<table>
<thead>
<tr>
<th>Title</th>
<th>Some isomorphism theorems of cohomology groups for completely integrable connections (Geometric methods in asymptotic analysis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Majima, Hideyuki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1997 (1014), 31-35</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/61591">http://hdl.handle.net/2433/61591</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Some isomorphism theorems of cohomology groups for completely integrable connections

お茶の水女子大学理学部数学科 真島 秀行
(Hideyuki Majima, Ochanomizu University)

1 Introduction

About 14 years ago, the author proved an isomorphism theorem between the cohomology group of complex of global meromorphic sections derived from a completely integrable connection and the cohomology group of kernel sheaf with values in the sheaf of functions asymptotically developable to the formal series $0$ for the connection ([5]). Recently, several researchers, who are interested in the intersection theory for differential equations with singular points, pushed him to prove the $C^\infty$ version (cf. [1], [2]). In this paper, firstly, we give a short review of the isomorphism theorem in asymptotic analysis and some examples with concrete calculation of basis for the cohomology groups. Secondly, we explain the $C^\infty$ version.

2 Isomorphism Theorem in Asymptotic Analysis

Let $M$ be a complex manifold and let $H$ be a divisor on $M$ at most normal crossing singularities. We denote by $\Omega^p(\ast H)$ the sheaf of germs of meromorphic $p$-forms which are holomorphic in $M - H$ and have poles on $H$ and denote by $S$ a locally free sheaf of $\mathcal{O}$-modules of rank $m$ on $M$. Put $\mathcal{S}\Omega^p(\ast H) = \Omega^p(\ast H) \otimes_\mathcal{O} S$ for $p = 0, \ldots, n$. For $p = 0$, instead of $\mathcal{S}\Omega^0(\ast H)$, we use frequently $\mathcal{S}(\ast H)$ of which the restriction to $U$, $\mathcal{S}(\ast H)|_U$ is isomorphic to

$$(\mathcal{O}(\ast H))^m|_U = (\mathcal{C}^m \otimes_\mathcal{C} \mathcal{O}(\ast H))|_U$$

and the isomorphism is denoted by $g_U$.

Let $\nabla$ be a connection on $\mathcal{S}\Omega^0(\ast H)$: $\nabla$ is an additive mapping of $\mathcal{S}\Omega^0(\ast H)$ into $\mathcal{S}\Omega^1(\ast H)$ satisfying "Leipnitz' rule"

$$\nabla(f \cdot u) = u \otimes df + f \cdot \nabla(u),$$

for all sections $f \in \Omega^0(\ast H)(U)$ and $u \in \mathcal{S}\Omega^1(\ast H)(U)$. We suppose that the connection is integrable, that is, the composite mapping

$$\nabla_2: \mathcal{S}\Omega^0(\ast H) \longrightarrow \mathcal{S}\Omega^1(\ast H) \longrightarrow \mathcal{S}\Omega^2(\ast H),$$
is a zero mapping.

If we take adequately an open covering \( \{ U_k \}_k \) on \( M \), then to give the connection \( \nabla \) means as follows: for each \( U_k \), the mapping
\[
g_{U_k} \circ \nabla \circ g_{U_k}^{-1} : \Omega^0(*H)(U_k)^m \rightarrow \Omega^1(*H)(U_k)^m,
\]
is induced by a mapping
\[
\nabla_k : \Omega^0(*H)(U_k)^m \rightarrow \Omega^1(*H)(U_k)^m,
\]
which is represented by \( (d + \Omega_k) \) under a generator system
\[
< e_{k,1}, \cdots, e_{k,m} >
\]
of \( (\mathcal{O}(U_k))^m \), i.e.
\[
\nabla_k(< e_{k,1}, \cdots, e_{k,m} > u) = < e_{k,1}, \cdots, e_{k,m} > (du + \Omega_k u)
\]
where \( \Omega_k \) is an \( m \)-by-\( m \) matrix of meromorphic 1-forms on \( U_k \) at most with poles in \( U_k \cap H \).

Let \( x_1, \cdots, x_n \) be holomorphic local coordinates on \( U_k \) and suppose
\[
U_k \cap H = \{(x_1, \cdots, x_n) \mid x_1 \cdots x_{n'} = 0\},
\]
then \( \Omega_k \) is of the form
\[
\Omega_k = \sum_{i=1}^{n'} x^{-p_i} x_i^{-1} A_i(x) dx_i + \sum_{i=n'+1}^n x^{-p_i} A_i(x) dx_i,
\]
where \( p_i = (p_{i1}, \cdots, p_{in'}, 0, \cdots, 0) \in \mathbb{N}^n \) and \( A_i(x) \) is an \( m \)-by-\( m \) matrix of holomorphic functions in \( U_k \) for \( i = 1, \cdots, n \).

The connection \( \nabla \) is integrable if and only if, for \( k, d \Omega_k + \Omega_k \wedge \Omega_k = 0 \). For any \( k, k' \), denote by \( g_{kk'} \) the isomorphism
\[
g_{kk'} : (O(U_k \cap U_{k'}))^m \rightarrow (O(U_k \cap U_{k'}))^m,
\]
induced by the isomorphism
\[
g_{U_k} g_{U_k'}^{-1} : (O|_{U_k \cap U_{k'}})^m \rightarrow (O|_{U_k \cap U_{k'}})^m.
\]
Then, by using the generator systems, \( g_{kk'} \) is represented by \( G_{kk'} \) a matrix of elements in \( O(U_k \cap U_{k'}) \), i.e.
\[
g_{kk'} < e_{k,1}, \cdots, e_{k,m} > = < e_{k,1}, \cdots, e_{k,m} > G_{kk'},
\]
and
\[
\Omega_k = G_{kk'}^{-1} d G_{kk'} + G_{kk'}^{-1} \Omega_k G_{kk'},
\]
in $U_k \cap U_{k'}$.

Denote by $M^-$ the real blow-up of $M$ along $H$ and denote by $pr$ the natural projection from $M^-$ to $M$. Let $\mathcal{A}$ be the sheaf of germs of functions strongly asymptotically developable, and let $\mathcal{A}'$ and $\mathcal{A}_0'$ be the sheaves of germs of functions strongly asymptotically developable to $\mathcal{O}_{M|^H}$ and to 0, respectively, over the real blow-up $M^-$. Define the locally free $\mathcal{A}^-$ (resp. $\mathcal{A}'^-$)-sheaf $S^-\Omega^p(*)H$ (resp. $S'^-\Omega^p(*)H$) over the real blow-up $M^-$ by $S^-\Omega^p(*)H = \mathcal{A}^- \otimes_{pr} pr^*S\Omega^p(*)H$ (resp. $S'^-\Omega^p(*)H = \mathcal{A}'^- \otimes_{pr} pr^*S\Omega^p(*)H$), and the locally free $\mathcal{A}_0^-$ sheaf $S_0^-\Omega^p$ by $S_0^-\Omega^p = \mathcal{A}_0^- \otimes_{pr} pr^*S\Omega^p(*)H$ for $p = 0, \cdots, n$. Then, by a natural way, we obtain integrable connections

\[
\nabla^- : S^-(*)H \longrightarrow S^-\Omega^1(*)H),
\]
\[
\nabla'^- : S'^-(*)H \longrightarrow S'^-\Omega^1(*)H),
\]
and
\[
\nabla_0^- : S_0^- \longrightarrow S_0^-\Omega^1(*)H).
\]

For simplicity, we use also $\nabla$ instead of $\nabla^-$, $\nabla'^-$ and $\nabla_0^-$. By the integrability, we can consider the complexes of sheaves

\[
S^-(*)H \xrightarrow{\nabla} S^-\Omega^1(*)H \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} S^-\Omega^n(*)H \xrightarrow{\nabla} 0
\]
\[
S'^-(*)H \xrightarrow{\nabla} S'^-\Omega^1(*)H \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} S'^-\Omega^n(*)H \xrightarrow{\nabla} 0
\]
\[
S_0^- \xrightarrow{\nabla} S_0^-\Omega^1(*)H \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} S_0^-\Omega^n(*)H \xrightarrow{\nabla} 0.
\]

Suppose here that $\nabla$ satisfies the following condition: for any point $p \in H$, under the local representation of $\nabla$,

(H.1) $p_i = 0$ and $A_i(0)$ has no eigenvalue of integer for all $i \in [1, n]$,

or

(H.2)$p_{ii} > 0$ and $A_i(0)$ is invertible for all $i \in [1, n']$ or $p_i = 0$ and $A_i(0)$ has no eigenvalue of integer for all $i \in [1, n']$.

Then, we can assert

**Theorem 1.** If the assumption (H.1) is satisfied for any point in $H$, then the above three sequences are exact. If (H.1) or (H.2) is satisfied for any point in $H$, then the above sequences are exact except the second.

Moreover, we consider the complex $(\Gamma(M^-, S^-\Omega^*(*)H), \nabla)$ of global sections:

\[
S^-(*)(H)(M^-)^m \xrightarrow{\nabla} S^-\Omega^1(*)(H)(M^-)^m \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} S^-\Omega^1(*)(H)(M^-)^m \xrightarrow{\nabla} 0.
\]

Then, we can prove

**Theorem 2.** If $H^1(M, S) = 0$ and if (H.1) or (H.2) is satisfied for any point in $H$, then the following isomorphism is valid:

\[
H^1(\Gamma(M^-, S^-\Omega^*(*)H), \nabla) \cong H^1(M^-, \text{Ker}\nabla_0^-),
\]
where $\mathcal{K}er \nabla_0$ denote the sheaf of solutions of $\nabla_0^-$. Note that we have the natural isomorphism by the projection $pr$

\[ H^1(\Gamma(M, \Omega^\bullet(*H)), \nabla) \cong H^1(\Gamma(M^-, \mathcal{S}^-\Omega^\bullet(*H)) \]

and we can rewrite the theorem as

**Theorem 2'**. If $H^1(M, S) = 0$ and if (H.1) or (H.2) is satisfied for any point in $H$, then the following isomorphism is valid:

\[ H^1(\Gamma(M, \Omega^\bullet(*H)), \nabla) \cong H^1(M^-, \mathcal{K}er \nabla^-_0). \]

**Example.** Consider the case where $M = \mathbb{P}^1_\mathbb{C}$, $H = \{\infty\}$ and $\nabla = d + x^{r-1}\wedge$. We can find the basis of $H^1(\Gamma(M, \Omega^\bullet(*H)), \nabla)$:

\[ H^1(\Gamma(M, \Omega^\bullet(*H)), \nabla) = \mathbb{C}\langle [dx], \cdots, [x^{r-2}dx] \rangle. \]

On the other hand, we can find the basis of $H^1(M^-, \mathcal{K}er \nabla^-_0)$ in the following manner. Let \{\$U_k|k=1,\cdots,r\}$ be the covering of $M - H$, where

\[ U_k = \{ x \in \mathbb{C} \mid |x| \geq R, \frac{(4k-5)\pi}{2r} < \arg x < \frac{(4k+1)\pi}{2r} \} \cup \{ x \in \mathbb{C} \mid |x| < R \} \]

for $k = 1, \cdots, r$. We put $U_{r+1} = U_1$ and for $k = 1, \cdots, r$, define 1-cocycles $\{f_{j,j+1}^{(k)}\} (j = 1, \cdots, r)$ by

\[ f_{j,j+1}^{(k)}(x) = \begin{cases} \exp\left(-\frac{1}{r}x^r\right), & (x \in U_j \cap U_{j+1}) (j = k) \\ 0, & (x \in U_j \cap U_{j+1}) (j \neq k) \end{cases} \]

Then, we have

\[ <\{f_{j,j+1}^{(k)}\}_{j=1,\cdots,r}, k = 1, \cdots, r > \]

as a basis of $H^1(M^-, \mathcal{K}er \nabla^-_0)$.

**3 Isomorphism Theorem in $C^\infty$ case**

We restrict here to treat the case of one variable. We give a $C^\infty$ version of isomorphism theorem of cohomology group. Let $M, H, \nabla$ be as above. Let $\mathcal{P}_0^{(j,h)}$ be the sheaf of germs of $C^\infty(j, h)$-forms infinitely flat on $H$ over $M$. Consider the following double complex of sheaves:

\[ \mathcal{P}_0^{(0,0)} \overset{\partial}{\rightarrow} \mathcal{P}_0^{(0,1)} \]

\[ \nabla \downarrow \quad \nabla \downarrow \]

\[ \mathcal{P}_0^{(1,0)} \overset{\partial}{\rightarrow} \mathcal{P}_0^{(1,1)} \]
and the complex of global sections

\[
\mathcal{P}_0^{(0,0)}(M) \xrightarrow{\partial} \mathcal{P}_0^{(0,1)}(M) \\
\nabla \downarrow \quad \nabla \downarrow
\]

\[
\mathcal{P}_0^{(1,0)}(M) \xrightarrow{\partial} \mathcal{P}_0^{(1,1)}(M)
\]

and the associated simple complex

\[
GC^\infty K^- : \mathcal{P}_0^{(0,0)}(M) \xrightarrow{\nabla + \partial} \mathcal{P}_0^{(0,1)}(M) \oplus \mathcal{P}_0^{(1,0)}(M) \xrightarrow{\nabla + \partial} \mathcal{P}_0^{(1,1)}(M) \rightarrow 0.
\]

Then, we know the following lemma formally due to Malgrange ([6]).

**Lemma 3.** We have the following isomorphism for \( j = 0, 1, 2 \):

\[
H^j(M^-, \text{Ker}\nabla^-) \cong H^j(GC^\infty K^-)
\]

By Theorem 2' and Lemma 3, we can derive the

**Theorem 4.** We have the following isomorphism:

\[
H^1(\Gamma(M, \Omega^*(*H)), \nabla) \cong H^1(GC^\infty K^-).
\]

参考文献


