Exact WKB analysis of anharmonic oscillators.

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Anharmonic oscillators have attracted much attention of physicists, particularly because of their relevance to the $\phi^4$-model in quantum field theory.

Here we show how to apply exact WKB analysis to the analysis of their eigenvalue problems, namely,

$$\left(-\frac{d^2}{dx^2} + \frac{1}{4}x^2(1 + \lambda x^{2N})\right)\psi = E(\lambda)\psi \quad (\lambda > 0),$$

$$\lim_{|x|\to\infty} \psi(x) = 0.$$

For $\lambda \ll 1$, an eigenvalue $E(\lambda)$ has a formal expansion with respect to $\lambda$ (the so-called Rayleigh-Schrödinger perturbation series);

$$E^K(\lambda) = K + \frac{1}{2} + \sum_{n=1}^{\infty} A^K_n \lambda^n \quad \text{where } K = 0, 1, 2, \cdots.$$

Our purpose is to determine the asymptotic behavior of $A^K_n$ for arbitrary $N$. The result is as follows:

$$A^K_n = \frac{(-1)^{n+1}N}{K!(2\pi)^{\frac{1}{2}}} 4^{(K+\frac{1}{2})/N} \frac{B\left(\frac{3}{2}, \frac{1}{N}\right)}{2^N} \Gamma(K + \frac{1}{2} + nN) \left(1 + O\left(\frac{1}{n}\right)\right), \quad (n \to \infty).$$

Here $B(x, y)$ denotes the Beta function.

For $N = 1$, this result was shown by Bender and Wu (Phys.Rev.D,7(1973)) in a heuristic manner. By making use of exact WKB analysis we verify the result rigorously.