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Embedding of Expansive Dynamical Systems into Message Systems and its Application to Entropy Theory

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1. Introduction

The concepts of entropy are useful in order to study the classification problems in topology and probability theory. As for the classification of shift transformations, we refer to Ornstein [6] and Sinai [7], and as for the classification of classical and quantum states, we refer to Ohya [5]. Topological entropy was introduced by Adler-Konheim-McAndrew [1] in order to study the classification of continuous mappings on topological spaces, and this concept has become a powerful tool for the theory of dynamical systems. Actually, the study of expansive dynamical systems is deeply related to the theory of topological entropy [2,3,8].

In this paper, we show that expansive dynamical systems can be embedded into some message systems, that is, into some message spaces with shift transformations. Moreover, we give upper bounds of the topological entropy of expansive mappings in terms of the ε-entropy which was introduced by Kolmogorov-Tihomirov [4].

2. Embedding of expansive dynamical systems into message systems

Throughout this paper, $\boldsymbol{N}$ and $\boldsymbol{Z}$ denote the set of all non-negative integers and the set of all integers, respectively.

Let (X,d) be a metric space, and T be a continuous mapping on X with values in X. Let ε be a positive number. Then T is said to be ε-expansive if, for any two distinct points x and y in X, there exists an $n \in \boldsymbol{N}$ satisfying $d(T^n x, T^n y) \geq \varepsilon$, where we identify T^n with the identity mapping on X if n=0. We call the dynamical system (X,d,T) an ε-expansive dynamical system.

Let m be a positive integer and M be the finite set consisting of integers which are greater than 0 and less than m+1. Endowing M with the discrete topology, M becomes a compact Hausdorff space. Let $M^{\boldsymbol{N}}$ be the set of all one-sided infinite sequences consisting of elements in M. Endowing $M^{\boldsymbol{N}}$ with the product topology, $M^{\boldsymbol{N}}$ becomes a compact Hausdorff space by Tychonoff's theorem. Let n be a non-negative integer, r be an element in $M^{\boldsymbol{N}}$, and $(r)_n$ be the n-th component of r. Then, the product topology on $M^{\boldsymbol{N}}$ is exactly equal to the topology which is determined by the following metric D:

$$D(r,s) = \sum_{n=0}^{\infty} \{1-\delta((r)_n,(s)_n)\}/m^n, \qquad r, s \in M^{\boldsymbol{N}},$$

where δ means Kronecker's delta. The mapping S on $M^{\boldsymbol{N}}$ with values in $M^{\boldsymbol{N}}$ is called the shift transformation if $(Sr)_n=(r)_{n+1}$ holds for all $r \in M^{\boldsymbol{N}}$ and all $n \in \boldsymbol{N}$. We call the dynamical system $(M^{\boldsymbol{N}},S)$ a message system.

Let X (resp. Y) be a topological space and T_X (resp. T_Y) be a continuous mapping on X (resp. Y) with values in X (resp. Y). Then we say that (Y,T_Y) can be embedded into (X,T_X) if there exist a subspace Z of X and a homeomorphism φ on Y with values in Z satisfying $\varphi^{-1}\circ T_X\circ\varphi=T_Y$.

Under the above notations and definitions, we obtain the following

Theorem 1. Let ε be a positive number, (X,d) be a compact and totally disconnected metric space and T be an ε-expansive continuous mapping on X with values in X. Then (X,d,T) can be embedded into a certain message system.

Proof. A subset P of X is said to be clopen if P is not only closed but also open. Since X is compact and totally disconnected, for some positive integer m, there exists a finite clopen partition $\{P_1,P_2,\cdots,P_m\}$ of X satisfying that the all diameters of elements in this partition are less than ε. Let M be the topological space consisting of integers which are greater than 0 and less than m+1 and M^N be the product topological space constructed by the methods which are previously stated. Moreover, for any $n\in N$, let φ_n be the mapping on X with values in M satisfying that $\varphi_n(x)=i$ if $T^n x\in P_i$ holds, and Φ be the mapping on X with values in M^N defined by

$$\Phi(x) = (\varphi_0(x),\varphi_1(x),\cdots), \qquad x \in X.$$

It is clear that Φ is injective because T is ε-expansive. While, for any p>0, there exists some positive integer n(p) satisfying

$$\sum_{k=n(p)+1}^{\infty} 1/m^k < p.$$

Since T is continuous, for any $x\in X$, there exists a positive number $\delta(x,p)$ satisfying the condition that $d(x,y)<\delta(x,p)$ implies

$$\varphi_i(x)=\varphi_i(y), \qquad 0 \le i \le n(p).$$

Therefore, for any $y \in X$ satisfying $d(x,y)<\delta(x,p)$,

$$D(\Phi(x),\Phi(y)) \le \sum_{k=n(p)+1}^{\infty} 1/m^k < p$$

holds. This result implies that Φ is continuous. Let $\Phi(X)$ be the range of Φ. Then, $\Phi(X)$ becomes a compact subspace of M^N by endowing it with the relative topology induced from M^N, because X is compact. Since Φ is a continuous bijection defined on X with values in $\Phi(X)$. Therefore, Φ is also a homeomorphism. It is trivial that $\Phi^{-1} \circ S \circ \Phi = T$ holds. This result concludes the proof of Theorem 1. Q.E.D.

3. An application to entropy theory

Let $\mathcal{A}$ be an open covering of X, and $N_X(\mathcal{A})$ be the number of open sets in a subcovering of minimal cardinality. Since X is compact, there exists a finite subcovering of $\mathcal{A}$. Therefore, we define the entropy of $\mathcal{A}$, which is denoted by $h_X(\mathcal{A})$, by the base-2 logarithm of $N_X(\mathcal{A})$. Especially, for any $\varepsilon>0$, we define an ε-covering of X by a family of open balls with radiuses ε whose union can cover X. Let $N_\varepsilon(X)$ be the number of elements in an ε-covering of minimal cardinality. Then, we define the ε-entropy of X , which is denoted by $H_\varepsilon(X)$, by the base-2 logarithm of $N_\varepsilon(X)$. For any $n \in \mathbb{N}$, we define $T^{-n}\mathcal{A}$ by $\{T^{-n}A; A \in \mathcal{A}\}$. For any two open coverings $\mathcal{A}$ and $\mathcal{B}$, we define the join of $\mathcal{A}$ and $\mathcal{B}$, which is denoted by $\mathcal{A} \vee \mathcal{B}$, by $\{A \cap B; A \in \mathcal{A}, B \in \mathcal{B}\}$. Then, the entropy of T with respect to $\mathcal{A}$, which is denoted by $h_X(T,\mathcal{A})$, is defined by

$$h_X(T,\mathcal{A}) = \lim_{n\to\infty} h_X\left(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{A}\right)/n.$$

Finally, the topological entropy of T, which is denoted by $h_X(T)$, is defined by

$$h_X(T) = \sup\{H(T,\mathcal{A});\ \mathcal{A} \text{ is an open covering of } X\},$$

where the supremum is taken over all open coverings $\mathcal{A}$ of X.

Under the above notations and definitions, we obtain the following

Theorem 2. Under the same assumptions as Theorem 1,

$$h_X(T) \leq \lim_{\delta \to +0} H_{\varepsilon/2-\delta}(X)$$

holds.

Proof. For any p>0 and any $x \in X$, $B_p(X)$ and $C_p(X)$ denote the open ball and the closed ball with radiuses p and centers x, respectively, that is,

$$B_p(x) = \{y \in X; d(x,y) < p\},$$

$$C_p(x) = \{y \in X; d(x,y) \leq p\}.$$

For any sufficiently small δ satisfying $0<\delta<\varepsilon/2$, it is clear that $\{B_{\varepsilon/2-\delta}(x); x \in X\}$ is an open covering. Since X is compact, we can choose a finite subcovering $\{B_{\varepsilon/2-\delta}(x_i); i=1,\cdots,N_{\varepsilon/2-\delta}(X)\}$. Therefore, for any $1 \leq i \leq N_{\varepsilon/2-\delta}(X)$, there exists a clopen set D_i satisfying

$$C_{\varepsilon/2-\delta}(x_i) \subset D_i \subset B_{\varepsilon/2}(x_i),$$

because X is totally disconnected. It is clear that

$$\bigcup_{i=1}^{N_{\varepsilon/2-\delta}(X)} D_i = X$$

holds. Using this finite clopen covering $\{D_i ; 1 \le i \le N_{\varepsilon/2-\delta}(X)\}$, we can construct easily a finite clopen partition of X satisfying that the number of elements in this partition is less than or equal to $N_{\varepsilon/2-\delta}(X)$. Let $\{P_i ; 1 \le i \le N_{\varepsilon/2-\delta}(X)\}$ be such a partition. Then, applying the proof of Theorem 1 to this partition, we can construct the message system (M^{N}, S), where $M=\{1, \cdots, N_{\varepsilon/2-\delta}(X)\}$, and the continuous injective mapping Φ on X with values in M^{N} which enables that (X, d, T) can be embedded. Because of the definition of topological entropy and the fact that $\Phi(X)$ is a closed subset of M^{N}, we can easily prove that

$$h_X(T) \le h_{M^N}(S).$$

Moreover, it is known by Adler-Konheim-McAndrew [1] that

$$h_{M^N}(S) = \log N_{\varepsilon/2-\delta}(X) = H_{\varepsilon/2-\delta}(X)$$

holds. Since we can choose any sufficiently small $\delta>0$, the above two formulas conclude the proof of Theorem 2. Q.E.D.

4. Remarks

Let (X, d) be a metric space and T be a homeomorphism on X with values in X. Let ε be a positive number. Then T is said to be ε-expansive, if for any two distinct points x and y in X, there exists $n \in Z$ satisfying $d(T^n x, T^n y) > \varepsilon$. Then, the dynamical system (X, d, T) is also called an ε-expansive dynamical system.

Let m be a positive integer, M be the finite set consisting of integers which is greater than 0 and less than m+1, and M^Z be the set of all two-sided infinite sequences consisting of elements in M. Using the same methods as Section 2, we can endow M^Z with the product topology by endowing M with the discrete topology, and we can make M^Z a compact Hausdorff space. The mapping S on M^Z with values in M^Z is also called the shift transformation if $(Sr)_n=(r)_{n+1}$ holds for all $r\in M^Z$ and all $n\in Z$. The dynamical system (M^Z, S) is also called a message system.

Under the above definitions and notations in this section, using the same methods as Theorem 1 and Theorem 2, we can prove that any ε-expansive dynamical system (X, d, T) can be embedded into a certain message system if X is compact and totally disconnected, and that the same inequality as Theorem 2 holds.

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