Some isomorphism theorems of cohomology groups for completely integrable connections (Geometric methods in asymptotic analysis)

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Some isomorphism theorems of cohomology groups for completely integrable connections

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1 Introduction

About 14 years ago, the author proved an isomorphism theorem between the cohomology group of complex of global meromorphic sections derived from a completely integrable connection and the cohomology group of kernel sheaf with values in the sheaf of functions asymptotically developable to the formal series 0 for the connection ([5]). Recently, several researchers, who are interested in the intersection theory for differential equations with singular points, pushed him to prove the $C^\infty$ version (cf. [1], [2]). In this paper, firstly, we give a short review of the isomorphism theorem in asymptotic analysis and some examples with concrete calculation of basis for the cohomology groups. Secondly, we explain the $C^\infty$ version.

2 Isomorphism Theorem in Asymptotic Analysis

Let $M$ be a complex manifold and let $H$ be a divisor on $M$ at most normal crossing singularities. we denote by $\mathcal{O}(\ast H)$ the sheaf of germs of meromorphic $p$-forms which are holomorphic in $M - H$ and have poles on $H$ and denote by $S$ a locally free sheaf of $\mathcal{O}$-modules of rank $m$ on $M$. Put $S\mathcal{O}(\ast H) = \mathcal{O}(\ast H) \otimes \mathcal{O} S$ for $p = 0, \ldots, n$. For $p = 0$, instead of $S\mathcal{O}(\ast H)$, we use frequently $S(\ast H)$ of which the restriction to $U$, $S(\ast H)|_U$ is isomorphic to

$$\mathcal{O}(\ast H)^m|_U = (\mathcal{O} \otimes \mathcal{O}(\ast H))|_U$$

and the isomorphism is denoted by $g_U$.

Let $\nabla$ be a connection on $S\mathcal{O}(\ast H)$: $\nabla$ is an additive mapping of $S\mathcal{O}(\ast H)$ into $S\mathcal{O}|_{\ast H}$ satisfying "Leipnitz' rule"

$$\nabla(f \cdot u) = u \otimes df + f \cdot \nabla(u),$$

for all sections $f \in \mathcal{O}(\ast H)(U)$ and $u \in S\mathcal{O}|_{\ast H}(U)$. We suppose that the connection is integrable, that is, the composite mapping

$$\nabla_2 : S\mathcal{O}(\ast H) \longrightarrow S\mathcal{O}|_{\ast H} \longrightarrow S\mathcal{O}(\ast H),$$
is a zero mapping.

If we take adequately an open covering \( \{ U_k \}_k \) on \( M \), then to give the connection \( \nabla \) means as follows: for each \( U_k \), the mapping
\[
g_{U_k} \circ \nabla \circ g_{U_k}^{-1} : \Omega^0(*)H(U_k)^m \longrightarrow \Omega^1(*)H(U_k)^m,
\]
is induced by a mapping
\[
\nabla_k : \Omega^0(*)H(U_k)^m \longrightarrow \Omega^1(*)H(U_k)^m,
\]
which is represented by \((d + \Omega_k)\) under a generator system
\[
< e_{k,1}, \ldots, e_{k,m} >
\]
of \((\mathcal{O}(U_k))^m\), i.e.
\[
\nabla_k(< e_{k,1}, \ldots, e_{k,m} > u) = < e_{k,1}, \ldots, e_{k,m} > (du + \Omega_k u)
\]
where \( \Omega_k \) is an \( m \)-by-\( m \) matrix of meromorphic 1-forms on \( U_k \) at most with poles in \( U_k \cap H \).

Let \( x_1, \ldots, x_n \) be holomorphic local coordinates on \( U_k \) and suppose
\[
U_k \cap H = \{(x_1, \ldots, x_n) | x_1 \cdots x_{n'} = 0\},
\]
then \( \Omega_k \) is of the form
\[
\Omega_k = \sum_{i=1}^{n'} x^{-p_i} x_i^{-1} A_i(x) dx_i + \sum_{i=n'+1}^{n} x^{-p_i} A_i(x) dx_i,
\]
where \( p_i = (p_{i1}, \ldots, p_{in'}, 0, \ldots, 0) \in \mathbb{N}^n \) and \( A_i(x) \) is an \( m \)-by-\( m \) matrix of holomorphic functions in \( U_k \) for \( i = 1, \ldots, n \).

The connection \( \nabla \) is integrable if and only if, for \( k, d\Omega_k + \Omega_k \wedge \Omega_k = 0 \). For any \( k, k' \), denote by \( g_{kk'} \) the isomorphism
\[
g_{kk'} : (\mathcal{O}(U_k \cap U_{k'}))^m \longrightarrow (\mathcal{O}(U_k \cap U_{k'}))^m,
\]
induced by the isomorphism
\[
g_{U_k} g_{U_{k'}}^{-1} : (\mathcal{O}|_{U_k \cap U_{k'}})^m \longrightarrow (\mathcal{O}|_{U_k \cap U_{k'}})^m.
\]
Then, by using the generator systems, \( g_{kk'} \) is represented by \( G_{kk'} \) a matrix of elements in \( \mathcal{O}(U_k \cap U_{k'}) \), i.e.
\[
g_{kk'} < e_{k,1}, \ldots, e_{k,m} > = < e_{k,1}, \ldots, e_{k,m} > G_{kk'},
\]
and
\[
\Omega_k = G_{kk'}^{-1} dG_{kk'} + G_{kk'}^{-1} \Omega_k G_{kk'},
\]
in $U_k \cap U_{k'}$.

Denote by $M^-$ the real blow-up of $M$ along $H$ and denote by $pr$ the natural projection from $M^-$ to $M$. Let $\mathcal{A}^-$ be the sheaf of germs of functions strongly asymptotically developable, and let $\mathcal{A}'^-$ and $\mathcal{A}'_0$ be the sheaves of germs of functions strongly asymptotically developable to $\mathcal{O}_{M|H}$ and to 0, respectively, over the real blow-up $M^-$. Define the locally free $\mathcal{A}^-$ (resp. $\mathcal{A}'^-$)-sheaf $S^-\Omega^p(*H)$ (resp. $S'^-\Omega^p(*H)$) over the real blow-up $M$ by $S^-\Omega^p(*H) = \mathcal{A}^- \otimes_{pr^*\mathcal{O}} pr^*S\Omega^p(*H)$ (resp. $S'^-\Omega^p(*H) = \mathcal{A}'^- \otimes_{pr^*\mathcal{O}} pr^*S\Omega^p(*H)$), and the locally free $\mathcal{A}'_0$-sheaf $S'_{0}^-\Omega^p$ by $S'_{0}^-\Omega^p = \mathcal{A}'_0 \otimes_{pr^*\mathcal{O}} pr^*S\Omega^p(*H)$ for $p = 0, \cdots, n$. Then, by a natural way, we obtain integrable connections

$$\nabla^- : S^-(*H) \longrightarrow S^-\Omega^1(*H),$$

$$\nabla'^- : S'^-(*H) \longrightarrow S'^-\Omega^1(*H),$$

and

$$\nabla'_0^- : S'_{0}^- \longrightarrow S'_{0}^-\Omega^1(*H).$$

For simplicity, we use also $\nabla$ instead of $\nabla^-$, $\nabla'^-$ and $\nabla'_0$. By the integrability, we can consider the complexes of sheaves

$$S^-(*H) \xrightarrow{\nabla} S^-\Omega^1(*H) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} S^-\Omega^n(*H) \xrightarrow{\nabla} 0,$$

$$S'^-(*H) \xrightarrow{\nabla} S'^-\Omega^1(*H) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} S'^-\Omega^n(*H) \xrightarrow{\nabla} 0,$$

$$S'_{0}^- \xrightarrow{\nabla} S'_{0}^-\Omega^1(*H) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} S'_{0}^-\Omega^n(*H) \xrightarrow{\nabla} 0.$$

Suppose here that $\nabla$ satisfies the following condition: for any point $p \in H$, under the local representation of $\nabla$,

(H.1) $p_i = 0$ and $A_i(0)$ has no eigenvalue of integer for all $i \in [1, n]$,

or

(H.2) $p_{ii} > 0$ and $A_i(0)$ is invertible for all $i \in [1, n']$ or $p_i = 0$ and $A_i(0)$ has no eigenvalue of integer for all $i \in [1, n']$.

Then, we can assert

**Theorem 1.** If the assumption (H.1) is satisfied for any point in $H$, then the above three sequences are exact. If (H.1) or (H.2) is satisfied for any point in $H$, then the above sequences are exact except the second.

Moreover, we consider the complex $(\Gamma(M^-, S^-\Omega^p(*H)), \nabla)$ of global sections:

$$S^-(*H)(M^-)^m \xrightarrow{\nabla} S^-\Omega^1(*H)(M^-)^m \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} S^-\Omega^n(*H)(M^-)^m \xrightarrow{\nabla} 0.$$

Then, we can prove

**Theorem 2.** If $H^1(M, S) = 0$ and if (H.1) or (H.2) is satisfied for any point in $H$, then the following isomorphism is valid:

$$H^1(\Gamma(M^-, S^-\Omega^p(*H)), \nabla) \cong H^1(M^-, \text{Ker}\nabla'_0),$$
where $\mathcal{K}er\nabla_0$ denote the sheaf of solutions of $\nabla_0^-$. Note that we have the natural isomorphism by the projection $pr$

$$H^1(\Gamma(M, \Omega^*(\ast H)), \nabla) \cong H^1(\Gamma(M^-, S^- \Omega^*(\ast H))$$

and we can rewrite the theorem as

**Theorem 2'**. If $H^1(M, S) = 0$ and if (H.1) or (H.2) is satisfied for any point in $H$, then the following isomorphism is valid:

$$H^1(\Gamma(M, S\Omega^*(\ast H)), \nabla) \cong H^1(M^-, \mathcal{K}er\nabla_0^-).$$

**Example.** Consider the case where $M = \mathbb{P}_C^1$, $H = \{\infty\}$ and $\nabla = d + x^{r-1}\wedge$. We can find the basis of $H^1(\Gamma(M, \Omega^*(\ast H)), \nabla)$:

$$H^1(\Gamma(M, \Omega^*(\ast H)), \nabla) = \mathbb{C}_1[dx], \ldots, [x^{r-2}dx].$$

On the other hand, we can find the basis of $H^1(M^-, \mathcal{K}er\nabla_0^-)$ in the following manner. Let $\{U_k | k = 1, \cdots, r\}$ be the covering of $M - H$, where

$$U_k = \{x \in \mathbb{C} | x| \geq R, (\frac{4k-5}{2})\pi < \arg x < (\frac{4k+1}{2})\pi \} \cup \{x \in \mathbb{C} | x| < R\}$$

for $k = 1, \cdots, r$. We put $U_{r+1} = U_1$ and for $k = 1, \cdots, r$, define 1-cocycles $\{f_{j,j+1}^{(k)}(x)\}$ by

$$f_{j,j+1}^{(k)}(x) = \begin{cases} \exp(-\frac{1}{r}x^r), & (x \in U_j \cap U_{j+1}) (j = k) \\ 0, & (x \in U_j \cap U_{j+1}) (j \neq k) \end{cases}$$

Then, we have

$$<\{f_{j,j+1}^{(k)}\}_{j=1,\cdots,r}, k = 1, \cdots, r>$$

as a basis of $H^1(M^-, \mathcal{K}er\nabla_0^-)$.

3 **Isomorphism Theorem in $C^\infty$ case**

We restrict here to treat the case of one variable. We give a $C^\infty$ version of isomorphism theorem of cohomology group. Let $M$, $H$, $\nabla$ be as above. Let $\mathcal{P}_0^{(j,h)}$ be the sheaf of germs of $C^\infty(j,h)$—forms infinitely flat on $H$ over $M$. Consider the following double complex of sheaves:

$$\begin{array}{c}
\mathcal{P}_0^{(0,0)} \rightarrow \mathcal{P}_0^{(0,1)} \\
\nabla \downarrow \\
\mathcal{P}_0^{(1,0)} \rightarrow \mathcal{P}_0^{(1,1)}
\end{array}$$

where $\mathcal{K}er\nabla_0$ denote the sheaf of solutions of $\nabla_0^-$. Note that we have the natural isomorphism by the projection $pr$

$$H^1(\Gamma(M, \Omega^*(\ast H)), \nabla) \cong H^1(\Gamma(M^-, S^- \Omega^*(\ast H))$$

and we can rewrite the theorem as

**Theorem 2'**. If $H^1(M, S) = 0$ and if (H.1) or (H.2) is satisfied for any point in $H$, then the following isomorphism is valid:

$$H^1(\Gamma(M, S\Omega^*(\ast H)), \nabla) \cong H^1(M^-, \mathcal{K}er\nabla_0^-).$$

**Example.** Consider the case where $M = \mathbb{P}_C^1$, $H = \{\infty\}$ and $\nabla = d + x^{r-1}\wedge$. We can find the basis of $H^1(\Gamma(M, \Omega^*(\ast H)), \nabla)$:

$$H^1(\Gamma(M, \Omega^*(\ast H)), \nabla) = \mathbb{C}_1[dx], \ldots, [x^{r-2}dx].$$

On the other hand, we can find the basis of $H^1(M^-, \mathcal{K}er\nabla_0^-)$ in the following manner. Let $\{U_k | k = 1, \cdots, r\}$ be the covering of $M - H$, where

$$U_k = \{x \in \mathbb{C} | x| \geq R, (\frac{4k-5}{2})\pi < \arg x < (\frac{4k+1}{2})\pi \} \cup \{x \in \mathbb{C} | x| < R\}$$

for $k = 1, \cdots, r$. We put $U_{r+1} = U_1$ and for $k = 1, \cdots, r$, define 1-cocycles $\{f_{j,j+1}^{(k)}(x)\}$ by

$$f_{j,j+1}^{(k)}(x) = \begin{cases} \exp(-\frac{1}{r}x^r), & (x \in U_j \cap U_{j+1}) (j = k) \\ 0, & (x \in U_j \cap U_{j+1}) (j \neq k) \end{cases}$$

Then, we have

$$<\{f_{j,j+1}^{(k)}\}_{j=1,\cdots,r}, k = 1, \cdots, r>$$

as a basis of $H^1(M^-, \mathcal{K}er\nabla_0^-)$.
and the complex of global sections
\[ \mathcal{P}_0^{(0,0)}(M) \xrightarrow{\partial} \mathcal{P}_0^{(0,1)}(M) \]
\[ \nabla \downarrow \quad \nabla \downarrow \]
\[ \mathcal{P}_0^{(1,0)}(M) \xrightarrow{\partial} \mathcal{P}_0^{(1,1)}(M) \]
and the associated simple complex
\[ GC^\infty K^- : \mathcal{P}_0^{(0,0)}(M) \xrightarrow{\nabla + \partial} \mathcal{P}_0^{(0,1)}(M) \oplus \mathcal{P}_0^{(1,0)}(M) \xrightarrow{\nabla + \partial} \mathcal{P}_0^{(1,1)}(M) \rightarrow 0. \]

Then, we know the following lemma formally due to Malgrange ([6]).

**Lemma 3.** We have the following isomorphism for \( j = 0, 1, 2 \):
\[ H^j(M^-, \text{Ker} \nabla^-) \cong H^j(GC^\infty K^-) \]

By Theorem 2' and Lemma 3, we can derive the

**Theorem 4.** We have the following isomorphism:
\[ H^1(\Gamma(M, \Omega^\cdot(*H)), \nabla) \cong H^1(GC^\infty K^-) \]

参考文献


