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A dependence domain for a class of micro-differential equations with involutive double characteristics

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1 Statement of the Main Theorem

Let M be a real analytic manifold with a complex neighborhood X. Let P be a microdifferential operator defined in a neighborhood U in $T^*X$of $\dot{q}\in T_M^*X\setminus M$. We assume that the characteristic variety of P satisfies

$$\mathrm{Char(P)}\subset\{q\in U;\ p_1(q)\cdot p_2(q)=0\}$$

with homogeneous holomorphic functions p_1 and p_2 on U. We assume that

$$p_1\text{and}\ \ p_2 \text{ are real valued on } T_M^*X, \tag{1}$$

$$dp_1\wedge dp_2\wedge\omega_X(q)\neq 0 \text{ if } p_1(q)=p_2(q)=0, \tag{2}$$

$$\{p_1,p_2\}(q)=0 \quad\text{if}\ \ p_1(q)=p_2(q)=0. \tag{3}$$

Here ω_X is the canonical 1-form of T^*X, and $\{\cdot,\cdot\}$ the Poisson bracket on T^*X.

In this situation, we can define a regular involutive submanifold V by

$$V=\{q\in U;\ p_1(q)=p_2(q)=0\}.$$

We assume, for simplicity, that $\dot{q}\in V$. Moreover Γ denotes the canonical leaf of V passing through $\dot{q}$.

A set K in Γ is called, in this article, a Γ-rectangle if there exists an injective real analytic map

$$\Phi:\ [0,1]\times[0,1]\longrightarrow\Gamma$$

with the following three properties.

- $\Phi([0,1]\times[0,1])=K$

- $\Phi(\cdot, t)$ is an integral curve of the Hamiltonian vector field H_{p_1} for any fixed $t \in [0, 1]$.
- $\Phi(s, \cdot)$ is an integral curve of the Hamiltonian vector field H_{p_2} for any fixed $s \in [0, 1]$.

We give, in this situation,

Theorem 1.1 *There exists an open neighborhood U_0 of $\dot{q}$ in Γ with the property that for any Γ-rectangle K contained in U_0 with the four vertices q_0, q_1, q_2, and q_3 and for any microfuntion solution u to $Pu = 0$ on K,*

$$q_1,\ q_2,\ q_3 \notin \operatorname{supp}(u) \Longrightarrow q_0 \notin \operatorname{supp}(u).$$

This theorem can be deduced from the model case given in the next section, where several remarks are also given.

2 Theorem in the model case

Let M be an open subset of $\mathbf{R}^n$ with a complex neighborhood X in $\mathbf{C}^n$ ($n \geq 3$). We take a coordinate system of M (resp. X) as $x = (x_1, \cdots, x_n)$ (resp. $z = (z_1, \cdots, z_n)$). Then $(x; \sqrt{-1}\xi \cdot dx)$ (resp. $(z; \zeta \cdot dz)$) denotes a point in $T^*_M X$ (resp. T^*X) with $\xi = (\xi_1, \cdots, \xi_n)$ (resp. $\zeta = (\zeta_1, \cdots, \zeta_n)$).

We take a point $q_0 = (0; \sqrt{-1}dx_n) \in T^*_M X$. Let P be a microdifferential operator defined in a neighborhood of q_0 whose principal symbol is of the form

$$\zeta_1^{m_1} \zeta_2^{m_2}$$

with $m_1, m_2 \geq 1$. We define an involutive manifold V of $T^*_M X$ by

$$V = \{(x; \sqrt{-1}\xi \cdot dx); \xi_1 = \xi_2 = 0\}$$

and denote by Γ the leaf of V passing through the point q_0. We take a rectangle K on Γ defined by

$$K = \{(x_1, x_2, x'' = 0; \sqrt{-1}dx_n); 0 \leq x_1 \leq t_1, \quad 0 \leq x_2 \leq t_2\}.$$

The vertices of K are denoted by

$$q_0, q_1 = (t_1, 0, 0; \sqrt{-1}dx_n), q_2 = (0, t_2, x'' = 0; \sqrt{-1}dx_n),$$
$$q_3 = (t_1, t_2, x'' = 0; \sqrt{-1}dx_n).$$

Here $x'' = (x_3, \cdots, x_n)$. Then we have

Theorem 2.1 *Let u be a microfunction defined in a neighborhood of K. We assume that u satisfies*

$$Pu = 0$$

and that the three points q_1, q_2, q_3 are not in supp(u)*:*

$$q_1, q_2, q_3 \notin \text{supp}\ (u).$$

Then

$$q_0 \notin \text{supp}(u).$$

Remark 2.1 *The phenomenon in the above theorem was first observed by Y. Okada[O] for C^∞ wavefront set of microdistribution solutions. His result concerns with the case $m_1 = m_2 = 1$ under a Levi condition on the lower order term of P. He employed a microlocal version of Goursat problem in the complex domain.*

Remark 2.2 *It is inevitable to assume the condition $q_3 \notin$* supp(u) *in Theorem 2.1 For example, we define a hyperfunction*

$$u(x_1, x_2, x_3) = (Y(x_1) - Y(x_2)) \cdot \delta(x_3),$$

which is a solution to $D_1D_2u = 0$. Then we have, for a positive constant $t > 0$,

$$q_1 = (t, -t, 0; \sqrt{-1}dx_3), q_2 = (-t, t, 0; \sqrt{-1}dx_3) \notin \text{supp}(u),$$

but

$$q_0 = (t, t, 0; \sqrt{-1}dx_3), q_2 = (-t, -t, 0; \sqrt{-1}dx_3) \in \text{supp}(u).$$

To give an implication of Theorem 1.1, we recall a result obtained by N. Tose [T2].

Theorem 2.2 *Let u be a microfunction solution to $Pu = 0$ on an open subset U of Γ. Then there exist a family $\{b_\lambda^{(1)}\}_{\lambda\in\Lambda_1}$ of integral curves on Γ of $\partial/\partial x_1$ and another family $\{b_\lambda^{(2)}\}_{\lambda\in\Lambda_2}$ of integral curves on Γ of $\partial/\partial x_2$ which satisfy the property that* supp(u) *has unique continuation property on the set*

$$\Omega = U \setminus \left(\bigcup_{\lambda\in\Lambda_1} b_\lambda^{(1)} \cup \bigcup_{\lambda\in\Lambda_2} b_\lambda^{(2)} \right).$$

More precisely, if a point $q \in \Omega$ is not in supp(u)*, then the connected component of Ω containing q is disjoint with* supp(u)*.*

In the situation of Theorem 2.2, we take a point

$$\dot{q} = (s_1, s_2, x'' = 0; \sqrt{-1}dx_n) \in \Gamma.$$

We assume that, for a neighborhood U_1 of $\dot{q}$, the only one integral curve $b^{(1)}_{\lambda_1}$ of $\partial/\partial x_1$ and the only one $b^{(2)}_{\lambda_2}$ of $\partial/\partial x_2$ pass U_1. We assume, for simplicity, that the both two curves pass $\dot{q}$:

$$\dot{q} \in b^{(j)}_{\lambda_j} \quad (j = 1, 2).$$

We assume that

$$\mathrm{supp}(u) \cap U_1 \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n);\ x_1 < s_1,\ x_2 > s_2\} = \emptyset$$

and that

$$\mathrm{supp}(u) \cap U_1 \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n);\ x_1 > s_1,\ x_2 < s_2\} = \emptyset.$$

In this situation, if a point

$$\dot{q}' \in \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n);\ x_1 < s_1,\ x_2 < s_2\}$$

does not belong to supp(u), then it follows from Theorem 2.1 that

$$\mathrm{supp}(u) \cap \{(x_1, x_2, x'' = 0, \sqrt{-1}dx_n);\ x_1 > s_1,\ x_2 > s_2\} = \emptyset.$$

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