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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1997), 1013: 207-224</td>
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<td>Issue Date</td>
<td>1997-09</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/61557">http://hdl.handle.net/2433/61557</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Strong and Weak Coupling Limits of Interaction Models of Quantum Fields and Particles

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1 INTRODUCTION

Asymptotic behaviors of scaling Hamiltonians which describe interactions of particles and quantized fields are considered. In a mathematical formulation, interaction Hamiltonians of the particles and the quantized fields are described by the theory of self-adjoint operators acting in the tensor product of two Hilbert spaces over the complex field C. Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two Hilbert spaces. We define a self-adjoint operator \( \mathbf{H} \) acting in the tensor product of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \), by

\[
\mathbf{H} = H_1 \otimes I + \alpha H_{\text{int}} + I \otimes H_2.
\]

Here \( H_1 \) and \( H_2 \) are self-adjoint operators in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively, \( H_{\text{int}} \) is a symmetric operator in \( \mathcal{H} \) and \( \alpha \in \mathbb{R} \) is a coupling constant. Then, for the given self-adjoint operator \( \mathbf{H} \), we define "\( \beta \)-coupling Hamiltonian, \( \mathbf{H}_{\beta}(\Lambda) \)" by

\[
\mathbf{H}_{\beta}(\Lambda) = H_1 \otimes I + \Lambda \alpha H_{\text{int}} + \Lambda^\beta I \otimes H_2, \quad 1 \leq \beta.
\]

Introducing a renormalization \( E_{\beta}(\Lambda) \) which goes to infinity or minus infinity as \( \Lambda \to \infty \) in some sense, we want to investigate the following asymptotic behaviors

\[
s - \lim_{\Lambda \to \infty} e^{-it(\mathbf{H}_{\beta}(\Lambda) - E_{\beta}(\Lambda))} = \mathbf{U} \left( e^{-itH_{\text{eff}} \otimes \mathbf{P}} \right) \mathbf{U}^{-1}, \quad t \in \mathbb{R}.
\]

Here \( H_{\text{eff}} \) is a self-adjoint operator in \( \mathcal{H}_1 \), which is called "effective Hamiltonian", \( \mathbf{U} \) is a unitary operator in \( \mathcal{H} \) and \( \mathbf{P} \) a projection operator onto a one-dimensional subspace in \( \mathcal{H}_2 \). It
seems to be useful to readers to collect some background ingredient. Motivation of this paper is [1] and [3]. In [1], in order to give an interpretation of a physical phenomenon "Lamb shift" without formal perturbation theory, A.Arai elaborates a scaling limit of the Pauli-Fierz model. The scaling limit corresponds to the case $\beta = 1$ in (1.1). In [3], E.B.Davies studies a scaling limit of the Nelson model to derive a Schrödinger Hamiltonian (effective Hamiltonian) with a scalar potential. The scaling limit corresponds to the case $\beta = 2$ in (1.1). In this paper, we deal with the Nelson model [2,3,4,5,8,10], the Pauli-Fierz model [1,6,7,8,9,10] and the spin-boson model [1]. Thus considering scaling limits as in (1.2) for these models is an extension of those considered in [1,2,3,6,7,10]. We organize this paper as follows. In section 2, we overview an abstract theory of a scaling limit of self-adjoint operators. In section 3,4 and 5, we study the Nelson model, the Pauli-Fierz model and the spin-boson model, respectively. In section 6, we give some remarks.

2 FUNDAMENTAL FACTS

2.1 An abstract Boson Fock space

In this subsection we define an abstract Boson Fock space and basic notations. For a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$, we denote the scalar product by $\langle f, g \rangle_{\mathcal{H}}$ and the associated norm by $\|f\|_{\mathcal{H}}$, where the scalar product is linear in $g$ and antilinear in $f$. For the tempered distributions $f$ and $g$, the notation $\bar{f}$ denotes the complex conjugate of $f$, and $\hat{f}$ (resp.$\check{g}$) the Fourier transform of $f$ (resp.the inverse Fourier transform of $g$). We denote the domain of an operator $A$ by $D(A)$. The Boson Fock space over the Hilbert space $\mathcal{H}$ is defined by

$$\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} [\otimes^n_{\mathbb{C}} \mathcal{H}],$$

where $\otimes^n_{\mathbb{C}} \mathcal{H}, n \geq 1$, denotes the n-fold symmetric tensor product of $\mathcal{H}$, and $\otimes^0_{\mathbb{C}} \mathcal{H} = \mathbb{C}$. Define $\Omega_{\mathcal{H}} = \{1, 0, 0, \ldots\}$. Let the annihilation and creation operators in the Boson Fock space denoted by $a_{\mathcal{H}}(f), f \in \mathcal{H}$ and $a_{\mathcal{H}}^\dagger(g), g \in \mathcal{H}$, respectively. It is well known that

$$\mathcal{F}_{\mathcal{H}}^\infty \equiv \text{L} \{a_{\mathcal{H}}^\dagger(f_1)\ldots a_{\mathcal{H}}^\dagger(f_n)\Omega, \Omega, f_j \in \mathcal{H}, j = 1, \ldots, n, n \geq 1\}$$

is dense in $\mathcal{F}_{\mathcal{H}}$, where $\text{L}$ denotes the linear hull of the vectors in $\{\ldots\}$. The annihilation and the creation operators in the Boson Fock space satisfy the following canonical commutation
relations on $\mathcal{F}_H^\infty$:

$$[a_H(f), a_H^d(g)] = \langle \hat{f}, g \rangle_H,$$

$$[a_H^d(f), a_H^d(g)] = 0,$$

where $a_H^d$ means $a_H$ or $a_H^d$. Let $h$ be a self-adjoint operator in $\mathcal{H}$. Define $d\Gamma_H(h)$ by

$$d\Gamma_H(h)\Omega = 0,$$

$$d\Gamma_H(h)a_H^d(f_1)\ldots a_H^d(f_n)\Omega_H = \sum_{j=1}^n a_H^d(f_1)\ldots a_H^d(hf_j)\ldots a_H^d(f_n)\Omega_H, f_j \in D(h).$$

Then $d\Gamma_H(h)$ is essentially self-adjoint. Let us use the same notation as $d\Gamma_H(h)$ for its self-adjoint extension.

### 2.2 An abstract theory of a scaling limit

We overview an abstract theory of a scaling limit of self-adjoint operators acting in a tensor product Hilbert space established in [1] with a little modification. Let $\mathcal{K}$ be a Hilbert space and put $\mathcal{X} = \mathcal{H} \otimes \mathcal{K}$. Suppose that an operator, $A$ (resp.$B$), is a nonnegative self-adjoint operator in $\mathcal{H}$ (resp.$\mathcal{K}$) and $\text{Ker}B = \{kG|k \in \mathbb{C}, ||G||_{\mathcal{X}} = 1\}$. Set the projection operator onto $\text{Ker}B$ by $P_B$. We suppose that a family of self-adjoint operators, $\{C_\Lambda\}_{\Lambda > 0}$, in $\mathcal{X}$ admits the following conditions:

1. For any $\epsilon > 0$, there exists $\Lambda_0$ so that, for all $\Lambda > \Lambda_0$, $D(C_\Lambda) \supset D(A \otimes I + \Lambda I \otimes B)$ with

   $$||C_\Lambda \Phi||_{\mathcal{X}} \leq \epsilon ||(A \otimes I + \Lambda I \otimes B)\Phi|| + b(\epsilon) ||\Phi||_{\mathcal{X}}, \Phi \in D(A \otimes I) \cap D(I \otimes B),$$

   where $b(\epsilon) > 0$ is a constant independent of $\Lambda > \Lambda_0$.

2. There exists a symmetric operator $C$ in $\mathcal{X}$ so that $D(C) \supset D(A) \otimes \text{Ker}B$ and, for $z \in \mathbb{C} \setminus \mathbb{R}$,

   $$s - \lim_{\Lambda \to \infty} C_\Lambda(A \otimes I + \Lambda I \otimes B - z)^{-1} = C \left\{(A - z)^{-1} \otimes P_B\right\}.$$

We define an operator $E_G(C)$ with the domain $D(E_G(C)) = D(A)$ by

$$\langle f, E_G(C)g \rangle_H = \langle f \otimes G, C(g \otimes G) \rangle_{\mathcal{X}}, f \in \mathcal{H}, g \in D(A).$$
We call $E_G(C)$ "the partial expectation of $C$ with respect to $G$". Set

$$K_{eff} = A + E_G(C).$$

The following proposition is fundamental in this paper.

**Proposition 2.1** ([1, Theorem 2.1]) Let operators $A, B, C_\Lambda$, and $C$ be as above. Then

1. For $\Lambda > \Lambda_0$, $K_\Lambda = A \otimes I + \Lambda I \otimes B + C_\Lambda$ is self-adjoint on $D(A \otimes I) \cap D(I \otimes B)$ and uniformly bounded from below. Moreover $E_G(C)$ is infinitesimally small with respect to $A$, i.e., $K_{eff}$ is self-adjoint on $D(A)$.

2. For $z \in \mathbb{C} \setminus \mathbb{R}$

$$s - \lim_{\Lambda \to \infty} (K_\Lambda - z)^{-1} = (K_{eff} - z)^{-1} \otimes P_B.$$  \hspace{1cm} (2.1)

Finally we note a fundamental fact.

**Proposition 2.2** Let $K_\Lambda$ and $K_{eff}$ satisfy (2.1). Then

$$s - \lim_{\Lambda \to \infty} e^{-itK_\Lambda} = e^{-itK_{\infty}} \otimes P_B.$$  \hspace{1cm} (3.1)

**Proof:** See [1, Theorem 2.2] \hspace{1cm} \Box

By Proposition 2.2, it is enough to show strong resolvent limits of $\beta$-coupling Hamiltonian to investigate (1.2).

**3 THE NELSON MODEL**

**3.1 The Nelson model**

In this section, we consider the Nelson Hamiltonian with an ultraviolet cut-off function $\hat{\phi}$ and with a finite number of nonrelativistic particles. Fix the number of the nonrelativistic particles $N$. For the mathematical generality, suppose that the dimension of the space in which the nonrelativistic particles move is $d \geq 1$. (This assumption remains throughout this paper.) We use the following identification

$$\mathcal{F}_N \equiv L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}_{L^2(\mathbb{R}^d)} \cong L^2(\mathbb{R}^{dN}; \mathcal{F}_{L^2(\mathbb{R}^d)}).$$
For notational simplicity, we write the annihilation or creation operators by $a^\dagger(f)$ instead of $a^\dagger_{L^2(\mathbb{R}^d)}(f)$ in sections 3 and 5. We define a time-zero scalar field $\phi(\hat{f})$ by

$$
\phi(\hat{f}) = \frac{1}{\sqrt{2}} \left\{ a\left(\frac{\hat{f}}{\sqrt{\omega}}\right) + a\left(\frac{\overline{\hat{f}}}{\sqrt{\omega}}\right) \right\}.
$$

Here $\omega = \omega(k) = \sqrt{k^2 + \mu^2}$, $\mu \geq 0$. In this section we require that $\varrho$ is a real valued even function, $\varrho(k) = \varrho(-k)$, with

$$
\frac{\hat{\varrho}}{\omega \sqrt{\omega}}, \frac{\hat{\varrho}}{\sqrt{\omega}} \in L^2(\mathbb{R}^d).
$$

(3. 1)

For each $x = (x^1, \ldots, x^N) \in \mathbb{R}^{dN}$, $x^j \in \mathbb{R}^d$, $j = 1, \ldots, N$, we set

$$
\tilde{\varrho}(x) = \frac{1}{\sqrt{(2\pi)^d}} \sum_{j=1}^{N} \hat{\varrho}(k)e^{-ikx^j}.
$$

We define

$$
H_I(\tilde{\varrho}) \equiv \phi\left(\tilde{\varrho}(\cdot)\right).
$$

For the multiplication operator $\omega$ in $L^2(\mathbb{R}^d)$ with the maximal domain, we set $d\Gamma_{L^2(\mathbb{R}^d)}(\omega) \equiv H_b$. Define an operator in $\mathcal{F}_N$ by

$$
H_\beta^\Lambda(\tilde{\varrho}, \Lambda) = -\frac{1}{2m} \Delta_N \otimes I - \Lambda g H_I(\tilde{\varrho}) + \Lambda^\beta I \otimes H_b, \quad 1 \leq \beta < \infty,
$$

where $g \in \mathbb{R}$ is a coupling constant, $m > 0$ a mass of the nonrelativistic particles, $\Delta_N$ the Laplacian in $L^2(\mathbb{R}^{dN})$ and $\Lambda > 0$ a scaling parameter. Moreover we put a decoupled Hamiltonian $H_{\beta, N}(\Lambda)$ by

$$
H_{\beta, N}(\Lambda) = -\frac{1}{2m} \Delta_N \otimes I + \Lambda^\beta I \otimes H_b.
$$

We define a class of the set of multiplication operators in $L^2(\mathbb{R}^{dN})$. A multiplication operator $V$ is in a class, $\mathcal{M}_\pm(N)$, if and only if $V$ is infinitesimally small with respect to $-\Delta_N$.

**Proposition 3.1** ([2]) For $\Lambda > 0$ and $V \in \mathcal{M}_\pm(N)$, $H_\beta^\Lambda(\tilde{\varrho}, \Lambda) + V \otimes I$ is self-adjoint on $D(H_\beta^\Lambda(\Lambda))$ and bounded from below. Moreover it is essentially self-adjoint on any core for $H_\beta^\Lambda(\Lambda)$.
In the case of $\beta = 2$, following proposition is well known.

**Proposition 3.2** ([2,3], $\beta = 2$) Let $V \in \mathcal{M}_\pm(N)$. Then

$$s - \lim_{\Lambda \to \infty} e^{-it(H_N^2(\hat{\vartheta}, \Lambda) + V \otimes I)} = e^{-it(-\frac{1}{2m} \Delta_N + V + g^2 V(\hat{\vartheta})) \otimes P_N}.$$  

Here $P_N$ is the projection operator onto the subspace in $\mathcal{F}$, spanned by the vector $\Omega_{L^2(\mathbb{R}^4)}$.

### 3.2 The case of $\beta = 1$

Put $C_0^\infty(\mathbb{R}^{dN}) \otimes \mathcal{F}_{L^2(\mathbb{R}^4)}^\infty \equiv \mathcal{F}_N^\infty$, where $\otimes$ denotes the algebraic tensor product. We perform a unitary transformation

$$\mathcal{U}(g) = \exp \left( \frac{g}{\sqrt{2}} \left\{ a^\dagger \left( \frac{\hat{\partial}(\cdot)}{\omega \sqrt{\omega}} \right) - a \left( \frac{\bar{\partial}(\cdot)}{\omega \sqrt{\omega}} \right) \right\} \right)$$

with the following result:

**Proposition 3.3** The unitary operator $\mathcal{U}(\Lambda^{1-\beta} g)$ maps $\mathcal{F}_N^\infty$ into $D(H_N^\beta(\hat{\vartheta}, \Lambda))$ with

$$\mathcal{U}(\Lambda^{1-\beta} g)^{-1}(H_N^\beta(\hat{\vartheta}, \Lambda) + V \otimes I) \mathcal{U}(\Lambda^{1-\beta} g)$$

$$= \frac{1}{2m} \sum_{j=1}^N \left( p^j \otimes I - g\Lambda^{1-\beta} \phi_j \right)^2 + g^2 \Lambda^{2-\beta} V(\hat{\vartheta}) \otimes I + \Lambda^\beta I \otimes H_b + V \otimes I, \quad (3.2)$$

on $\mathcal{F}_N^\infty$, where $p^j = (-i \frac{\partial}{\partial x_1}, ..., -i \frac{\partial}{\partial x_d})$, $\phi_j = (\phi \left( \hat{\partial}^\dagger_1 (\cdot) \right), ..., \phi \left( \hat{\partial}^\dagger_d (\cdot) \right))$, $j = 1, ..., N$, and

$$\hat{\partial}^\dagger_\mu(x) = \hat{\partial}^\dagger_\mu(x, k) = \frac{1}{\sqrt{(2\pi)^d}} \frac{\hat{\partial}(k)e^{-ikx}\kappa_\mu}{\omega(k)}, \mu = 1, ..., d,$$

$$V(\hat{\vartheta}) = V(\hat{\vartheta}, x) = -\frac{1}{2(2\pi)^d} \left\| \sum_{j=1}^N \frac{\hat{\partial}e^{-ikx}}{\omega} \right\|_{L^2(\mathbb{R}^4)}^2.$$  

Moreover, for sufficiently large $\Lambda > 0$, the right hand side (R.H.S.) of (3.2) is self-adjoint on $D(H_N^\beta(\Lambda))$ and the equation (3.2) can be extended to the equation on $D(H_N^\beta(\Lambda))$.

Proposition 3.3 implies that the following equation holds, for $V \in \mathcal{M}_\pm(N)$ and sufficiently large $\Lambda > 0$;

$$\mathcal{U}(\Lambda^{1-\beta} g)^{-1} \left( H_N^\beta(\hat{\vartheta}, \Lambda) - g^2 \Lambda^{2-\beta} V(\hat{\vartheta}) \otimes I + V \otimes I \right) \mathcal{U}(\Lambda^{1-\beta} g)$$

$$= \frac{1}{2m} \sum_{j=1}^N \left( p^j \otimes I - g\Lambda^{1-\beta} \phi_j \right)^2 + V \otimes I + \Lambda^\beta I \otimes H_b. \quad (3.3)$$
In this subsection we set $\beta = 1$. Then we define a symmetric operator $Q(\hat{z})$, which is independent of $\Lambda$, by

\[
R.H.S. \text{ of (3.3)} = H^2_N(\Lambda) + Q(\hat{z}).
\]

Lemma 3.4 Let $V \in \mathcal{M}_\pm(N)$. Then, for any $\varepsilon > 0$, there exists $\Lambda_0$ and $b(\varepsilon) > 0$ so that, for all $\Lambda > \Lambda_0$, $D(Q(\hat{z})) \supset D(H^2_N(\Lambda))$ with

\[
\|Q(\hat{z})\Phi\|_{\mathcal{F}_N} \leq \varepsilon \|H^2_N(\Lambda)\Phi\|_{\mathcal{F}_N} + b(\varepsilon) \|\Phi\|_{\mathcal{F}_N}, \Phi \in D(H_F).
\]

Moreover $D(Q(\hat{z})) \supset D(-\Delta_N) \otimes \text{Ker} H_b$ with, for $z \in \mathbb{C} \setminus [0, \infty)$,

\[
s - \lim_{\Lambda \to \infty} Q(\hat{z}) \left( H^2_N(\Lambda) - z \right)^{-1} = Q(\hat{z}) \left[ \left( -\frac{1}{2m} \Delta_N - z \right)^{-1} \otimes P_N \right]. \tag{3.5}
\]

Proof: The proof of (3.4) follows from fundamental estimates with respect to $a^1$ and $H_b$.

By (3.4), for any $\varepsilon > 0$, taking sufficiently large $\Lambda > 0$, we see that

\[
\|Q(\hat{z}) \left( H^2_N(\Lambda) - z \right)^{-1} - \left( -\frac{1}{2m} \Delta_N - z \right)^{-1} \otimes P_N \|_{\mathcal{F}_N} \leq \varepsilon \|H^2_N(\Lambda) \left( H^2_N(\Lambda) - z \right)^{-1} - \left( -\frac{1}{2m} \Delta_N - z \right)^{-1} \otimes P_N \|_{\mathcal{F}_N} + b(\varepsilon) \|\Phi\|_{\mathcal{F}_N}.
\]

Taking $\Lambda \to \infty$ on the both sides above, we have

\[
\lim_{\Lambda \to \infty} \|Q(\hat{z}) \left( H^2_N(\Lambda) - z \right)^{-1} - \left( -\frac{1}{2m} \Delta_N - z \right)^{-1} \otimes P_N \|_{\mathcal{F}_N} \leq \varepsilon \|(I \otimes I - I \otimes P_N) \Phi\|_{\mathcal{F}_N}.
\]

Since $\varepsilon > 0$ is arbitrary, (3.5) follows.

\[
\text{Theorem 3.5 (}\beta = 1\text{)} \text{ Let } V \in \mathcal{M}_\pm(N) \text{ and } z \in \mathbb{C} \setminus \mathbb{R}. \text{ Put } \omega_0 = \omega_0(k) = |k| \text{ and}
\]

\[
\delta(\hat{z}) = \frac{1}{2(2\pi)^d} \left\| \frac{\partial \omega_0}{\omega \sqrt{\omega}} \right\|^2_{L^2(\mathbb{R}^d)}.
\]

Then

\[
s - \lim_{\Lambda \to \infty} \left( H^2_N(\hat{z}, \Lambda) - g^2 \Lambda^{2-\beta} V(\hat{z}) \otimes I + V \otimes I - z \right)^{-1}
\]

\[
= U(g) \left\{ \left( -\frac{1}{2m} \Delta_N + g^2 N \delta(\hat{z}) + V - z \right)^{-1} \otimes P_N \right\} U(g)^{-1}. \tag{3.6}
\]
**Proof:** From (3.3) it follows that

\[
\left( H_N^\beta(\hat{\rho}, \Lambda) - g^2 \Lambda^{2-\beta} V(\hat{\rho}) \otimes I + V \otimes I - z \right)^{-1} = \mathcal{U}(\Lambda^{1-\beta} g) \left( H_N^\beta(\Lambda) + Q(\hat{\rho}) - z \right)^{-1} \mathcal{U}(\Lambda^{1-\beta} g)^{-1}.
\]

By the fact that \( \mathcal{U}(\Lambda^{1-\beta} g) \) is independent of \( \Lambda \), it is enough to show that

\[
s - \lim_{\Lambda \to \infty} \left( H_N^\beta(\Lambda) + Q(\hat{\rho}) - z \right)^{-1} = \left( -\frac{1}{2m} \Delta_N + g^2 N \delta(\hat{\rho}) + V - z \right)^{-1} \otimes P_N.
\]

Since the partial expectation of \( Q(\hat{\rho}) \) with respect to \( \Omega_{L^2(\mathbb{R}^d)} \) is

\[
E_{\Omega_{L^2(\mathbb{R}^d)}}(Q(\hat{\rho})) = g^2 \sum_{j=1}^N \sum_{\mu=1}^3 \frac{1}{2} \left\| \frac{\partial j}{\sqrt{\omega}} \right\|_{L^2(\mathbb{R}^d)}^2 + V = g^2 N \delta(\hat{\rho}) + V,
\]

it follows (3.6) from Lemma 3.4 and Proposition 2.1 with the following correspondence:

\[
A = -\frac{1}{2m} \Delta_N, \quad B = H_b, \quad C_\Lambda = C = Q(\hat{\rho}), \quad G = \Omega_{L^2(\mathbb{R}^d)}.
\]

\( \square \)

### 3.3 The case of \( 1 < \beta < 2, 2 < \beta \)

First we study the case of \( 1 < \beta < 2 \). We put the R.H.S. of (3.3) by

\[
R.H.S. of \ (3.3) = H_N^\beta(\Lambda) + Q^1(\hat{\rho}, \Lambda).
\]  
(3.7)

Similar to (3.4) and (3.5), one can see that, for \( V \in \mathcal{M}_+(N) \) and any \( \epsilon > 0 \), there exists \( \Lambda_0 \) and \( b(\epsilon) > 0 \) so that, for all \( \Lambda > \Lambda_0 \), \( D(Q^1(\hat{\rho}, \Lambda)) \supset D(H_N^\beta(\Lambda)) \) with

\[
\left\| Q^1(\hat{\rho}, \Lambda) \Phi \right\|_{\mathcal{F}_N} \leq \epsilon \left\| H_N^\beta(\Lambda) \Phi \right\|_{\mathcal{F}_N} + b(\epsilon) \left\| \Phi \right\|_{\mathcal{F}_N}, \Phi \in D(H_N^\beta(\Lambda)).
\]  
(3.8)

Moreover \( D(Q^1(\hat{\rho}, \Lambda)) \supset D(-\Delta_N) \otimes \text{Ker} H_b \) and, for \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
s - \lim_{\Lambda \to \infty} Q^1(\hat{\rho}, \Lambda) \left( H_N^\beta(\Lambda) - z \right)^{-1} = \left[ V \left( -\frac{1}{2m} \Delta_N - z \right)^{-1} \right] \otimes P_N.
\]  
(3.9)

Note that, for \( \beta > 1 \),

\[
s - \lim_{\Lambda \to \infty} \mathcal{U} \left( \frac{g}{\Lambda^{\beta-1}} \right) = I.
\]  
(3.10)

Hence we prove the following theorem.
Theorem 3.6 (1 < $\beta$ < 2) Let $V \in \mathcal{M}_\pm(N)$, $z \in \mathbb{C} \setminus \mathbb{R}$. Then
\[
 s - \lim_{\lambda \to \infty} \left( H_N^\lambda(\hat{\theta}, \Lambda) - g^2 \Lambda^{2-\beta} V(\hat{\varphi}) \otimes I + V \otimes I - z \right)^{-1} = \left( -\frac{1}{2m} \Delta_N + V - z \right)^{-1} \otimes P_N.
\]

Proof: Since the partial expectation of $V \otimes I$ with respect to $\Omega_{L^2(\mathbb{R}^d)}$ is $E_{\Omega_{L^2(\mathbb{R}^d)}}(V \otimes I) = V$, from (3.8), (3.9), (3.10) and Proposition 2.1, theorem follows with the following correspondence:
\[
 A = -\frac{1}{2m} \Delta_N, \quad B = H_b, \quad C = Q^1(\hat{\theta}, \Lambda), \quad C = V \otimes I \quad G = \Omega_{L^2(\mathbb{R}^d)}.
\]

\[\square\]

Secondly we study the case of $2 < \beta$. In this case, note that we do not need to subtract the renormalization $\Lambda^{2-\beta}V(\hat{\varphi}) \otimes I$. Put the R.H.S. of (3.2) by
\[
 R.H.S. \text{ of } (3.2) = H_N^\lambda(\Lambda) + Q^2(\hat{\varphi}, \Lambda).
\]

By the same argument as that of the case of $1 < \beta < 2$ with $Q^1(\Lambda, \hat{\varphi})$ replaced by $Q^2(\Lambda, \hat{\varphi})$, one can easily prove the following theorem.

Theorem 3.7 (2 < $\beta$) Let $V \in \mathcal{M}_\pm(N)$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then
\[
 s - \lim_{\lambda \to \infty} \left( H_N^\lambda(\hat{\theta}, \Lambda) + V \otimes I - z \right)^{-1} = \left( -\frac{1}{2m} \Delta_N + V - z \right)^{-1} \otimes P_N.
\]

4 THE PAULI-FIERZ MODEL

4.1 The Pauli-Fierz model

In this section, we study the Pauli-Fierz model in quantum electrodynamics with the dipole approximation. Let
\[
 \mathcal{W} = \underbrace{L^2(\mathbb{R}^d) \oplus \ldots \oplus L^2(\mathbb{R}^d)}_{d-1}.
\]

For $(0 \oplus \ldots \oplus 0) \in \mathcal{W}$, we set $a_{\mathcal{W}}^d(0 \oplus \ldots \oplus 0) = a^{(r)}(f)$. We write
\[
a_{\mathcal{W}}^{(r)}(g) = \int a_{\mathcal{W}}^{(r)}(k) g(k) dk, \quad r = 1, \ldots, d-1.
\]
Let $e^r : \mathbb{R}^d \to \mathbb{R}^d, r = 1, \ldots, d - 1$, be measurable functions so that

\begin{align*}
(1) & \ e^r(k) \cdot k = 0, \quad r = 1, \ldots, d - 1, \\
(2) & \ e^r(k) \cdot e^s(k) = \delta_{r,s}.
\end{align*}

We denote the $\mu$-th component of $e^r$ by $e^r_\mu, \mu = 1, \ldots, d$. The quantized smeared radiation field $A_\mu(\hat{f}, x)$ with $f$ in the Coulomb gauge, and the conjugate momentum $\Pi_\mu(\hat{f}, x), \mu = 1, \ldots, d, x \in \mathbb{R}^d$, are defined by

\begin{align*}
A_\mu(\hat{f}, x) &= \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \int \left\{ \frac{a^{(r)}_W(k) e^r_\mu(k) \hat{f}(k) e^{-ikx}}{\sqrt{\omega(k)}} + a^{(r)}_W(k) e^r_\mu(k) \hat{f}(k) e^{ikx} \right\} \text{dk}, \\
\Pi_\mu(\hat{f}, x) &= \frac{i}{\sqrt{2}} \sum_{r=1}^{d-1} \int \left\{ a^{(r)}_W(k) \sqrt{\omega(k)} e^r_\mu(k) \hat{f}(k) e^{-ikx} - a^{(r)}_W(k) \sqrt{\omega(k)} e^r_\mu(k) \hat{f}(k) e^{ikx} \right\} \text{dk}.
\end{align*}

Here $\hat{g}(k) = g(-k)$. We define the free Hamiltonian in $\mathcal{F}_W$ by

$$
d\Gamma_W(\omega \oplus \ldots \oplus \omega) = H_{EM}.
$$

We require that $\hat{g}$ satisfies (3.1) and $\rho$ is real-valued rotation invariant function throughout this section. Then the Pauli-Fierz Hamiltonian with the ultraviolet cut-off function $\hat{g}$ and with $N$-nonrelativistic particles is defined as an operator acting in

\begin{align*}
\mathcal{L}_N = L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}_W \cong L^2(\mathbb{R}^{dN}; \mathcal{F}_W),
\end{align*}

by

\begin{align}
\frac{1}{2m} \sum_{j=1}^{N} \left( p_j \otimes I - e A(\hat{\rho}, \cdot) \right)^2 + I \otimes H_{EM}, \tag{4.1}
\end{align}

where $e \in \mathbb{R}$ is a coupling constant and $A(\hat{\rho}, \cdot) = (A_1(\hat{\rho}, \cdot), \ldots, A_d(\hat{\rho}, \cdot))$. Introducing the polarization vector $e^r$, which corresponds to taking the Coulomb gauge, we see that, on a suitable dense domain,

$$
[p_j \otimes I, A(\hat{\rho}, \cdot)] = 0.
$$

Then formally we may rewrite (4.1) by

\begin{align*}
-\frac{1}{2m} \Delta_N \otimes I - \frac{e}{m} \sum_{j=1}^{N} \sum_{\mu=1}^{d} (p_\mu \otimes I) A_\mu(\hat{\rho}, \cdot) + \frac{e^2 N}{2m} A^2(\hat{\rho}, \cdot) + I \otimes H_{EM}.
\end{align*}

Here, for simplicity, we introduce the following assumptions to the Pauli-Fierz Hamiltonian:
(1) The self-interaction term $A^2(\partial, \cdot)$ is neglected.

(2) We introduce the dipole approximation, i.e., $A(\partial, x)$ is replaced by $A(\partial, 0)$.

Then, putting $A(\partial, 0) = I \otimes A(\partial)$, our Hamiltonian is as follows:

$$H_{EM}^\beta(\partial, \Lambda) = -\frac{1}{2m} \Delta_N \otimes I - \Lambda e H_{I}^{EM}(\partial) + \Lambda^\beta I \otimes H_{EM},$$

where

$$H_{I}^{EM}(\partial) = \frac{1}{m} \sum_{j=1}^{N} \sum_{\mu=1}^{d} P^j_\mu \otimes A_\mu(\partial).$$

Put

$$H_{EM}^\beta(\Lambda) = -\frac{1}{2m} \Delta_N \otimes I + \Lambda^\beta I \otimes H_{EM}.$$

**Theorem 4.1 ([1])** Let $V \in \mathcal{M}_{\pm}(N)$. Then the operator $H_{EM}^\beta(\partial, \Lambda) + V \otimes I$ is self-adjoint on $D(H_{EM}^\beta(\Lambda))$ and bounded from below. Moreover it is essentially self-adjoint on any core for $H_{EM}^\beta(\Lambda)$.

We define a unitary operator by

$$S(\epsilon) = \exp \left( -ie \left( \sum_{j=1}^{N} \sum_{\mu=1}^{d} \frac{1}{m} P^j_\mu \otimes \Pi_\mu \left( \frac{\partial}{\omega^2} \right) \right) \right),$$

where we put $\Pi_\mu(0, f) = I \otimes \Pi_\mu(f)$.

**Lemma 4.2 ([1])** Let $V \in \mathcal{M}_{\pm}(N)$. Then the unitary operator $S(\epsilon)$ maps $D(H_{EM}^\beta(\partial, \Lambda))$ onto itself with

$$S(\Lambda^{1-\beta}e)^{-1}(H_{EM}^\beta(\partial, \Lambda) + V \otimes I)S(\Lambda^{1-\beta}e)$$

$$= -\left( \frac{1}{2m} + \Lambda^{2-\beta} \frac{e^2}{2M} \right) \Delta_N \otimes I + \Lambda^\beta I \otimes H_{EM} + V_{\beta}(\partial, \Lambda),$$

(4.2)

where

$$\frac{1}{2M} = \frac{d-1}{d} \left( \frac{1}{m} \right) \left\| \frac{\partial}{\omega} \right\|^2_{L^2(\mathbb{R}^d)}, \quad V_{\beta}(\partial, \Lambda) = S(\Lambda^{1-\beta}e)^{-1}(V \otimes I)S(\Lambda^{1-\beta}e).$$
To obtain the scaling limit of the case of $\beta = 1$, we need to fix a $dN \times dN$-matrix $T$ so that

$$
T \left( \begin{array}{ccc}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1
\end{array} \right) T^{-1} = \left( \begin{array}{ccc}
N & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & 0 & \vdots \\
0 & 0 & \ldots & 0
\end{array} \right).
$$

Then, for a multiplication operator $V$ in $L^2(\mathbb{R}^{dN})$, we put

$$
V_{\text{eff}}^\delta(x) = \left(2\pi C_N(\hat{\beta})\right)^{-\frac{d}{2}} \int_{\mathbb{R}^d} dy V(T^{-1}(y,(Tx)_1,\ldots,(Tx)_N)) e^{-\frac{\| \hat{\beta}(x) - \gamma \|^2}{2C_N(\hat{\beta})}},
$$

where

$$
C_N(\hat{\beta}) = \frac{d - 1}{2d} \left( \frac{\epsilon}{m} \right)^2 \int_{\mathbb{R}^{dN}} dk \frac{\| \hat{\beta}(k) \|^2}{\omega(k)^3},
$$

and $(Tx)_j \in \mathbb{R}^d, j = 1,\ldots,N$, denotes the $j$-th element of $Tx \in \mathbb{R}^{dN}$. In the case of $\beta = 1$, the following proposition is well known.

**Proposition 4.3** ([1,6,7], $\beta = 1$) Let $V \in \mathcal{M}_\pm(N)$ with

$$
|V|_{\text{eff}}^\delta(x) < \infty, a.e.x \in \mathbb{R}^{dN}, \quad |V|_{\text{eff}}^\delta \in L^1_\text{loc}(\mathbb{R}^{dN}).
$$

Then $-\frac{1}{2m} \Delta_N + V_{\text{eff}}^\delta$ is self-adjoint on $D(-\Delta_N)$ with, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$
s - \lim_{\Lambda \to \infty} \left( H_{\text{EM}}^\beta(\hat{\beta},\Lambda) + V \otimes I + \Lambda^{2-\beta} \frac{e^2}{2M} \Delta_N \otimes I - z \right)^{-1}
= S(e) \left\{ \left( -\frac{1}{2m} \Delta_N + V_{\text{eff}}^\delta - z \right)^{-1} \otimes P_{EM} \right\} S(e)^{-1},
$$

where $P_{EM}$ is the projection operator onto the subspace $\{k\Omega_W | k \in \mathbb{C} \} \subset \mathcal{F}_W$.

### 4.2 The case of $\beta = 2$

Put

$$
\tilde{H}_{\text{EM}}^\beta(\Lambda) = -\left( \frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N \otimes I + \Lambda^\beta I \otimes H_{\text{EM}}.
$$
Lemma 4.4 Let $V \in \mathcal{M}_\pm(N)$. Then, for any $\epsilon > 0$, there exists $\Lambda_0$ and $b(\epsilon) > 0$ so that, for all $\Lambda > \Lambda_0$, $D(V_{\beta}(\hat{\omega}, \Lambda)) \supset D(\tilde{H}_{EM}^0(\Lambda))$ with

$$
\|V_{\beta}(\hat{\omega}) \Phi\|_{\mathcal{L}_N} \leq \epsilon \|\tilde{H}_{EM}^0(\Lambda) \Phi\|_{\mathcal{L}_N} + b(\epsilon) \|\Phi\|_{\mathcal{L}_N}, \Phi \in D(\tilde{H}_{EM}^0(\Lambda)).
$$

(4.3)

Moreover, for $z \in \mathbb{C} \setminus [0, \infty)$,

$$
\begin{align*}
S - \lim_{\Lambda \to \infty} V_{\beta}(\hat{\omega}, \Lambda) \left( \frac{\tilde{H}_{EM}^0(\Lambda) - z}{-\Delta_N} \right)^{-1} = & \left( V - \left( \frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N - z \right)^{-1} \otimes P_{EM}, \quad (4.4)
\end{align*}
$$

Proof: Since $V$ is infinitesimally small with respect to $-\Delta_N$ and $-\Delta_N$ commutes $S(\epsilon)$, one can derive (4.3). Put $\left( \frac{\tilde{H}_{EM}^0(\Lambda) - z}{-\Delta_N} \right)^{-1} = K_{\Lambda} \left( -\left( \frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N - z \right)^{-1} \otimes P_{EM} = K_{\infty}$ and $S(\Lambda^{1-\epsilon}) = S_{\Lambda}$. Note that, for $\beta > 1$,

$$
S - \lim_{\Lambda \to \infty} S(\Lambda^{1-\epsilon}) = I,
$$

By (4.3), for any $\epsilon > 0$, taking sufficiently large $\Lambda > 0$, we have

$$
\begin{align*}
\|V_{\beta}(\hat{\omega}, \Lambda)K_{\Lambda} \Phi - (V \otimes I)K_{\infty} \Phi\|_{\mathcal{L}_N} & \leq \epsilon \|\Delta_N(K_{\Lambda} - K_{\infty}) \Phi\|_{\mathcal{L}_N} + \epsilon \|\Delta_N(S_{\Lambda}K_{\infty} - K_{\infty}) \Phi\|_{\mathcal{L}_N} + b(\epsilon) \|\Delta_N(S_{\Lambda}K_{\infty} - K_{\infty}) \Phi\|_{\mathcal{L}_N} + b(\epsilon) \|\Delta_N(S_{\Lambda}^{-1} - I)(V \otimes I)K_{\infty} \Phi\|_{\mathcal{L}_N}.
\end{align*}
$$

Taking $\Lambda \to \infty$ on the both sides above, we have

$$
\lim_{\Lambda \to \infty} \|V_{\beta}(\hat{\omega}, \Lambda)K_{\Lambda} \Phi - (V \otimes I)K_{\infty} \Phi\|_{\mathcal{L}_N} \leq \epsilon \|(I \otimes I - I \otimes P_{EM}) \Phi\|_{\mathcal{L}_N}.
$$

Since $\epsilon > 0$ is arbitrary, (4.4) follows.

\[\square\]

Theorem 4.5 $(\beta = 2)$ Let $V \in \mathcal{M}_\pm(N)$. Then, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$
S - \lim_{\Lambda \to \infty} \left( \frac{\tilde{H}_{EM}^0(\Lambda) + V \otimes I - z}{-\Delta_N + V - z} \right)^{-1} = \left\{ -\left( \frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N + V - z \right\}^{-1} \otimes P_{EM}.
$$

(4.5)

Proof: By virtue of (4.2), we see that, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$
\left( \frac{\tilde{H}_{EM}^0(\Lambda) + V \otimes I - z}{-\Delta_N + V - z} \right)^{-1} = S(\Lambda^{1-\epsilon}) \left( \frac{\tilde{H}_{EM}^0(\Lambda) + V_{\beta}(\hat{\omega}, \Lambda) - z}{-\Delta_N + V - z} \right)^{-1} S(\Lambda^{1-\epsilon})^{-1},
$$

and the partial expectation of $V \otimes I$ with respect to $\Omega_W$ is $E_{\Omega_W}(V \otimes I) = V$. Hence, it follows (4.5) from Lemma 4.4 and Proposition 2.1 with the following correspondence:

$$
A = -\left( \frac{1}{2m} + \frac{e^2}{2M} \right) \Delta_N, B = H_{EM}, C(\Lambda) = V_{\beta}(\hat{\omega}, \Lambda), C = V \otimes I, G = \Omega_W.
$$

\[\square\]
4.3 The case of $1 < \beta < 2$, $2 < \beta$

For the case of $1 < \beta < 2$, by (4.2), we should subtract the term $-\Lambda^{2-\beta} \frac{e^4}{2M} \Delta_N \otimes I$ from the original Hamiltonian $H_{EM}^\beta(q, \Lambda)$, and for the case of $\beta > 2$, we do not need any renormalization. Hence, the similar argument of the cases of $\beta = 2$ and $\beta = 1$ gives asymptotic behaviors of $H_{EM}^\beta(q, \Lambda)$. See Fig. 6.2.

5 THE SPIN-BOSON MODEL

5.1 The spin-boson model

In this section we study the spin-boson model. The total Hamiltonian of the spin-boson model is defined as an operator acting in the Hilbert space

$$L_{SB} = \mathcal{C}^2 \otimes \mathcal{F}_{L^2(\mathbb{R}^d)} \cong \mathcal{F}_{L^2(\mathbb{R}^d)} \otimes \mathcal{F}_{L^2(\mathbb{R}^d)},$$

by

$$H_{SB}^\beta(\lambda, \Lambda) = \nu \sigma_1 + \Lambda \sigma_3 \otimes \left(a^\dagger(\lambda) + a(\lambda)\right) + \Lambda^\beta I \otimes H_{SB}.$$ 

Here $H_{SB} = d\Gamma_{L^2(\mathbb{R}^d)}(\omega)$, $\nu > 0$, $\Lambda \in L^2(\mathbb{R}^d)$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

In what follows, we assume that

$$\lambda, \frac{\lambda}{\sqrt{\omega}}, \frac{\lambda}{\omega} \in L^2(\mathbb{R}^d).$$

**Theorem 5.1** ([1]) The operator $H_{SB}(\lambda, \Lambda)$ is self-adjoint on $D(I \otimes H_{SB})$ and bounded from below. Moreover essentially self-adjoint on any core for $I \otimes H_{SB}$.

We define a unitary operator by

$$T(\lambda) = \begin{pmatrix} e^{+\{a^\dagger(q) - a(q)\}} & 0 \\ 0 & e^{-\{a^\dagger(q) - a(q)\}} \end{pmatrix} \equiv \begin{pmatrix} T_+(\lambda) & 0 \\ 0 & T_-(\lambda) \end{pmatrix}.$$ 

In the case of $\beta = 1$, following proposition is well known.
Proposition 5.2 ([1], $\beta = 1$) Let $F(\lambda) = \langle \Omega_{L^2(\mathbb{R}^d)}, T_+(\lambda) \Omega_{L^2(\mathbb{R}^d)} \rangle_{\mathcal{F}_{L^2(\mathbb{R}^d)}}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Then
\[
 s - \lim_{\lambda \to \infty} \left( H_{SB}^\beta(\lambda, \Lambda) - \Lambda^{2-\beta} E_{SB} - z \right)^{-1} = T(\lambda) \left\{ (\nu F(\lambda) \sigma_1 - z)^{-1} \otimes P_{SB} \right\} T(\lambda)^{-1}.
\]
Here $P_{SB}$ is the projection operator onto $\{ k\Omega | k \in \mathbb{C} \} \subset \mathcal{F}_{L^2(\mathbb{R}^d)}$ and $E_{SB} = - \| \frac{\lambda}{\sqrt{\omega}} \|_{L^2(\mathbb{R}^d)}^2$.

5.2 the case of $\beta = 2$

It is well known and easily checked that $T(\Lambda^{1-\beta})$ maps $D(I \otimes H_{SB})$ onto itself with
\[
 T^{-1}(\Lambda^{1-\beta}) H_{SB}^\beta(\lambda, \Lambda) T(\Lambda^{1-\beta}) = \nu \begin{pmatrix} 0 & T^2(\Lambda^{1-\beta}) \\ T^2(\Lambda^{1-\beta}) & 0 \end{pmatrix} + \Lambda^{\beta} I \otimes H_{SB} + \Lambda^{2-\beta} E_{SB}. \tag{5.1}
\]

Theorem 5.3 ($\beta = 2$) Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then
\[
 s - \lim_{\lambda \to \infty} \left( H_{SB}^\beta(\lambda, \Lambda) - z \right)^{-1} = (\nu \sigma_1 + E_{SB} - z)^{-1} \otimes P_{SB}. \tag{5.2}
\]

Proof: We see that, by (5.1),
\[
 T^{-1}(\Lambda^{1-\beta}) H_{SB}^\beta(\lambda, \Lambda) - z)^{-1} T(\Lambda^{1-\beta}) = \left\{ \nu \begin{pmatrix} 0 & T^2(\Lambda^{1-\beta}) \\ T^2(\Lambda^{1-\beta}) & 0 \end{pmatrix} + \Lambda^{\beta} I \otimes H_{SB} + \Lambda^{2-\beta} E_{SB} - z \right\}^{-1}.
\]
It is easily seen that $s - \lim_{\lambda \to \infty} T(\Lambda^{1-\beta}) = I$ and
\[
 s - \lim_{\lambda \to \infty} \nu \begin{pmatrix} 0 & T^2(\Lambda^{1-\beta}) \\ T^2(\Lambda^{1-\beta}) & 0 \end{pmatrix} (\Lambda^{2-\beta} E_{SB} + \Lambda^{\beta} I \otimes H_{SB} - z)^{-1} = [\nu \sigma_1 (E_{SB} - z)^{-1}] \otimes P_{SB}.
\]
Hence, with the following correspondence:
\[
 A = E_{SB}, B = H_{SB}, C(\Lambda) = \nu \begin{pmatrix} 0 & T^2(\Lambda^{1-\beta}) \\ T^2(\Lambda^{1-\beta}) & 0 \end{pmatrix}, C = \nu \sigma_1, G = \Omega_{L^2(\mathbb{R}^d)},
\]
one can easily check the conditions with respect to $C(\Lambda)$ and $C$ in section 2. Since the partial expectation of $\nu \sigma_1 \otimes I$ with respect to $\Omega_{L^2(\mathbb{R}^d)}$ is $E_{\Omega_{L^2(\mathbb{R}^d)}}(\nu \sigma_1 \otimes I) = \nu \sigma_1$, we get (5.2) by Proposition 2.1. \qed
5.3 The case of $1 < \beta < 2, 2 < \beta$

For the case of $1 < \beta < 2$, by (5.1), we should subtract the term $\Lambda^{2-\beta} E_{SB}$ from the original Hamiltonian $H_{SB}^{\beta}(\Lambda, \Lambda)$, and for the case of $\beta > 2$, we do not need any renormalization. Hence, the similar argument of the cases of $\beta = 2$ and $\beta = 1$ gives an asymptotic behaviors of $H_{SB}^{\beta}(\Lambda, \Lambda)$. See Fig 6.3.

6 CONCLUDING REMARKS

(1) In section 4, we studied the Pauli-Fierz model neglected the terms $A^2(\hat{\alpha}, \cdot)$. In [6,7], we studied the Pauli-Fierz Hamiltonian with the terms $A^2(\hat{\alpha}, \cdot)$. By the same method developed in [6,7], we can investigate the following scaling Hamiltonians:

\[
-\frac{1}{2m} \Delta_N \otimes I - \Lambda e H_{EM}^E(\hat{\alpha}) + \Lambda I \otimes H_{EM} + \frac{e^2 N}{2m} A^2(\hat{\alpha}, \cdot) + V \otimes I, \quad (6.1)
\]

\[
-\frac{1}{2m} \Delta_N \otimes I - \Lambda e H_{EM}^E(\hat{\alpha}) + \Lambda^2 I \otimes H_{EM} + \frac{e^2 N}{2m} A^2(\hat{\alpha}, \cdot) + V \otimes I. \quad (6.2)
\]

Introducing different renormalizations from those given in this paper, we can get effective Hamiltonians of (6.1) and (6.2).

(2) In the case of $0 < \beta < 1$, we need delicate discussions of asymptotic behaviors of unitary operators $U(\Lambda^{1-\beta} g), S(\Lambda^{1-\beta} e)$ and $T(\Lambda^{1-\beta} \Lambda)$ as $\Lambda \to \infty$. We omit the discussions.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Effective Hamiltonian</th>
<th>Unitary operator</th>
<th>Renormalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta &gt; 2$</td>
<td>$-\frac{1}{2m} \Delta_N + V$</td>
<td>$I$</td>
<td>0</td>
</tr>
<tr>
<td>$\beta = 2$</td>
<td>$-\frac{1}{2m} \Delta_N + g^2 V(\hat{\alpha}) + V$</td>
<td>$I$</td>
<td>0</td>
</tr>
<tr>
<td>$1 &lt; \beta &lt; 2$</td>
<td>$-\frac{1}{2m} \Delta_N + V$</td>
<td>$I$</td>
<td>$g^2 \Lambda^{2-\beta} V(\hat{\alpha})$</td>
</tr>
<tr>
<td>$\beta = 1$</td>
<td>$-\frac{1}{2m} \Delta_N + g^2 N \delta(\hat{\alpha}) + V$</td>
<td>$U(g)$</td>
<td>$g^2 \Lambda V(\hat{\alpha})$</td>
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</tbody>
</table>

Fig 6.1 $\beta$-coupling Nelson model $H_{N}^{\beta}(\hat{\alpha}, \Lambda)$
<table>
<thead>
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<th>Unitary operator</th>
<th>Renormalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta &gt; 2$</td>
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<td>$I$</td>
<td>0</td>
</tr>
<tr>
<td>$\beta = 2$</td>
<td>$-\left(\frac{1}{2m} + \frac{\varepsilon^2}{2M}\right)\Delta_N + V$</td>
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<td>0</td>
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<tr>
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<td>$I$</td>
<td>$-\Lambda^{2-\beta}\frac{\varepsilon^2}{2M}\Delta_N$</td>
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<tr>
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<td>$-\frac{1}{2m}\Delta_N + V_{\text{eff}}$</td>
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<td>$-\Lambda\frac{\varepsilon^2}{2M}\Delta_N$</td>
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Fig. 6.2 $\beta$-coupling Pauli-Fierz model $H_{EM}^\beta(\delta, \Lambda)$

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<th>Renormalization</th>
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<tr>
<td>$\beta &gt; 2$</td>
<td>$\nu\sigma_1$</td>
<td>$I$</td>
<td>0</td>
</tr>
<tr>
<td>$\beta = 2$</td>
<td>$\nu\sigma_1 + E_{SB}$</td>
<td>$I$</td>
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<tr>
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<td>$\Lambda E_{SB}$</td>
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Fig. 6.3 $\beta$-coupling spin-boson model $H_{SB}^\beta(\lambda, \Lambda)$

**REFERENCES**


