

DIFFERENTIAL EQUATIONS ON FILTERED MANIFOLDS

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A filtered manifold is a differential manifold M equipped with a tangential filtration $F = \{F^p\}_{p \in \mathbb{Z}}$ satisfying the following conditions:

- o) F^p is a subbundle of the tangent bundle TM of M ,
- i) $F^p \subset F^{p+1}$,
- ii) $F^0 = 0$, $\bigcup_{p \in \mathbb{Z}} F^p = TM$,
- iii) $[F^p, F^q] \subset F^{p+q}$, for all $p, q \in \mathbb{Z}$,

where \underline{F}^p denotes the sheaf of the germs of sections of F^p .

As the first order approximation of a filtered manifold M at a point $x \in M$, we can associate to each x a nilpotent graded Lie algebra grF_x by setting

$$grF_x = \bigoplus F_x^p / F_x^{p-1}$$

with a natural bracket operation induced from that of vector fields.

The category of the filtered manifolds (M, F) and the functor grF are natural generalizations of those of differential manifolds and tangent bundles. This refinement from the abelians to the nilpotents opens new perspectives in geometry and analysis. Nilpotent geometry in the above sense was initiated by N. Tanaka and has been developed by himself and T. Morimoto et al, to provide us with powerful methods in geometry especially in studying geometric structures.

In this talk I would like to discuss, as a first step to nilpotent analysis, a general theory of differential equations on filtered manifolds, which leads us unexpectedly to a remarkable existence theorem of analytic solutions to a system of non-linear partial differential equations possibly with singularities. The story consists of three parts:

1. Formal theory. Let (M, F) be a filtered manifold. Our key idea to study differential equations on M is to use the weighted order of differential operators: A vector field X on M is said to be of weighted order $\leq k$ if X is a section of F^k . This definition of weighted order is immediately extended to any differential operators on M and gives rise to the notion of weighted jet bundle $\widehat{J}^k E$ of weighted order k for a vector bundle E on M . Then a system of PDE's on M of weighted order k may be formulated as a submanifold R of $\widehat{J}^k E$. The first task is to study the compatibility conditions for R . Generalizing the formal theory developed by Spencer and Goldschmidt et al (abelian case) to our nilpotent case, we have a general criterion for formal integrability, namely the notion of *weighted involutivity*.

2. Formal Gevrey solution. As well-known, by Cartan-Kähler theorem, an analytic system of PDE's has an analytic local solution if it is involutive. However, it turns out that it is not the case for weighted involutive systems. What we have found instead is that *for a weighted involutive analytic system there exists a formal solution satisfying certain estimates called formal Gevrey (weaker than convergence).*

For the sake of simplicity, let us work on a standard filtered manifold, that is a Lie group N whose Lie algebra \mathfrak{n} is a graded nilpotent Lie algebra: $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$. Choose a basis $\{X_1, \dots, X_n\}$ of \mathfrak{n} such that $\{X_{d(p-1)+1}, \dots, X_{d(p)}\}$ is a basis of \mathfrak{n}_p for all p , where $d(p) = \sum_{i=1}^p \dim \mathfrak{n}_i$. We define a weight function $w : \{1, \dots, n\} \rightarrow \{1, \dots, \mu\}$ by the condition: $X_i \in \mathfrak{n}_{w(i)}$, for all i . For $I = (i_1, \dots, i_l) \in \{1, \dots, n\}^l$, we set

$$X_I = X_{i_1} \cdots X_{i_l}, \quad w(X_I) = w(I) = \sum_{a=1}^l w(i_a).$$

We shall regard X_I as left invariant differential operator on N of weighted order $w(I)$. A formal function F at $o \in N$ is called *formal Gevrey* if there exist positive constants C, ρ such that

$$|(X_I F)(o)| \leq C w(I)! \rho^{w(I)} \quad \text{for all } I$$

To prove the existence of formal Gevrey solution we use a non-commutative version of privileged neighborhood theorem (the commutative version was first obtained by Grauert and then ameliorated and often used by Malgrange).

3. Analytic solution. A formal Gevrey function on N is, roughly speaking, analytic only in the directions of the distribution \mathfrak{n}_1 . Through studying a subriemannian geometry of the distribution \mathfrak{n}_1 and encountering with a deep Gabrièlov's theorem in analytic geometry, we are led to the following unexpected conclusion that *if the Lie algebra \mathfrak{n} is generated by \mathfrak{n}_1 (Hörmander condition) then every formal Gevrey function on N is analytic.*

As a corollary we have: *A weighted involutive analytic system on a graded nilpotent Lie group has always an analytic solution if the Lie algebra \mathfrak{n} is generated by \mathfrak{n}_1 .*

It should be remarked that the class of the weighted involutive systems contains a wide class of systems of PDE's with singularities.

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