'(z_0))}{1 + b\omega(z_0)}\right|^\delta = |b|^\sigma \left|k + 1 - \frac{k}{1 + b\omega(z_0)}\right|^\delta,$$

which contradicts the condition (2.20). Hence we have $J_{0,z}^{\alpha,\beta,\eta}f(z) \in C_1(b)$.

**Theorem 7.** Let the function $f(z)$ defined by (1.1) be in the class $A$ and satisfy

(2.22) $\text{Re} \left(\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)}\right) > \frac{|2b - 1| - 1}{2(2b - 1) + 1}$ if $b - \frac{1}{2} < \frac{1}{2}$

or

(2.23) $\text{Re} \left(\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)}\right) < \frac{|2b - 1| - 1}{2(2b - 1) + 1}$ if $b - \frac{1}{2} > \frac{1}{2}$

for some $g(z) \in \mathcal{K}_0(c)$. Then $f(z)$ belongs to the class $C_0(b)$.

**Proof.** Let us introduce the function $\omega(z)$ by

(2.24) $1 + \frac{1}{b} \left(\frac{f'(z)}{g'(z)} - 1\right) = \frac{1 + \omega(z)}{1 - \omega(z)}$

for some $g(z) \in \mathcal{K}_0(c)$ and $f(z) \in A$. Differentiating both side of (2.24) logarithmically, we obtain

$$zf''(z) - zg''(z) = \frac{(2b - 1)z\omega'(z)}{1 + (2b - 1)\omega(z)} + \frac{z\omega'(z)}{1 - \omega(z)}.$$

Suppose that there exists $z_0 \in \mathcal{U}$ such that (2.13) holds true for the function $\omega(z)$ in (2.24). Then, letting $\omega(z_0) = e^{i\theta}$ and $2b - 1 = |2b - 1|e^{i\phi}$, and using Lemma 1, we have

$$\text{Re} \left(\frac{z_0f''(z_0)}{f'(z_0)} - \frac{z_0g''(z_0)}{g'(z_0)}\right) = \text{Re} \left(\frac{(2b - 1)k\omega(z_0)}{1 + (2b - 1)\omega(z_0)}\right) + \text{Re} \left(\frac{k\omega(z_0)}{1 - \omega(z_0)}\right)$$

$$= \frac{k|2b - 1|(|2b - 1| + \cos(\theta + \phi))}{1 + |2b - 1|^2 + 2|2b - 1|\cos(\theta + \phi)} - \frac{k}{2}.$$
for \( k \geq 1 \) and \( z_0 \in \mathcal{U} \). Hence, let
\[
h(t) = \frac{|2b-1| + t}{1 + |2b-1|^2 + 2|2b-1|t} \quad (-1 \leq t \leq 1).
\]
If \(|b - 1/2| \leq 1/2\), then \( h(t) \) is monotone increasing and
\[
\text{Re} \left( \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g''(z_0)}{g'(z_0)} \right) \leq \frac{|2b-1|k}{|2b-1|+1} - \frac{k}{2}
\]
\[
\leq \frac{|2b-1| - 1}{2(|2b-1|+1)}.
\]
If, on the other hand, \(|b - 1/2| \geq 1/2\), then \( h(t) \) is monotone decreasing and
\[
\text{Re} \left( \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g''(z_0)}{g'(z_0)} \right) \geq \frac{|2b-1|k}{|2b-1|+1} - \frac{k}{2}
\]
\[
\geq \frac{|2b-1| - 1}{2(|2b-1|+1)}.
\]
These contradict (2.22) and (2.23), which evidently completes the proof of Theorem 6.

**Corollary 8.** Let the function \( f(z) \) defined by (1.1) be in the class \( A \) and satisfy
\[
(2.25) \quad \text{Re} \left( \frac{z(J_{0,z}^\alpha,\beta,\eta(z))''}{(J_{0,z}^\alpha,\beta,\eta(z))'} - \frac{z g''(z)}{g'(z)} \right) > \frac{|2b-1| - 1}{2(|2b-1|+1)} \quad \text{if } |b - 1/2| < \frac{1}{2}
\]
or
\[
(2.26) \quad \text{Re} \left( \frac{z(J_{0,z}^\alpha,\beta,\eta(z))''}{(J_{0,z}^\alpha,\beta,\eta(z))'} - \frac{z g''(z)}{g'(z)} \right) < \frac{|2b-1| - 1}{2(|2b-1|+1)} \quad \text{if } |b - 1/2| > \frac{1}{2}
\]
for some \( g(z) \in \mathcal{K}_0(c) \). Suppose also that \( \alpha, \beta, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \) and \( \min\{\text{Re}(\alpha + \eta), \text{Re}(-\beta + \eta), \text{Re}(-\beta)\} > -2 \). Then \( J_{0,z}^\alpha,\beta,\eta(z) \) belongs to the class \( C_0(b) \).

**Theorem 8.** Let the function \( f(z) \) defined by (1.1) be in the class \( A \) and satisfy
\[
(2.27) \quad \text{Re} \left( \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) > \frac{-|2b-1| - 3}{2(|2b-1|+1)} \quad \text{if } |b - 1/2| \leq \frac{1}{2}
\]
or
\[
(2.28) \quad \text{Re} \left( \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < \frac{-|2b-1| - 3}{2(|2b-1|+1)} \quad \text{if } |b - 1/2| > \frac{1}{2}
\]
Then \( f(z) \) belongs to the class \( S_0^* (b) \).

**Proof.** The proof of Theorem 8 runs parallel to that of Theorem 7 with
\[
1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) = \frac{1+\omega(z)}{1-\omega(z)},
\]
and we omit the details involved.
Corollary 9. Let the function $f(z)$ defined by (1.1) be in the class $A$ and satisfy

\begin{equation}
\Re \left( \frac{z(J_{0,z}^{\alpha,\beta,\eta}(z))'' - z(J_{0,z}^{\alpha,\beta,\eta}(z))'}{J_{0,z}^{\alpha,\beta,\eta}(z)} \right) > \frac{-|2b-1|-3}{2(|2b-1|+1)} \quad \text{if} \quad |b - \frac{1}{2}| \leq \frac{1}{2}
\end{equation}

or

\begin{equation}
\Re \left( \frac{z(J_{0,z}^{\alpha,\beta,\eta}(z))'' - z(J_{0,z}^{\alpha,\beta,\eta}(z))'}{J_{0,z}^{\alpha,\beta,\eta}(z)} \right) < \frac{-|2b-1|-3}{2(|2b-1|+1)} \quad \text{if} \quad |b - \frac{1}{2}| > \frac{1}{2}.
\end{equation}

Suppose also that $\alpha, \beta, \eta \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $\min\{\Re(\alpha+\eta), \Re(-\beta+\eta), \Re(-\beta)\} > -2$. Then $J_{0,z}^{\alpha,\beta,\eta}(z)$ belongs to the class $S_{0}^{*}(b)$.

REFERENCES