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# Strong and Weak Coupling Limits of Interaction Models of Quantum Fields and Particles

HIROSHIMA FUMIO
Hokkaido University

## 1 INTRODUCTION

Asymptotic behaviors of scaling Hamiltonians which describe interactions of particles and quantized fields are considered. In a mathematical formulation, interaction Hamiltonians of the particles and the quantized fields are described by the theory of self-adjoint operators acting in the tensor product of two Hilbert spaces over the complex field $\mathbb{C}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. We define a self-adjoint operator $\mathbf{H}$ acting in the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$, by

$$\mathbf{H}=H_1\otimes I+\alpha H_{int}+I\otimes H_2.$$

Here $H_1$ and $H_2$ are self-adjoint operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $H_{int}$ is a symmetric operator in $\mathcal{H}$ and $\alpha\in\mathbb{R}$ is a coupling constant. Then, for the given self-adjoint operator $\mathbf{H}$, we define "$\beta$-coupling Hamiltonian, $\mathbf{H}_\beta(\Lambda)$", by

$$\mathbf{H}_\beta(\Lambda)=H_1\otimes I+\Lambda\alpha H_{int}+\Lambda^\beta I\otimes H_2,\quad 1\leq\beta. \tag{1. 1}$$

Introducing a renormalization $E_\beta(\Lambda)$ which goes to infinity or minus infinity as $\Lambda\to\infty$ in some sense, we want to investigate the following asymptotic behaviors

$$s-\lim_{\Lambda\to\infty}e^{-it\left(\mathbf{H}_\beta(\Lambda)-E_\beta(\Lambda)\right)}=\mathbf{U}\left(e^{-itH_{eff}}\otimes\mathbf{P}\right)\mathbf{U}^{-1},\quad t\in\mathbb{R}. \tag{1. 2}$$

Here $H_{eff}$ is a self-adjoint operator in $\mathcal{H}_1$, which is called "effective Hamiltonian", $\mathbf{U}$ is a unitary operator in $\mathcal{H}$ and $\mathbf{P}$ a projection operator onto a one-dimensional subspace in $\mathcal{H}_2$. It seems to be useful to readers to collect some background ingredient. Motivation of this paper is [1] and [3]. In [1], in order to give an interpretation of a physical phenomenon "Lamb shift" without formal perturbation theory, A.Arai elaborates a scaling limit of the Pauli-Fierz model. The scaling limit corresponds to the case $\beta = 1$ in (1.1). In [3], E.B.Davies studies a scaling limit of the Nelson model to derive a Schrödinger Hamiltonian (effective Hamiltonian) with a scalar potential. The scaling limit corresponds to the case $\beta = 2$ in (1.1). In this paper, we deal with the Nelson model [2,3,4,5,8,10], the Pauli-Fierz model [1,6,7,8,9,10] and the spin-boson model [1]. Thus considering scaling limits as in (1.2) for these models is an extension of those considered in [1,2,3,6,7,10]. We organize this paper as follows. In section 2, we overview an abstract theory of a scaling limit of self-adjoint operators. In section 3,4 and 5, we study the Nelson model, the Pauli-Fierz model and the spin-boson model,respectively. In section 6, we give some remarks.

# 2 FUNDAMENTAL FACTS

## 2.1 An abstract Boson Fock space

In this subsection we define an abstract Boson Fock space and basic notations. For a separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$ , we denote the scalar product by $< f, g >_{\mathcal{H}}$ and the associated norm by $||f||_{\mathcal{H}}$, where the scalar product is linear in $g$ and antilinear in $f$. For the tempered distributions $f$ and $g$, the notation $\bar{f}$ denotes the complex conjugate of $f$, and $\hat{f}$ (resp.$\check{g}$) the Fourier transform of $f$ (resp.the inverse Fourier transform of $g$). We denote the domain of an operator $A$ by $D(A)$. The Boson Fock space over the Hilbert space $\mathcal{H}$ is defined by

$$\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} [\otimes_s^n \mathcal{H}],$$

where $\otimes_s^n \mathcal{H}, n \geq 1$, denotes the $n$-fold symmetric tensor product of $\mathcal{H}$, and $\otimes_s^0 \mathcal{H} = \mathbb{C}$. Define $\Omega_{\mathcal{H}} = \{1, 0, 0, ..\}$. Let the annihilation and creation operators in the Boson Fock space denoted by $a_{\mathcal{H}}(f), f \in \mathcal{H}$ and $a_{\mathcal{H}}^{\dagger}(g), g \in \mathcal{H}$, respectively. It is well known that

$$\mathcal{F}_{\mathcal{H}}^{\infty} \equiv \mathbf{L}\left\{a_{\mathcal{H}}^{\dagger}(f_1)...a_{\mathcal{H}}^{\dagger}(f_n)\Omega, \Omega | f_j \in \mathcal{H}, j = 1, ..., n, n \geq 1\right\}$$

is dense in $\mathcal{F}_{\mathcal{H}}$, where $\mathbf{L}$ denotes the linear hull of the vectors in $\{...\}$. The annihilation and the creation operators in the Boson Fock space satisfy the following canonical commutation relations on $\mathcal{F}_{\mathcal{H}}^{\infty}$:

$$\begin{aligned}[][a_{\mathcal{H}}(f), a_{\mathcal{H}}^{\dagger}(g)] &= <\bar{f}, g>_{\mathcal{H}}, \\ [a_{\mathcal{H}}^{\sharp}(f), a_{\mathcal{H}}^{\sharp}(g)] &= 0,\end{aligned}$$

where $a_{\mathcal{H}}^{\sharp}$ means $a_{\mathcal{H}}$ or $a_{\mathcal{H}}^{\dagger}$. Let $h$ be a self-adjoint operator in $\mathcal{H}$. Define $d\Gamma_{\mathcal{H}}(h)$ by

$$d\Gamma_{\mathcal{H}}(h)\Omega = 0,$$
$$d\Gamma_{\mathcal{H}}(h)a_{\mathcal{H}}^{\dagger}(f_1)...a_{\mathcal{H}}^{\dagger}(f_n)\Omega_{\mathcal{H}} = \sum_{j=1}^{n} a_{\mathcal{H}}^{\dagger}(f_1)...a_{\mathcal{H}}^{\dagger}(hf_j)...a_{\mathcal{H}}^{\dagger}(f_n)\Omega_{\mathcal{H}}, f_j \in D(h).$$

Then $d\Gamma_{\mathcal{H}}(h)$ is essentially self-adjoint. Let us use the same notation as $d\Gamma_{\mathcal{H}}(h)$ for its self-adjoint extension.

## 2.2 An abstract theory of a scaling limit

We overview an abstract theory of a scaling limit of self-adjoint operators acting in a tensor product Hilbert space established in [1] with a little modification. Let $\mathcal{K}$ be a Hilbert space and put $\mathcal{X} = \mathcal{H} \otimes \mathcal{K}$. Suppose that an operator, $A$ (resp.$B$), is a nonnegative self-adjoint operator in $\mathcal{H}$ (resp.$\mathcal{K}$) and $\mathrm{Ker}B = \{kG | k \in \mathbb{C}, ||G||_{\mathcal{X}} = 1\}$. Set the projection operator onto $\mathrm{Ker}B$ by $P_B$. We suppose that a family of self-adjoint operators, $\{C_\Lambda\}_{\Lambda>0}$, in $\mathcal{X}$ admits the following conditions:

**(1)** For any $\epsilon > 0$, there exists $\Lambda_0$ so that, for all $\Lambda > \Lambda_0$, $D(C_\Lambda) \supset D(A \otimes I + \Lambda I \otimes B)$ with

$$||C_\Lambda \Phi||_{\mathcal{X}} \leq \epsilon ||(A \otimes I + \Lambda I \otimes B)\Phi|| + b(\epsilon) ||\Phi||_{\mathcal{X}}, \Phi \in D(A \otimes I) \cap D(I \otimes B),$$

where $b(\epsilon) > 0$ is a constant independent of $\Lambda > \Lambda_0$.

**(2)** There exists a symmetric operator $C$ in $\mathcal{X}$ so that $D(C) \supset D(A) \otimes \mathrm{Ker}B$ and, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$s - \lim_{\Lambda \to \infty} C_\Lambda (A \otimes I + \Lambda I \otimes B - z)^{-1} = C\left\{(A - z)^{-1} \otimes P_B\right\}.$$

We define an operator $E_G(C)$ with the domain $D(E_G(C)) = D(A)$ by

$$\langle f, E_G(C) g \rangle_{\mathcal{H}} = \langle f \otimes G, C(g \otimes G) \rangle_{\mathcal{X}}, \quad f \in \mathcal{H}, g \in D(A).$$

We call $E_G(C)$ "the partial expectation of $C$ with respect to $G$". Set

$$K_{eff} = A + E_G(C).$$

The following proposition is fundamental in this paper.

**Proposition 2.1 ([1,Theorem 2.1])** *Let operators $A$,$B$,$C_\Lambda$, and $C$ be as above. Then*

**(1)** *For $\Lambda > \Lambda_0$, $K_\Lambda = A \otimes I + \Lambda I \otimes B + C_\Lambda$ is self-adjoint on $D(A \otimes I) \cap D(I \otimes B)$ and* uniformly *bounded from below. Moreover $E_G(C)$ is infinitesimally small with respect to $A$, i.e., $K_{eff}$ is self-adjoint on $D(A)$.*

**(2)** *For $z \in \mathbb{C} \setminus \mathbb{R}$*

$$s - \lim_{\Lambda \to \infty} (K_\Lambda - z)^{-1} = (K_{eff} - z)^{-1} \otimes P_B. \tag{2. 1}$$

Finally we note a fundamental fact.

**Proposition 2.2** *Let $K_\Lambda$ and $K_{eff}$ satisfy (2.1). Then*

$$s - \lim_{\Lambda \to \infty} e^{-itK_\Lambda} = e^{-itK_\infty} \otimes P_B.$$

**Proof:** See [1,Theorem 2.2] □

By Proposition 2.2, it is enough to show strong resolvent limits of $\beta$-coupling Hamiltonian to investigate (1.2).

# 3 THE NELSON MODEL

## 3.1 The Nelson model

In this section, we consider the Nelson Hamiltonian with an ultraviolet cut-off function $\hat{\varrho}$ and with a finite number of nonrelativistic particles. Fix the number of the nonrelativistic particles $N$. For the mathematical generality, suppose that the dimension of the space in which the nonrelativistic particles move is $d \geq 1$. (This assumption remains throughout this paper.) We use the following identification

$$\mathcal{F}_N \equiv L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}_{L^2(\mathbb{R}^d)} \cong L^2(\mathbb{R}^{dN}; \mathcal{F}_{L^2(\mathbb{R}^d)}).$$

For notational simplicity, we write the annihilation or creation operators by $a^\sharp(f)$ instead of $a^\sharp_{L^2(\mathbb{R}^d)}(f)$ in sections 3 and 5. We define a time-zero scalar field $\phi(\hat{f})$ by

$$\phi(\hat{f}) = \frac{1}{\sqrt{2}}\left\{a^\dagger\left(\frac{\hat{f}}{\sqrt{\omega}}\right) + a\left(\frac{\overline{\hat{f}}}{\sqrt{\omega}}\right)\right\}.$$

Here $\omega = \omega(k) = \sqrt{k^2+\mu^2}$, $\mu \geq 0$. In this section we require that $\varrho$ is a real valued even function, $\varrho(k) = \varrho(-k)$, with

$$\frac{\hat{\varrho}}{\omega\sqrt{\omega}}, \frac{\hat{\varrho}}{\omega}, \frac{\hat{\varrho}}{\sqrt{\omega}} \in L^2(\mathbb{R}^d). \tag{3. 1}$$

For each $x = (x^1, ..., x^N) \in \mathbb{R}^{dN}, x^j \in \mathbb{R}^d, j = 1, ..., N$, we set

$$\tilde{\hat{\varrho}}(x) = \frac{1}{\sqrt{(2\pi)^d}}\sum_{j=1}^{N}\hat{\varrho}(k)e^{-ikx^j}.$$

We define

$$H_I(\hat{\varrho}) \equiv \phi\left(\tilde{\hat{\varrho}}(\cdot)\right).$$

For the multiplication operator $\omega$ in $L^2(\mathbb{R}^d)$ with the maximal domain, we set $d\Gamma_{L^2(\mathbb{R}^d)}(\omega) \equiv H_b$. Define an operator in $\mathcal{F}_N$ by

$$H_N^\beta(\hat{\varrho}, \Lambda) = -\frac{1}{2m}\Delta_N \otimes I - \Lambda g H_I(\hat{\varrho}) + \Lambda^\beta I \otimes H_b, \quad 1 \leq \beta < \infty,$$

where $g \in \mathbb{R}$ is a coupling constant, $m > 0$ a mass of the nonrelativistic particles, $\Delta_N$ the Laplacian in $L^2(\mathbb{R}^{dN})$ and $\Lambda > 0$ a scaling parameter. Moreover we put a decoupled Hamiltonian $H_{\beta,N}(\Lambda)$ by

$$H_N^\beta(\Lambda) = -\frac{1}{2m}\Delta_N \otimes I + \Lambda^\beta I \otimes H_b.$$

We define a class of the set of multiplication operators in $L^2(\mathbb{R}^{dN})$. A multiplication operator $V$ is in a class, $\mathcal{M}_\pm(N)$, if and only if $V$ is infinitesimally small with respect to $-\Delta_N$.

**Proposition 3.1** ([2]) *For $\Lambda > 0$ and $V \in \mathcal{M}_\pm(N)$, $H_N^\beta(\hat{\varrho}, \Lambda) + V \otimes I$ is self-adjoint on $D(H_N^\beta(\Lambda))$ and bounded from below. Moreover it is essentially self-adjoint on any core for $H_N^\beta(\Lambda)$.*

In the case of $\beta = 2$, following proposition is well known.

**Proposition 3.2 ([2,3],$\beta = 2$)** *Let $V \in \mathcal{M}_{\pm}(N)$. Then*

$$s - \lim_{\Lambda \to \infty} e^{-it(H_N^\beta(\hat{\varrho},\Lambda)+V\otimes I)} = e^{-it(-\frac{1}{2m}\Delta_N + V + g^2 V(\hat{\varrho}))} \otimes P_N.$$

*Here $P_N$ is the projection operator onto the subspace in $\mathcal{F}$, spanned by the vector $\Omega_{L^2(\mathbb{R}^d)}$.*

## 3.2 The case of $\beta = 1$

Put $C_0^\infty(\mathbb{R}^{dN})\hat{\otimes}\mathcal{F}_{L^2(\mathbb{R}^d)}^\infty \equiv \mathcal{F}_N^\infty$, where $\hat{\otimes}$ denotes the algebraic tensor product. We perform a unitary transformation

$$\mathcal{U}(g) = \exp\left(\frac{g}{\sqrt{2}}\left\{a^\dagger\left(\frac{\tilde{\hat{\varrho}}(\cdot)}{\omega\sqrt{\omega}}\right) - a\left(\frac{\overline{\tilde{\hat{\varrho}}(\cdot)}}{\omega\sqrt{\omega}}\right)\right\}\right)$$

with the following result:

**Proposition 3.3** *The unitary operator $\mathcal{U}(\Lambda^{1-\beta}g)$ maps $\mathcal{F}_N^\infty$ into $D(H_N^\beta(\hat{\varrho},\Lambda)))$ with*

$$\begin{aligned}
&\mathcal{U}(\Lambda^{1-\beta}g)^{-1}(H_N^\beta(\hat{\varrho},\Lambda) + V\otimes I)\mathcal{U}(\Lambda^{1-\beta}g) \\
&= \frac{1}{2m}\sum_{j=1}^N \left(\mathbf{p}^j \otimes I - g\Lambda^{1-\beta}\phi_j\right)^2 + g^2\Lambda^{2-\beta}V(\hat{\varrho})\otimes I + \Lambda^\beta I \otimes H_b + V\otimes I, \quad (3.\ 2)
\end{aligned}$$

*on $\mathcal{F}_N^\infty$, where $\mathbf{p}^j = (-i\frac{\partial}{\partial x_1^j}, ..., -i\frac{\partial}{\partial x_d^j})$, $\phi_j = (\phi\left(\hat{\varrho}_1^j(\cdot)\right), ..., \phi\left(\hat{\varrho}_d^j(\cdot)\right))$, $j = 1, ..., N$, and*

$$\begin{aligned}
\hat{\varrho}_\mu^j(x) &= \hat{\varrho}_\mu^j(x,k) = \frac{1}{\sqrt{(2\pi)^d}}\frac{\hat{\varrho}(k)e^{-ikx^j}k_\mu}{\omega(k)}, \mu = 1, ..., d, \\
V(\hat{\varrho}) &= V(\hat{\varrho}, x) = -\frac{1}{2(2\pi)^d}\left\|\sum_{j=1}^N \frac{\hat{\varrho}e^{-ikx^j}}{\omega}\right\|_{L^2(\mathbb{R}^d)}^2.
\end{aligned}$$

*Moreover, for sufficiently large $\Lambda > 0$, the right hand side (R.H.S.) of (3.2) is self-adjoint on $D(H_N^\beta(\Lambda))$ and the equation (3.2) can be extended to the equation on $D(H_N^\beta(\Lambda))$.*

Proposition 3.3 implies that the following equation holds, for $V \in \mathcal{M}_{\pm}(N)$ and sufficiently large $\Lambda > 0$;

$$\begin{aligned}
&\mathcal{U}(\Lambda^{1-\beta}g)^{-1}\left(H_N^\beta(\hat{\varrho},\Lambda) - g^2\Lambda^{2-\beta}V(\hat{\varrho})\otimes I + V\otimes I\right)\mathcal{U}(\Lambda^{1-\beta}g) \\
&= \frac{1}{2m}\sum_{j=1}^N \left(\mathbf{p}^j \otimes I - g\Lambda^{1-\beta}\phi_j\right)^2 + V\otimes I + \Lambda^\beta I \otimes H_b. \quad (3.\ 3)
\end{aligned}$$

In this subsection we set $\beta = 1$. Then we define a symmetric operator $Q(\hat{\varrho})$, which is independent of $\Lambda$, by

$$R.H.S.\ of(3.3) = H_N^\beta(\Lambda) + Q(\hat{\varrho}).$$

**Lemma 3.4** *Let $V \in \mathcal{M}_\pm(N)$. Then, for any $\epsilon > 0$, there exists $\Lambda_0$ and $b(\epsilon) > 0$ so that, for all $\Lambda > \Lambda_0$, $D(Q(\hat{\varrho})) \supset D(H_N^\beta(\Lambda))$ with*

$$\|Q(\hat{\varrho})\Phi\|_{\mathcal{F}_N} \le \epsilon \left\|H_N^\beta(\Lambda)\Phi\right\|_{\mathcal{F}_N} + b(\epsilon)\|\Phi\|_{\mathcal{F}_N}, \Phi \in D(H_F). \tag{3. 4}$$

*Moreover $D(Q(\hat{\varrho})) \supset D(-\Delta_N)\hat{\otimes}\mathrm{Ker}H_b$ with, for $z \in \mathbb{C} \setminus [0,\infty)$,*

$$s - \lim_{\Lambda\to\infty} Q(\hat{\varrho})\left(H_N^\beta(\Lambda) - z\right)^{-1} = Q(\hat{\varrho})\left[\left(-\frac{1}{2m}\Delta_N - z\right)^{-1} \otimes P_N\right]. \tag{3. 5}$$

**Proof:** The proof of (3.4) follows from fundamental estimates with respect to $a^\sharp$ and $H_b$. By (3.4), for any $\epsilon > 0$, taking sufficiently large $\Lambda > 0$, we see that

$$\begin{aligned}
&\left\|Q(\hat{\varrho})\left\{\left(H_N^\beta(\Lambda) - z\right)^{-1} - \left(-\frac{1}{2m}\Delta_N - z\right)^{-1} \otimes P_N\right\}\Phi\right\|_{\mathcal{F}_N}\\
&\le \epsilon\left\|H_N^\beta(\Lambda)\left\{\left(H_N^\beta(\Lambda) - z\right)^{-1} - \left(-\frac{1}{2m}\Delta_N - z\right)^{-1} \otimes P_N\right\}\Phi\right\|_{\mathcal{F}_N}\\
&+b(\epsilon)\left\|\left\{\left(H_N^\beta(\Lambda) - z\right)^{-1} - \left(-\frac{1}{2m}\Delta_N - z\right)^{-1} \otimes P_N\right\}\Phi\right\|_{\mathcal{F}_N}.
\end{aligned}$$

Taking $\Lambda \to \infty$ on the both sides above, we have

$$\begin{aligned}
&\lim_{\Lambda\to\infty}\left\|Q(\hat{\varrho})\left\{\left(H_N^\beta(\Lambda) - z\right)^{-1} - \left(-\frac{1}{2m}\Delta_N - z\right)^{-1} \otimes P_N\right\}\Phi\right\|_{\mathcal{F}_N}\\
&\le \epsilon\|(I \otimes I - I \otimes P_N)\Phi\|_{\mathcal{F}_N}.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, (3.5) follows. □

**Theorem 3.5** ($\beta = 1$) *Let $V \in \mathcal{M}_\pm(N)$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Put $\omega_0 = \omega_0(k) = |k|$ and*

$$\delta(\hat{\varrho}) = \frac{1}{2(2\pi)^d}\left\|\frac{\hat{\varrho}\omega_0}{\omega\sqrt{\omega}}\right\|^2_{L^2(\mathbb{R}^d)}.$$

*Then*

$$\begin{aligned}
&s - \lim_{\Lambda\to\infty}\left(H_N^\beta(\hat{\varrho},\Lambda) - g^2\Lambda^{2-\beta}V(\hat{\varrho}) \otimes I + V \otimes I - z\right)^{-1}\\
&= \mathcal{U}(g)\left\{\left(-\frac{1}{2m}\Delta_N + g^2N\delta(\hat{\varrho}) + V - z\right)^{-1} \otimes P_N\right\}\mathcal{U}(g)^{-1}.
\end{aligned} \tag{3. 6}$$

**Proof:** From (3.3) it follows that

$$\begin{aligned}&\left(H_N^\beta(\hat{\varrho},\Lambda) - g^2\Lambda^{2-\beta}V(\hat{\varrho})\otimes I + V\otimes I - z\right)^{-1}\\ &= \mathcal{U}(\Lambda^{1-\beta}g)\left(H_N^\beta(\Lambda) + Q(\hat{\varrho}) - z\right)^{-1}\mathcal{U}(\Lambda^{1-\beta}g)^{-1}.\end{aligned}$$

By the fact that $\mathcal{U}(\Lambda^{1-\beta}g)$ is independent of $\Lambda$, it is enough to show that

$$s-\lim_{\Lambda\to\infty}\left(H_N^\beta(\Lambda) + Q(\hat{\rho}) - z\right)^{-1} = \left(-\frac{1}{2m}\Delta_N + g^2N\delta(\hat{\varrho}) + V - z\right)^{-1}\otimes P_N.$$

Since the partial expectation of $Q(\hat{\varrho})$ with respect to $\Omega_{L^2(\mathbb{R}^d)}$ is

$$\begin{aligned} E_{\Omega_{L^2(\mathbb{R}^d)}}(Q(\hat{\varrho})) &= g^2\sum_{j=1}^{N}\sum_{\mu=1}^{3}\frac{1}{2}\left\|\frac{\hat{\varrho}_\mu^j(\cdot)}{\sqrt{\omega}}\right\|_{L^2(\mathbb{R}^d)}^2 + V\\ &= g^2N\delta(\hat{\varrho}) + V,\end{aligned}$$

it follows (3.6) from Lemma 3.4 and Proposition 2.1 with the following correspondence :

$$A = -\frac{1}{2m}\Delta_N, \quad B = H_b, \quad C_\Lambda = C = Q(\hat{\varrho}), \quad G = \Omega_{L^2(\mathbb{R}^d)}.$$

□

## 3.3 The case of $1 < \beta < 2$, $2 < \beta$

First we study the case of $1 < \beta < 2$. We put the R.H.S. of (3.3) by

$$R.H.S.of\ (3.3) = H_N^\beta(\Lambda) + Q^1(\hat{\varrho},\Lambda). \tag{3. 7}$$

Similar to (3.4) and (3.5), one can see that, for $V \in \mathcal{M}_\pm(N)$ and any $\epsilon > 0$, there exists $\Lambda_0$ and $b(\epsilon) > 0$ so that, for all $\Lambda > \Lambda_0$, $D(Q^1(\hat{\varrho},\Lambda)) \supset D(H_N^\beta(\Lambda))$ with

$$\left\|Q^1(\hat{\varrho},\Lambda)\Phi\right\|_{\mathcal{F}_N} \le \epsilon\left\|H_N^\beta(\Lambda)\Phi\right\|_{\mathcal{F}_N} + b(\epsilon)\left\|\Phi\right\|_{\mathcal{F}_N}, \Phi \in D(H_N^\beta(\Lambda)). \tag{3. 8}$$

Moreover $D(Q^1(\hat{\varrho},\Lambda)) \supset D(-\Delta_N)\hat{\otimes}\mathrm{Ker}H_b$ and, for $z \in \mathbb{C}\setminus\mathbb{R}$,

$$s-\lim_{\Lambda\to\infty} Q^1(\hat{\varrho},\Lambda)\left(H_N^\beta(\Lambda) - z\right)^{-1} = \left[V\left(-\frac{1}{2m}\Delta_N - z\right)^{-1}\right]\otimes P_N. \tag{3. 9}$$

Note that, for $\beta > 1$,

$$s-\lim_{\Lambda\to\infty}\mathcal{U}\left(\frac{g}{\Lambda^{\beta-1}}\right) = I. \tag{3. 10}$$

Hence we prove the following theorem

**Theorem 3.6** $(1 < \beta < 2\ )$ *Let* $V \in \mathcal{M}_{\pm}(N)$*,* $z \in \mathbb{C} \setminus \mathbb{R}$*. Then*

$$s - \lim_{\Lambda \to \infty} \left( H_N^{\beta}(\hat{\varrho}, \Lambda) - g^2 \Lambda^{2-\beta} V(\hat{\varrho}) \otimes I + V \otimes I - z \right)^{-1} = \left( -\frac{1}{2m} \Delta_N + V - z \right)^{-1} \otimes P_N.$$

**Proof:** Since the partial expectation of $V \otimes I$ with respect to $\Omega_{L^2(\mathbb{R}^d)}$ is $E_{\Omega_{L^2(\mathbb{R}^d)}}(V \otimes I) = V$, from (3.8), (3.9), (3.10) and Proposition 2.1, theorem follows with the following correspondence:

$$A = -\frac{1}{2m} \Delta_N, \quad B = H_b, \quad C_\Lambda = Q^1(\hat{\varrho}, \Lambda), \quad C = V \otimes I \quad G = \Omega_{L^2(\mathbb{R}^d)}.$$

□

Secondly we study the case of $2 < \beta$. In this case, note that we do not need to subtract the renormalization $\Lambda^{2-\beta} V(\hat{\varrho}) \otimes I$. Put the R.H.S. of (3.2) by

$$R.H.S. of\ (3.2) = H_N^{\beta}(\Lambda) + Q^2(\hat{\varrho}, \Lambda).$$

By the same argument as that of the case of $1 < \beta < 2$ with $Q^1(\Lambda, \hat{\varrho})$ replaced by $Q^2(\Lambda, \hat{\varrho})$, one can easily prove the following theorem.

**Theorem 3.7** $(2 < \beta)$ *Let* $V \in \mathcal{M}_{\pm}(N)$ *and* $z \in \mathbb{C} \setminus \mathbb{R}$*. Then*

$$s - \lim_{\Lambda \to \infty} \left( H_N^{\beta}(\hat{\varrho}, \Lambda) + V \otimes I - z \right)^{-1} = \left( -\frac{1}{2m} \Delta_N + V - z \right)^{-1} \otimes P_N.$$

# 4 THE PAULI-FIERZ MODEL

## 4.1 The Pauli-Fierz model

In this section, we study the Pauli-Fierz model in quantum electrodynamics with the dipole approximation. Let

$$\mathcal{W} = \underbrace{L^2(\mathbb{R}^d) \oplus \ldots \oplus L^2(\mathbb{R}^d)}_{d-1}.$$

For $(0 \oplus \ldots . \underbrace{f}_{the\ r-th} \ldots \oplus 0) \in \mathcal{W}$, we set $a_{\mathcal{W}}^{\sharp}(0 \oplus \ldots . \underbrace{f}_{the\ r-th} \ldots \oplus 0) = a^{\sharp(r)}(f)$. We write

$$a_{\mathcal{W}}^{\sharp(r)}(g) = \int a_{\mathcal{W}}^{\sharp(r)}(k) g(k) dk, \quad r = 1, \ldots, d-1.$$

Let $e^r : \mathbb{R}^d \to \mathbb{R}^d, r = 1, ..., d-1$, be measurable functions so that

$$(1) e^r(k) \cdot k = 0, \quad r = 1, ..., d-1, \quad (2) e^r(k) \cdot e^s(k) = \delta_{rs}.$$

We denote the $\mu$-th component of $e^r$ by $e^r_\mu, \mu = 1, ..., d$. The quantized smeared radiation field $A_\mu(\hat{f}, x)$ with $f$ in the Coulomb gauge, and the conjugate momentum $\Pi_\mu(\hat{f}, x)$, $\mu = 1, ..., d, x \in \mathbb{R}^d$, are defined by

$$A_\mu(\hat{f}, x) = \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \int \left\{ a^{\dagger(r)}_{\mathcal{W}}(k) \frac{e^r_\mu(k) \tilde{\hat{f}}(k) e^{-ikx}}{\sqrt{\omega(k)}} + a^{(r)}_{\mathcal{W}}(k) \frac{e^r_\mu(k) \hat{f}(k) e^{ikx}}{\sqrt{\omega(k)}} \right\} dk,$$

$$\Pi_\mu(\hat{f}, x) = \frac{i}{\sqrt{2}} \sum_{r=1}^{d-1} \int \left\{ a^{\dagger(r)}_{\mathcal{W}}(k) \sqrt{\omega(k)} e^r_\mu(k) \tilde{\hat{f}}(k) e^{-ikx} - a^{(r)}_{\mathcal{W}}(k) \sqrt{\omega(k)} e^r_\mu(k) \hat{f}(k) e^{ikx} \right\} dk.$$

Here $\tilde{g}(k) = g(-k)$. We define the free Hamiltonian in $\mathcal{F}_{\mathcal{W}}$ by

$$d\Gamma_{\mathcal{W}}(\underbrace{\omega \oplus ... \oplus \omega}_{d-1}) = H_{EM}.$$

We require that $\hat{\varrho}$ satisfies (3.1) and $\varrho$ is real-valued rotation invariant function throughout this section. Then the Pauli-Fierz Hamiltonian with the ultraviolet cut-off function $\hat{\varrho}$ and with $N$-nonrelativistic particles is defined as an operator acting in

$$\mathcal{L}_N = L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}_{\mathcal{W}} \cong L^2(\mathbb{R}^{dN}; \mathcal{F}_{\mathcal{W}}),$$

by

$$\frac{1}{2m} \sum_{j=1}^{N} \left( \mathbf{p}^j \otimes I - eA(\hat{\varrho}, \cdot) \right)^2 + I \otimes H_{EM}, \tag{4. 1}$$

where $e \in \mathbb{R}$ is a coupling constant and $A(\hat{\varrho}, \cdot) = (A_1(\hat{\varrho}, \cdot), ..., A_d(\hat{\varrho}, \cdot))$. Introducing the polarization vector $e^r$, which corresponds to taking the Coulomb gauge, we see that, on a suitable dense domain,

$$[\mathbf{p}^j \otimes I, A(\hat{\varrho}, \cdot)] = 0.$$

Then formally we may rewrite (4.1) by

$$-\frac{1}{2m} \Delta_N \otimes I - \frac{e}{m} \sum_{j=1}^{N} \sum_{\mu=1}^{d} (\mathbf{p}^j_\mu \otimes I) A_\mu(\hat{\varrho}, \cdot) + \frac{e^2 N}{2m} A^2(\hat{\varrho}, \cdot) + I \otimes H_{EM}.$$

Here, for simplicity, we introduce the following assumptions to the Pauli-Fierz Hamiltonian:

**(1)** The self-interaction term $A^2(\hat{\varrho},\cdot)$ is neglected.

**(2)** We introduce the dipole approximation,i.e.,$A(\hat{\varrho},x)$ is replaced by $A(\hat{\varrho},0)$.

Then, putting $A(\hat{\varrho},0)=I\otimes A(\hat{\varrho})$, our Hamiltonian is as follows:

$$H_{EM}^{\beta}(\hat{\varrho},\Lambda)=-\frac{1}{2m}\Delta_N\otimes I-\Lambda eH_I^{EM}(\hat{\varrho})+\Lambda^{\beta}I\otimes H_{EM},$$

where

$$H_I^{EM}(\hat{\varrho})=\frac{1}{m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\mathbf{p}_{\mu}^{j}\otimes A_{\mu}(\hat{\varrho}).$$

Put

$$H_{EM}^{\beta}(\Lambda)=-\frac{1}{2m}\Delta_N\otimes I+\Lambda^{\beta}I\otimes H_{EM}.$$

**Theorem 4.1** **([1])** *Let $V\in\mathcal{M}_{\pm}(N)$. Then the operator $H_{EM}^{\beta}(\hat{\varrho},\Lambda)+V\otimes I$ is self-adjoint on $D(H_{EM}^{\beta}(\Lambda))$ and bounded from below. Moreover it is essentially self-adjoint on any core for $H_{EM}^{\beta}(\Lambda)$.*

We define a unitary operator by

$$S(e)=\exp\left(-ie\left(\sum_{j=1}^{N}\sum_{\mu=1}^{d}\frac{1}{m}\mathbf{p}_{\mu}^{j}\otimes\Pi_{\mu}\left(\frac{\hat{\varrho}}{\omega^2}\right)\right)\right),$$

where we put $\Pi_{\mu}(0,f)=I\otimes\Pi_{\mu}(f)$.

**Lemma 4.2** **([1])** *Let $V\in\mathcal{M}_{\pm}(N)$. Then the unitary operator $S(e)$ maps $D(H_{EM}^{\beta}(\hat{\varrho},\Lambda))$ onto itself with*

$$\begin{aligned}&S(\Lambda^{1-\beta}e)^{-1}(H_{EM}^{\beta}(\hat{\varrho},\Lambda)+V\otimes I)S(\Lambda^{1-\beta}e)\\&=-\left(\frac{1}{2m}+\Lambda^{2-\beta}\frac{e^2}{2M}\right)\Delta_N\otimes I+\Lambda^{\beta}I\otimes H_{EM}+V_{\beta}(\hat{\varrho},\Lambda),\end{aligned}\tag{4. 2}$$

*where*

$$\frac{1}{2M}=\frac{d-1}{d}\left(\frac{1}{m}\right)^2\left\|\frac{\hat{\varrho}}{\omega}\right\|_{L^2(\mathbb{R}^d)}^2,\quad V_{\beta}(\hat{\varrho},\Lambda)=S(\Lambda^{1-\beta}e)^{-1}(V\otimes I)S(\Lambda^{1-\beta}e).$$

To obtain the scaling limit of the case of $\beta = 1$, we need to fix a $dN \times dN$-matrix $\mathbf{T}$ so that

$$\mathbf{T}\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & 1 & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}\mathbf{T}^{-1} = \begin{pmatrix} N & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then, for a multiplication operator $V$ in $L^2(\mathbb{R}^{dN})$, we put

$$V_{eff}^{\hat{\varrho}}(x) = (2\pi C_N(\hat{\varrho}))^{-\frac{d}{2}} \int_{\mathbb{R}^d} dy V\left(\mathbf{T}^{-1}(y, (\mathbf{T}x)_2, ..., (\mathbf{T}x)_N)\right) e^{-\frac{|(\mathbf{T}x)_1 - y|^2}{2C_N(\hat{\varrho})}},$$

where

$$C_N(\hat{\varrho}) = \frac{d-1}{2d}\left(\frac{e}{m}\right)^2 \int_{\mathbb{R}^{dN}} dk \frac{|\hat{\varrho}(k)|^2}{\omega(k)^3},$$

and $(\mathbf{T}x)_j \in \mathbb{R}^d, j = 1, ..., N$, denotes the $j$-th element of $\mathbf{T}x \in \mathbb{R}^{dN}$. In the case of $\beta = 1$, the following proposition is well known.

**Proposition 4.3 ([1,6,7],$\beta = 1$)** *Let $V \in \mathcal{M}_{\pm}(N)$ with*

$$|V|_{eff}^{\hat{\varrho}}(x) < \infty, a.e. x \in \mathbb{R}^{dN}, \quad |V|_{eff}^{\hat{\varrho}} \in L^1_{loc}(\mathbb{R}^{dN}).$$

*Then $-\frac{1}{2m}\Delta_N + V_{eff}^{\hat{\varrho}}$ is self-adjoint on $D(-\Delta_N)$ with, for $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$s - \lim_{\Lambda \to \infty} \left(H_{EM}^{\beta}(\hat{\varrho}, \Lambda) + V \otimes I + \Lambda^{2-\beta}\frac{e^2}{2M}\Delta_N \otimes I - z\right)^{-1}$$
$$= S(e)\left\{\left(-\frac{1}{2m}\Delta_N + V_{eff}^{\hat{\varrho}} - z\right)^{-1} \otimes P_{EM}\right\} S(e)^{-1},$$

*where $P_{EM}$ is the projection operator onto the subspace $\{k\Omega_W | k \in \mathbb{C}\} \subset \mathcal{F}_W$.*

## 4.2 The case of $\beta = 2$

Put

$$\widetilde{H_{EM}^{\beta}}(\Lambda) = -\left(\frac{1}{2m} + \frac{e^2}{2M}\right)\Delta_N \otimes I + \Lambda^{\beta} I \otimes H_{EM}.$$

**Lemma 4.4** *Let $V \in \mathcal{M}_{\pm}(N)$. Then, for any $\epsilon > 0$, there exists $\Lambda_0$ and $b(\epsilon) > 0$ so that, for all $\Lambda > \Lambda_0$, $D(V_\beta(\hat{\varrho}, \Lambda)) \supset D(\widetilde{H^\beta_{EM}}(\Lambda))$ with*

$$||V_\beta(\hat{\varrho})\Phi||_{\mathcal{L}_N} \leq \epsilon \left\|\widetilde{H^\beta_{EM}}(\Lambda)\Phi\right\|_{\mathcal{L}_N} + b(\epsilon)\,||\Phi||_{\mathcal{L}_N}, \Phi \in D(\widetilde{H^\beta_{EM}}(\Lambda)). \tag{4. 3}$$

*Moreover, for $z \in \mathbb{C} \setminus [0, \infty)$,*

$$s - \lim_{\Lambda \to \infty} V_\beta(\hat{\varrho}, \Lambda)\left(\widetilde{H^\beta_{EM}}(\Lambda) - z\right)^{-1} = \left[V\left(-\left(\frac{1}{2m} + \frac{e^2}{2M}\right)\Delta_N - z\right)^{-1}\right] \otimes P_{EM}. \tag{4. 4}$$

**Proof:** Since $V$ is infinitesimally small with respect to $-\Delta_N$ and $-\Delta_N$ commutes $S(e)$, one can derive (4.3). Put $\left(\widetilde{H^\beta_{EM}}(\Lambda) - z\right)^{-1} = K_\Lambda$, $\left(-\left(\frac{1}{2m} + \frac{e^2}{2M}\right)\Delta_N - z\right)^{-1} \otimes P_{EM} = K_\infty$ and $S(\Lambda^{1-\beta}e) = S_\Lambda$. Note that, for $\beta > 1$,

$$s - \lim_{\Lambda \to \infty} S\left(\Lambda^{1-\beta}e\right) = I,$$

By (4.3), for any $\epsilon > 0$, taking sufficiently large $\Lambda > 0$, we have

$$\begin{aligned}
&||V_\beta(\hat{\varrho}, \Lambda)K_\Lambda\Phi - (V \otimes I)K_\infty\Phi|| \\
&\leq \epsilon\,||-\Delta_N(K_\Lambda - K_\infty)\Phi||_{\mathcal{L}_N} + \epsilon\,||-\Delta_N(S_\Lambda K_\infty - K_\infty)\Phi||_{\mathcal{L}_N} \\
&+ b(\epsilon)\,||(K_\Lambda - K_\infty)\Phi||_{\mathcal{L}_N} + b(\epsilon)\,||(S_\Lambda K_\infty - K_\infty)\Phi||_{\mathcal{L}_N} + \left\|(S_\Lambda^{-1} - I)(V \otimes I)K_\infty\Phi\right\|_{\mathcal{L}_N}.
\end{aligned}$$

Taking $\Lambda \to \infty$ on the both sides above, we have

$$\lim_{\Lambda \to \infty} ||V_\beta(\hat{\varrho}, \Lambda)K_\Lambda\Phi - (V \otimes I)K_\infty\Phi||_{\mathcal{L}_N} \leq \epsilon\,||(I \otimes I - I \otimes P_{EM})\Phi||_{\mathcal{L}_N}.$$

Since $\epsilon > 0$ is arbitrary, (4.4) follows. □

**Theorem 4.5** ($\beta = 2$) *Let $V \in \mathcal{M}_{\pm}(N)$. Then, for $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$s - \lim_{\Lambda \to \infty} \left(H^\beta_{EM}(\hat{\varrho}, \Lambda) + V \otimes I - z\right\}^{-1} = \left\{-\left(\frac{1}{2m} + \frac{e^2}{2M}\right)\Delta_N + V - z\right\}^{-1} \otimes P_{EM}. \tag{4. 5}$$

**Proof:** By virtue of (4.2), we see that, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\left(H^\beta_{EM}(\hat{\varrho}, \Lambda) + V \otimes I - z\right)^{-1} = S\left(\Lambda^{1-\beta}e\right)\left(\widetilde{H^\beta_{EM}}(\Lambda) + V_\beta(\hat{\varrho}, \Lambda) - z\right)^{-1} S\left(\Lambda^{1-\beta}e\right)^{-1},$$

and the partial expectation of $V \otimes I$ with respect to $\Omega_\mathcal{W}$ is $E_{\Omega_\mathcal{W}}(V \otimes I) = V$. Hence, it follows (4.5) from Lemma 4.4 and Proposition 2.1 with the following correspondence:

$$A = -\left(\frac{1}{2m} + \frac{e^2}{2M}\right)\Delta_N, B = H_{EM}, C(\Lambda) = V_\beta(\hat{\varrho}, \Lambda), C = V \otimes I, G = \Omega_\mathcal{W}.$$

□

### 4.3 The case of $1 < \beta < 2$, $2 < \beta$

For the case of $1 < \beta < 2$, by (4.2), we should subtract the term $-\Lambda^{2-\beta}\frac{e^2}{2M}\Delta_N \otimes I$ from the original Hamiltonian $H^{\beta}_{EM}(\hat{\varrho}, \Lambda)$, and for the case of $\beta > 2$, we do not need any renormalization. Hence,the similar argument of the cases of $\beta = 2$ and $\beta = 1$ gives an asymptotic behaviors of $H^{\beta}_{EM}(\hat{\varrho}, \Lambda)$. See Fig. 6.2.

# 5 THE SPIN-BOSON MODEL

## 5.1 The spin-boson model

In this section we study the spin-boson model. The total Hamiltonian of the spin-boson model is defined as an operator acting in the Hilbert space

$$\mathcal{L}_{SB} = \mathbb{C}^2 \otimes \mathcal{F}_{L^2(\mathbb{R}^d)} \cong \mathcal{F}_{L^2(\mathbb{R}^d)} \oplus \mathcal{F}_{L^2(\mathbb{R}^d)},$$

by

$$H^{\beta}_{SB}(\lambda, \Lambda) = \nu\sigma_1 + \Lambda\sigma_3 \otimes \left(a^{\dagger}(\bar{\lambda}) + a(\lambda)\right) + \Lambda^{\beta} I \otimes H_{SB}.$$

Here $H_{SB} = d\Gamma_{L^2(\mathbb{R}^d)}(\omega)$, $\nu > 0$, $\lambda \in L^2(\mathbb{R}^d)$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In what follows, we assume that

$$\lambda, \frac{\lambda}{\sqrt{\omega}}, \frac{\lambda}{\omega} \in L^2(\mathbb{R}^d).$$

**Theorem 5.1 ([1])** *The operator $H_{SB}(\lambda, \Lambda)$ is self-adjoint on $D(I \otimes H_{SB})$ and bounded from below. Moreover essentially self-adjoint on any core for $I \otimes H_{SB}$.*

We define a unitary operator by

$$\mathbf{T}(\lambda) = \begin{pmatrix} e^{+\{a^{\dagger}(\frac{\lambda}{\omega}) - a(\frac{\lambda}{\omega})\}} & 0 \\ 0 & e^{-\{a^{\dagger}(\frac{\lambda}{\omega}) - a(\frac{\lambda}{\omega})\}} \end{pmatrix} \equiv \begin{pmatrix} T_+(\lambda) & 0 \\ 0 & T_-(\lambda) \end{pmatrix}.$$

In the case of $\beta = 1$, following proposition is well known.

**Proposition 5.2** **([1],$\beta=1$)** *Let* $F(\lambda)=\left\langle \Omega_{L^2(\mathbb{R}^d)}, T_+(\lambda)\Omega_{L^2(\mathbb{R}^d)}\right\rangle_{\mathcal{F}_{L^2(\mathbb{R}^d)}}$ *and* $z\in\mathbb{C}\backslash\mathbb{R}$. *Then*

$$s-\lim_{\Lambda\to\infty}\left(H_{SB}^{\beta}(\lambda,\Lambda)-\Lambda^{2-\beta}E_{SB}-z\right)^{-1}=\mathbf{T}(\lambda)\left\{(\nu F(\lambda)\sigma_1-z)^{-1}\otimes P_{SB}\right\}\mathbf{T}(\lambda)^{-1}.$$

*Here* $P_{SB}$ *is the projection operator onto* $\{k\Omega|k\in\mathbb{C}\}\subset\mathcal{F}_{L^2(\mathbb{R}^d)}$ *and* $E_{SB}=-\left\|\frac{\lambda}{\sqrt{\omega}}\right\|^2_{L^2(\mathbb{R}^d)}$.

## 5.2 the case of $\beta=2$

It is well known and easily checked that $\mathbf{T}(\Lambda^{1-\beta}\lambda)$ maps $D(I\otimes H_{SB})$ onto itself with

$$\begin{aligned}&\mathbf{T}^{-1}(\Lambda^{1-\beta}\lambda)H_{SB}^{\beta}(\lambda)\mathbf{T}(\Lambda^{1-\beta}\lambda)\\ &=\nu\begin{pmatrix}0 & T_-^2(\Lambda^{1-\beta}\lambda)\\ T_+^2(\Lambda^{1-\beta}\lambda) & 0\end{pmatrix}+\Lambda^{\beta}I\otimes H_{SB}+\Lambda^{2-\beta}E_{SB}.\end{aligned}\tag{5. 1}$$

**Theorem 5.3** **($\beta=2$)** *Let* $z\in\mathbb{C}\setminus\mathbb{R}$. *Then*

$$s-\lim_{\Lambda\to\infty}\left(H_{SB}^{\beta}(\lambda,\Lambda)-z\right)^{-1}=(\nu\sigma_1+E_{SB}-z)^{-1}\otimes P_{SB}.\tag{5. 2}$$

**Proof:** We see that, by (5.1),

$$\begin{aligned}&\mathbf{T}^{-1}\left(\Lambda^{1-\beta}\lambda\right)(H_{SB}^{\beta}(\lambda,\Lambda)-z)^{-1}\mathbf{T}\left(\Lambda^{1-\beta}\lambda\right)\\ &=\left\{\nu\begin{pmatrix}0 & T_-^2\left(\Lambda^{1-\beta}\lambda\right)\\ T_+^2\left(\Lambda^{1-\beta}\lambda\right) & 0\end{pmatrix}+\Lambda^{\beta}I\otimes H_{SB}+\Lambda^{2-\beta}E_{SB}-z\right\}^{-1}.\end{aligned}$$

It is easily seen that $s-\lim_{\Lambda\to\infty}\mathbf{T}\left(\Lambda^{1-\beta}\lambda\right)=I$ and

$$\begin{aligned}&s-\lim_{\Lambda\to\infty}\nu\begin{pmatrix}0 & T_-^2\left(\Lambda^{1-\beta}\lambda\right)\\ T_+^2\left(\Lambda^{1-\beta}\lambda\right) & 0\end{pmatrix}\left(\Lambda^{2-\beta}E_{SB}+\Lambda^{\beta}I\otimes H_{SB}-z\right)^{-1}\\ &=\left[\nu\sigma_1(E_{SB}-z)^{-1}\right]\otimes P_{SB}.\end{aligned}$$

Hence, with the following correspondence:

$$A=E_{SB},B=H_{SB},C(\Lambda)=\nu\begin{pmatrix}0 & T_-^2\left(\Lambda^{1-\beta}\lambda\right)\\ T_+^2\left(\Lambda^{1-\beta}\lambda\right) & 0\end{pmatrix},C=\nu\sigma_1,G=\Omega_{L^2(\mathbb{R}^d)},$$

one can easily check the conditions with respect to $C(\Lambda)$ and $C$ in section 2. Since the partial expectation of $\nu\sigma_1\otimes I$ with respect to $\Omega\ L^2(\mathbb{R}^d)$ is $E_{\Omega_{L^2(\mathbb{R}^d)}}(\nu\sigma_1\otimes I)=\nu\sigma_1$, we get (5.2) by Proposition 2.1. □

### 5.3 The case of $1 < \beta < 2, 2 < \beta$

For the case of $1 < \beta < 2$, by (5.1), we should subtract the term $\Lambda^{2-\beta} E_{SB}$ from the original Hamiltonian $H^{\beta}_{SB}(\lambda, \Lambda)$, and for the case of $\beta > 2$, we do not need any renormalization. Hence,the similar argument of the cases of $\beta = 2$ and $\beta = 1$ gives an asymptotic behaviors of $H^{\beta}_{SB}(\lambda, \Lambda)$. See Fig 6.3.

## 6 CONCLUDING REMARKS

(**1**) In section 4, we studied the Pauli-Fierz model neglected the terms $A^2(\hat{\varrho}, \cdot)$. In [6,7], we studied the Pauli-Fierz Hamiltonian with the terms $A^2(\hat{\varrho}, \cdot)$. By the same method developed in [6,7], we can investigate the following scaling Hamiltonians:

$$-\frac{1}{2m}\Delta_N \otimes I - \Lambda e H_I^{EM}(\hat{\varrho}) + \Lambda I \otimes H_{EM} + \frac{e^2 N}{2m} A^2(\hat{\varrho}, \cdot) + V \otimes I, \tag{6. 1}$$

$$-\frac{1}{2m}\Delta_N \otimes I - \Lambda e H_I^{EM}(\hat{\varrho}) + \Lambda^2 I \otimes H_{EM} + \Lambda^2 \frac{e^2 N}{2m} A^2(\hat{\varrho}, \cdot) + V \otimes I. \tag{6. 2}$$

Introducing different renormalizations from those given in this paper, we can get effective Hamiltonians of (6.1) and (6.2).

(**2**) In the case of $0 < \beta < 1$, we need delicate discussions of asymptotic behaviors of unitary operators $\mathcal{U}(\Lambda^{1-\beta} g)$, $S(\Lambda^{1-\beta} e)$ and $\mathbf{T}(\Lambda^{1-\beta}\lambda)$ as $\Lambda \to \infty$. We omit the discussions.

| | Effective Hamiltonian | Unitary operator | Renormalization |
|---|---|---|---|
| $\beta > 2$ | $-\frac{1}{2m}\Delta_N + V$ | $I$ | 0 |
| $\beta = 2$ | $-\frac{1}{2m}\Delta_N + g^2 V(\hat{\varrho}) + V$ | $I$ | 0 |
| $1 < \beta < 2$ | $-\frac{1}{2m}\Delta_N + V$ | $I$ | $g^2 \Lambda^{2-\beta} V(\hat{\varrho})$ |
| $\beta = 1$ | $-\frac{1}{2m}\Delta_N + g^2 N \delta(\hat{\varrho}) + V$ | $\mathcal{U}(g)$ | $g^2 \Lambda V(\hat{\varrho})$ |

Fig 6.1 $\beta$-coupling Nelson model $H^{\beta}_N(\hat{\varrho}, \Lambda)$

| | Effective Hamiltonian | Unitary operator | Renormalization |
|---|---|---|---|
| $\beta > 2$ | $-\frac{1}{2m}\Delta_N + V$ | $I$ | 0 |
| $\beta = 2$ | $-\left(\frac{1}{2m} + \frac{e^2}{2M}\right)\Delta_N + V$ | $I$ | 0 |
| $1 < \beta < 2$ | $-\frac{1}{2m}\Delta_N + V$ | $I$ | $-\Lambda^{2-\beta}\frac{e^2}{2M}\Delta_N$ |
| $\beta = 1$ | $-\frac{1}{2m}\Delta_N + V_{eff}^{\hat{\varrho}}$ | $S(e)$ | $-\Lambda\frac{e^2}{2M}\Delta_N$ |

Fig.6.2 $\beta$-coupling Pauli-Fierz model $H_{EM}^{\beta}(\hat{\varrho}, \Lambda)$

| | Effective Hamiltonian | Unitary operator | Renormalization |
|---|---|---|---|
| $\beta > 2$ | $\nu\sigma_1$ | $I$ | 0 |
| $\beta = 2$ | $\nu\sigma_1 + E_{SB}$ | $I$ | 0 |
| $1 < \beta < 2$ | $\nu\sigma_1$ | $I$ | $\Lambda^{2-\beta}E_{SB}$ |
| $\beta = 1$ | $\nu F(\lambda)\sigma_1$ | $\mathbf{T}(\lambda)$ | $\Lambda E_{SB}$ |

Fig.6.3 $\beta$-coupling spin-boson model $H_{SB}^{\beta}(\lambda, \Lambda)$

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