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<td>YOSHIDA, Y.; YASUDA, M.; NAKAGAMI, J.; KURANO, M.</td>
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Kyoto University
On a property of fuzzy stopping times

Y. YOSHIDA, M. YASUDA, J. NAKAGAMI and M. KURANO

Faculty of Economics and Business Administration, Kitakyushu University, Kokuraminami, Kitakyushu 802, Japan. Faculty of Science, Chiba University, Faculty of Education, Yayoi-cho, Inage-ku, Chiba 263, Japan.

Abstract

This note is concerned with a fuzzy stopping time for a dynamic fuzzy system. A new class of fuzzy stopping times is introduced and constructed by subsets of α-cut for fuzzy states. The results are applied to the optimization of a corresponding problem with an additive weighting function.

Keywords: Fuzzy stopping times; Markov property; α-cuts of fuzzy sets; optimality.

1 Introduction and notations

The stopping time with fuzziness, which is called a fuzzy stopping time, is considered by our previous paper [11] in which optimization of a corresponding fuzzy problem is pursued by the constructive method.

In this note, we introduce a new class of fuzzy stopping times defined by subsets of the α-cuts of fuzzy states and we apply it to a fuzzy stopping problem with additive weighting functions as the scalarization of the fuzzy total rewards. As related works, refer to [1, 5, 6, 7, 15].

In the remainder of this section, a fuzzy stopping time for a fuzzy dynamic system is defined explicitly. A new class of fuzzy stopping time is introduced in Section 2 and its construction is discussed. These results are applied to the 'optimization' of a corresponding fuzzy stopping problem in Section 3. In Section 4, an example is given to illustrate the results.

Let $E, E_1, E_2$ be convex compact subsets of some Banach space. Throughout the paper, we will denote a fuzzy set and a fuzzy relation by their membership functions. For the theory of fuzzy sets, refer to Zadeh [16] and Novák [12]. A fuzzy set $\tilde{u}: E \mapsto [0, 1]$ is called convex if

$$\tilde{u}(\lambda x + (1 - \lambda)y) \geq \tilde{u}(x) \land \tilde{u}(y), \quad x, y \in E, \lambda \in [0, 1],$$

where $a \land b := \min\{a, b\}$ for real numbers $a, b$ (c.f. Chen-wei Xu [2]). Also, a fuzzy relation $\tilde{h}: E_1 \times E_2 \mapsto [0, 1]$ is called convex if

$$\tilde{h}(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \geq \tilde{h}(x_1, y_1) \land \tilde{h}(x_2, y_2)$$

for $x_1, x_2 \in E_1, y_1, y_2 \in E_2$ and $\lambda \in [0, 1]$.

Let $\mathcal{F}(E)$ be the set of all convex fuzzy sets, $\tilde{u}$, on $E$ whose membership functions are upper semi-continuous and have compact supports and the normality condition : $\sup_{x \in E} \tilde{u}(x) = 1$. The α-cut ($\alpha \in [0, 1]$) of the fuzzy set $\tilde{u}$ is defined by

$$\tilde{u}_\alpha := \{x \in E \mid \tilde{u}(x) \geq \alpha\} (\alpha > 0) \quad \text{and} \quad \tilde{u}_0 := \overline{\text{cl}}\{x \in E \mid \tilde{u}(x) > 0\},$$

where cl denotes the closure of a set. We denote by $\mathcal{C}(E)$ the collection of all compact convex subsets of $E$. Clearly, $\tilde{u} \in \mathcal{F}(E)$ means $\tilde{u}_\alpha \in \mathcal{C}(E)$ for all $\alpha \in [0, 1]$.

Let $\mathbb{R}$ be the set of all real numbers. We see, from the definition, that $\mathcal{C}(\mathbb{R})$ is the set of all bounded closed intervals in $\mathbb{R}$. The elements of $\mathcal{F}(\mathbb{R})$ are called fuzzy numbers. The addition and the scalar multiplication on $\mathcal{F}(\mathbb{R})$ are defined as follows (see Puri and Ralescu [13]): For $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbb{R})$ and $\lambda \geq 0$,

$$[\tilde{m} + \tilde{n}](x) := \sup_{x_1, x_2 \in \mathbb{R}, x_1 + x_2 = x} \{\tilde{m}(x_1) \land \tilde{n}(x_2)\} \quad (x \in \mathbb{R}) \quad (1.1)$$

and

$$(\lambda \tilde{m})(x) := \begin{cases} \tilde{m}(x/\lambda) & \text{if } \lambda > 0 \\ 1_0(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R}). \quad (1.2)$$

And hence

$$(\tilde{m} + \tilde{n})_\alpha = \tilde{m}_\alpha + \tilde{n}_\alpha \quad \text{and} \quad (\lambda \tilde{m})_\alpha = \lambda \tilde{m}_\alpha \quad (\alpha \in [0, 1]),$$
where $A + B := \{x + y \mid x \in A, y \in B\}$, $\lambda A := \{\lambda x \mid x \in A\}$, $A + \emptyset = \emptyset + A := A$ and $\lambda \emptyset := \emptyset$ for any non-empty closed intervals $A, B \in C(\mathbf{R})$. We use the following lemma.

**Lemma 1.1** (Chen-wei Xu [2]).

(i) For any $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbf{R})$ and $\lambda \geq 0$, it holds that $\tilde{m} + \tilde{n} \in \mathcal{F}(\mathbf{R})$.

(ii) Let $\tilde{u} \in \mathcal{F}(E_1)$ and $\tilde{p} \in \mathcal{F}(E_1 \times E_2)$. Then $\sup_{x \in E_1} \{\tilde{u}(x) \wedge \tilde{p}(x, \cdot)\} \in \mathcal{F}(E_2)$.

We consider the dynamic fuzzy system([9]), which is denoted by the elements $(S, \tilde{q})$ as follows.

**Definition 1.**

(i) The state space $S$ is a convex compact subset of some Banach space. In general, the system is fuzzy, so that the state of the system is called a fuzzy state and is denoted by an element of $\mathcal{F}(S)$.

(ii) The law of motion for the system is denoted by time-invariant fuzzy relations $\tilde{q} : S \times S \mapsto [0, 1]$, and assume that $\tilde{q} \in \mathcal{F}(S \times S)$.

If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}(S)$, the state is moved to a new fuzzy state $Q(\tilde{s})$ after unit time, where $Q : \mathcal{F}(S) \mapsto \mathcal{F}(S)$ is defined by

$$Q(\tilde{s})(y) := \sup_{x \in S} \{\tilde{s}(x) \wedge \tilde{q}(x, y)\} \quad (y \in S).$$

Note that the map $Q$ is well-defined by Lemma 1.1.

For the dynamic fuzzy system $(S, \tilde{q})$ with a given initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$, we can define a sequence of fuzzy states $\{\tilde{s}_t\}_{t=1}^\infty$ by

$$\tilde{s}_1 := \tilde{s} \quad \text{and} \quad \tilde{s}_{t+1} := Q(\tilde{s}_t) \quad (t \geq 1).$$

A fuzzy stopping time for this sequence $\{\tilde{s}_t\}_{t=1}^\infty$ is defined in the next section. In order to define a fuzzy stopping time, we need the following preliminaries.

Associated with the fuzzy relation $\tilde{q}$, the corresponding maps $Q_\alpha : C(S) \mapsto C(S) \ (\alpha \in [0, 1])$ are defined as follows: For $D \in C(S)$,

$$Q_\alpha(D) := \left\{ \begin{array}{ll} \{y \in S \mid \tilde{q}(x, y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0 \\
\{x \in S \mid \tilde{q}(x, y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0, \end{array} \right.$$  

From the assumption on $\tilde{q}$, the maps $Q_\alpha$ is well-defined. The iterates $Q_\alpha^t \ (t \geq 0)$ are defined by setting $Q_\alpha^0 := I($identity$)$ and iteratively,

$$Q_\alpha^{t+1} := Q_\alpha Q_\alpha^t \ (t \geq 0).$$

In the following lemma, which is easily verified by the idea in the proof of Kurano et al. [9, Lemma 1], the $\alpha$-cuts of $Q(\tilde{s})$ defined by (1.3) is specified using the maps $Q_\alpha$.

**Lemma 1.2** ([9, 10]). For any $\alpha \in [0, 1]$ and $\tilde{s} \in \mathcal{F}(S)$, we have:

$$Q(\tilde{s})_\alpha = Q_{\alpha}(\tilde{s}_\alpha);$$

(ii) $\tilde{s}_{t, \alpha} = Q_{\alpha}^{-t}(\tilde{s}_\alpha) \quad (t \geq 1),$

where $\tilde{s}_{t, \alpha} := (\tilde{s}_t)_\alpha$ and $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.4) with $\tilde{s}_1 = \tilde{s}$.

## 2 Fuzzy stopping times

In this section, we define a fuzzy stopping time to be discussed here. And a new class of fuzzy stopping times is introduced, which is constructed thorough subsets of $\alpha$-cuts of fuzzy states.

For the sake of simplicity, denote $\mathcal{F} := \mathcal{F}(S)$. Let $\mathbf{N} = \{1, 2, \cdots\}$ and $\mathcal{F}'$ a subset of $\mathcal{F}$.

**Definition 2** (cf.[11]). A fuzzy stopping time(FST) on $\mathcal{F}'$ is a fuzzy relation $\tilde{\sigma} : \mathcal{F}' \times \mathbf{N} \mapsto [0, 1]$ such that, for each fuzzy state $\tilde{s} \in \mathcal{F}'$, $\tilde{\sigma}(\tilde{s}, t)$ is non-increasing in $t$ and there exists a natural number $t(\tilde{s}) \in \mathbf{N}$ with $\tilde{\sigma}(\tilde{s}, t) = 0$ for all $t \geq t(\tilde{s})$. 


We note here that 0 represents 'stop' and 1 represents 'continue' in the grade of membership (cf.[11]). An FST $\tilde{\sigma}(\tilde{s}, \cdot)$ means the degree of 'continue' at time $t$ starting at a fuzzy state $\tilde{s} \in F'$. The set of all FSTs on $F'$ is denoted by $\Sigma(F')$. Assuming $Q(F') \subset F'$, an FST $\tilde{\sigma} \in \Sigma(F')$ is called Markov if there exist a mapping $\delta : F' \rightarrow [0, 1]$ satisfying

(i) $\delta(Q(\tilde{s})) \leq \delta(\tilde{s})$, and

(ii) $\tilde{\sigma}(\tilde{s}, t) = \delta(\tilde{s}_t)$ for all $\tilde{s} \in F'$ and $t \geq 1$,

where $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.4) with $\tilde{s}_1 = \tilde{s}$.

The above $\delta$ is called a support of $\tilde{\sigma}$. We consider ourselves with the construction of Markov FSTs. For this purpose, we assume the following condition holds.

**Condition A1.** For each $\alpha \in [0, 1]$, there exists a non-empty subset $K_\alpha$ of $C(S)$ satisfying

$$Q_{\alpha}(K_\alpha) \subset K_\alpha.$$  \hspace{1cm} (2.1)

Using this subset $K_\alpha$, we define a sequence of subsets $\{K^1_{\alpha}\}_{t=1}^\infty$ inductively by

$$K^1_{\alpha} := K_\alpha$$  \hspace{1cm} (2.2)

and for each $t \geq 2$,

$$K^t_{\alpha} := \{c \in C(S) \mid Q_\alpha(c) \in K^{t-1}_{\alpha}\}.$$  \hspace{1cm} (2.3)

Clearly, $K^t_{\alpha} = Q_\alpha^{-1}(K^{t-1}_{\alpha}) = Q_\alpha^{-(t-1)}(K_\alpha)$. Also, it holds from (2.1) that $K^t_{\alpha} \subset K^{t+1}_{\alpha} (t \geq 1)$.

To simplify our discussion, we assume the following condition holds henceforth.

**Condition A2.** For all $\alpha \in [0, 1]$, it holds that

$$C(S) = \bigcup_{t=1}^\infty K^t_{\alpha}.$$  \hspace{1cm} (2.4)

For $c \in C(S)$ and $\alpha \in [0, 1]$, define $\tilde{\sigma}_\alpha(c)$ by

$$\tilde{\sigma}_\alpha(c) := \min\{t \geq 1 \mid c \in K^t_{\alpha}\}.$$  \hspace{1cm} (2.5)

That is, it is the first entry time of $c \in C(S)$ with the grade $\alpha$. We define a restricted class $\hat{F} \subset F$ by

$$\hat{F} := \{\tilde{s} \in F \mid \tilde{\sigma}_\alpha(\tilde{s}_\alpha) \text{ is non-increasing in } \alpha \in [0, 1]\}.$$  \hspace{1cm} (2.6)

Using the class $\{\tilde{\sigma}_\alpha(\tilde{s}_\alpha) \mid \alpha \in [0, 1]\}$, for the restricted element $\tilde{s} \in \hat{F}$, let us construct

$$\hat{\sigma}(\tilde{s}, t) := \sup_{\alpha \in [0, 1]} \{\alpha \wedge 1_{D_\alpha}(t)\} \quad (t \geq 1),$$  \hspace{1cm} (2.7)

where $1_{D_\alpha}$ is the indicator of a set $D_\alpha = \{t \in \mathbb{N} \mid \tilde{\sigma}_\alpha(\tilde{s}_\alpha) > t\}$. This is the usual technique of constructing a corresponding fuzzy number from the class of level sets. Now let

$$\hat{\sigma}(\tilde{s}, \cdot)_\alpha := \min\{t \in \mathbb{N} \mid \hat{\sigma}(\tilde{s}, t) < \alpha\}.$$  \hspace{1cm} (2.8)

Then we obtain the following theorem.

**Theorem 2.1.**

(i) $\hat{\sigma}(\tilde{s}, \cdot)_\alpha = \tilde{\sigma}_\alpha(\tilde{s}_\alpha), \quad \tilde{s} \in \hat{F}, \; \alpha \in [0, 1]$;

(ii) $\hat{\sigma}$ is an FST on $\hat{F}$.
Proof. By (2.6) and (2.7), we have that \( \hat{\sigma}(\tilde{s}, t) \leq t \) is equivalent to \( \hat{\sigma}_\alpha(\tilde{s}_\alpha) \leq t \) for all \( t \geq 1 \). This fact shows (i). From Condition A2, there exists \( t^* \in \mathbb{N} \) with \( \hat{s}_0 \in \mathcal{K}_{t^*}^1 \). So, \( \hat{\sigma}_\alpha(\tilde{s}_\alpha) \leq \hat{s}_0(\tilde{s}_0) \leq t^* \) for all \( \alpha \in [0, 1] \), which shows by (2.5) that \( \hat{\sigma}(\tilde{s}, t) = 0 \) for all \( t \geq t^* \). Since \( \hat{\sigma}(\tilde{s}, t + 1) \leq \hat{\sigma}(\tilde{s}, t) \) holds clearly for \( t \geq 1 \) from the definition (2.6), we also obtain (ii). \( q.e.d. \)

In order to show the Markov property of \( \hat{\sigma} \), we need the following lemma.

**Lemma 2.1.** Let \( \tilde{s} \in \hat{\mathcal{F}} \). Then

(i) \( \hat{\sigma}(\tilde{s}, t) = \alpha \) if and only if, for any \( \epsilon > 0 \),

\[ \tilde{s}_{\alpha + \epsilon} \in \mathcal{K}_\alpha^t + \epsilon \quad \text{and} \quad \tilde{s}_{\alpha - \epsilon} \notin \mathcal{K}_\alpha^t - \epsilon; \]

(ii) \( \tilde{s}_t \in \hat{\mathcal{F}} \) (\( t \geq 1 \)).

**Proof.** By (2.6), \( \hat{\sigma}(\tilde{s}, t) = \sup\{ \alpha \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) > t \} \). So, (i) follows from (2.4). From Lemma 1.2(ii), for \( l \geq 1 \),

\[ \hat{\sigma}_\alpha((\tilde{s}_l)_\alpha) = \hat{\sigma}_\alpha(\tilde{s}_l) = \hat{\sigma}_\alpha(Q_{\alpha}^{l - 1}(\tilde{s}_\alpha)). \]

So, by (2.3) and (2.4),

\[ \hat{\sigma}_\alpha((\tilde{s}_l)_\alpha) = \min\{ t \geq 1 \mid Q_{\alpha}^{l - 1}(\tilde{s}_\alpha) \in \mathcal{K}_\alpha^t \} \]

\[ = \min\{ t \geq 1 \mid \tilde{s}_\alpha \in \mathcal{K}_\alpha^{t + l - 1} \} \]

\[ = \max\{ \hat{\sigma}_\alpha(\tilde{s}_\alpha) - (l - 1), 1 \}, \]

and it is non-increasing in \( \alpha \in [0, 1] \) since \( \tilde{s} \in \hat{\mathcal{F}} \). Therefore we obtain (ii). \( q.e.d. \)

**Theorem 2.2.** Let \( \tilde{s} \in \hat{\mathcal{F}} \). Then, \( \hat{\sigma} \) is a Markov FST with \( \hat{s} \).

**Proof.** Let \( \{ \tilde{s}_l \}_{l=1}^\infty \) be defined by (1.4) with \( \tilde{s}_1 = \tilde{s} \). First, we prove

\[ \hat{\sigma}(\tilde{s}, t + r) = \hat{\sigma}(\tilde{s}, t) \wedge \hat{\sigma}(\tilde{s}_{t+1}, r) \quad \text{for} \ t, r \in \mathbb{N}. \]  \[ (2.8) \]

Note that \( \hat{\sigma}(\tilde{s}_{t+1}, r) \) is well-defined from Lemma 2.1(ii). Let \( \alpha = \hat{\sigma}(\tilde{s}, t + r) \). From Lemma 2.1(i), we have

\[ \tilde{s}_{\alpha + \epsilon} \in \mathcal{K}_{\alpha + \epsilon}^t \quad \text{and} \quad \tilde{s}_{\alpha - \epsilon} \notin \mathcal{K}_{\alpha - \epsilon}^t \quad \text{for any} \ \epsilon > 0. \]

Noting \( Q_{\alpha}^t(K_{\alpha}^l) = \mathcal{K}_{\alpha}^{l - t} \) (\( 1 \leq t < l \)) and Lemma 1.2(ii), we obtain

\[ \tilde{s}_{t+1,\alpha + \epsilon} = Q_{\alpha + \epsilon}^t(\tilde{s}_{\alpha + \epsilon}) \in Q_{\alpha + \epsilon}^t(\mathcal{K}_{\alpha + \epsilon}^t) = \mathcal{K}_{\alpha + \epsilon}^t \]  \[ (2.9) \]

and

\[ \tilde{s}_{t+1,\alpha - \epsilon} = Q_{\alpha - \epsilon}^t(\tilde{s}_{\alpha - \epsilon}) \notin Q_{\alpha - \epsilon}^t(\mathcal{K}_{\alpha - \epsilon}^t) = \mathcal{K}_{\alpha - \epsilon}^t. \]  \[ (2.10) \]

Therefore, we get \( \hat{\sigma}(\tilde{s}_{t+1}, r) = \alpha \) from Lemma 2.1(i). Namely, \( \hat{\sigma}(\tilde{s}, t + r) = \hat{\sigma}(\tilde{s}_{t+1}, r) \). Since \( \hat{\sigma}(\tilde{s}, t + r) \leq \hat{\sigma}(\tilde{s}, t) \) from Theorem 2.1(iii), we obtain \( \hat{\sigma}(\tilde{s}, t) \wedge \hat{\sigma}(\tilde{s}_{t+1}, r) = \alpha \), and so (2.8) holds.

Next, we put \( \hat{\delta}(\tilde{s}) = \hat{\sigma}(\tilde{s}, 1) \) for \( \tilde{s} \in \hat{\mathcal{F}} \). From (2.8), we get

\[ \hat{\delta}(\tilde{s}, t) = \hat{\delta}(\tilde{s}, 1) \wedge \hat{\delta}(\tilde{s}_2, t - 1) \]

\[ = \hat{\delta}(\tilde{s}, 1) \wedge \hat{\delta}(\tilde{s}_2, 1) \wedge \hat{\delta}(\tilde{s}_3, t - 2) \]

\[ = \cdots \]

\[ = \bigwedge_{l=1}^t \hat{\delta}(\tilde{s}_l) \]

\[ = \hat{\delta}(\tilde{s}_1) \quad \text{for} \ t \in \mathbb{N}. \]

Since we also have \( \hat{\delta}(Q(\tilde{s})) \leq \hat{\delta}(\tilde{s}) \) from Theorem 2.1(ii), \( \hat{\sigma} \) is a Markov FST with \( \hat{s} \). \( q.e.d. \)
3 Applications to fuzzy stopping problem

In this section, applying the results in the previous section, we obtain the optimal FST for a fuzzy dynamic system with fuzzy rewards ([10]) when the weighting function is additive.

Firstly, we will formulate the stopping problem to be considered here. Let \( \tilde{r} : S \times \mathbb{R} \rightarrow [0, 1] \) be a fuzzy relation satisfying \( \tilde{r} \in \mathcal{F}(S \times \mathbb{R}) \). If the system is in a fuzzy state \( \tilde{s} \in \mathcal{F} \), the following fuzzy reward is earned:

\[
R(\tilde{s})(z) := \sup_{z \in S} \{ \overline{\tilde{r}}(x, z) \} , \quad z \in \mathbb{R}.
\]

Then we can define a sequence of fuzzy rewards \( \{ R(\tilde{s}_t) \}_{t=1}^{\infty} \), where \( \{ \tilde{s}_t \}_{t=1}^{\infty} \) is defined in (1.4) with the initial fuzzy state \( \tilde{s}_1 = \tilde{s} \). Let

\[
\varphi(\tilde{s}, t) := \sum_{t=1}^{t-1} R(\tilde{s}_t) \quad \text{for } t \in \mathbb{N}.
\]

We need the following lemma, which is proved in [9].

Lemma 3.1 ([9, 10]). For \( t \in \mathbb{N} \) and \( \alpha \geq 0 \),

\[
\varphi(\tilde{s}, t) = \sum_{t=1}^{t-1} R(\tilde{s}_t) = \sum_{t=1}^{t-1} R(\tilde{s}_t, \alpha)
\]

holds, where

\[
R(\tilde{s}_t, \alpha) := \left\{ \begin{array}{ll}
\{ z \in \mathbb{R} \mid \tilde{r}(x, z) \geq \alpha \text{ for some } z \in \tilde{s}_t \} & \text{for } \alpha > 0 \\
c\{ z \in \mathbb{R} \mid \tilde{r}(x, z) > 0 \text{ for some } z \in \tilde{s}_t \} & \text{for } \alpha = 0.
\end{array} \right.
\]

Let \( g : C(\mathbb{R}) \rightarrow \mathbb{R} \) be any additive map with \( g(\phi) = 0 \), that is,

\[
g(c' + c'') = g(c') + g(c'') \quad \text{for } c', c'' \in C(S).
\]

Adapting this \( g \) for a weighting function (see [4]), when an FST \( \hat{\sigma} \in \Sigma(\hat{\mathcal{F}}) \) and an initial fuzzy state \( \tilde{s} \in \hat{\mathcal{F}} \) are used, the scalarization of the total fuzzy reward is given by

\[
G(\tilde{s}, \hat{\sigma}) = \int_{0}^{1} g(\varphi(\tilde{s}, \hat{\sigma}, \alpha)) d\alpha = \int_{0}^{1} g(\sum_{t=1}^{t-1} R(\tilde{s}_t, \alpha)) d\alpha,
\]

where \( \sum_{t=1}^{t-1} R(\tilde{s}_t, \alpha) = \phi \) and \( \hat{\sigma}_t \) means \( \hat{\sigma}(\tilde{s}_t, \alpha) = \min\{ t \in \mathbb{N} \mid \hat{\sigma}(\tilde{s}_t, t) < \alpha \} \) for simplicity. Since \( \varphi(\tilde{s}, \hat{\sigma}, \alpha) \in C(\mathbb{R}) \) and the map \( \alpha \rightarrow g(\varphi(\tilde{s}, \hat{\sigma}, \alpha)) \) is left-continuous in \( \alpha \in (0, 1] \), therefore the right-hand integral of (3.3) is well-defined. For a given \( \mathcal{F}' \subset \mathcal{F} \), our objective is to maximize (3.3) over all FSTs \( \hat{\sigma} \in \Sigma(\mathcal{F}') \) for each initial fuzzy state \( \tilde{s} \in \mathcal{F}' \).

Definition 3. An FST \( \hat{\sigma} \) with \( \tilde{s} \in \mathcal{F}' \) is called an \( \tilde{s} \)-optimal if

\[
G(\tilde{s}, \hat{\sigma}) \leq G(\tilde{s}, \hat{\sigma}^*) \quad \text{for all } \hat{\sigma} \in \Sigma(\mathcal{F}')
\]

If \( \hat{\sigma}^* \) is \( \tilde{s} \)-optimal for all \( \tilde{s} \in \mathcal{F}' \), \( \hat{\sigma}^* \) is called optimal in \( \mathcal{F}' \).

Now we will seek a \( \tilde{s} \)-optimal or an optimal FST by using the results in the previous sections. For each \( \alpha \in (0, 1] \), let

\[
\mathcal{K}_\alpha(g) := \{ c \in C(S) \mid g(R(\alpha)) \leq 0 \}.
\]

Here we need the following Assumptions B1 and B2, which are assumed to hold henceforth.

Assumption B1 (Closedness).

\[
Q_\alpha(\mathcal{K}_\alpha(g)) \subset \mathcal{K}_\alpha(g) \quad \text{for all } \alpha \in (0, 1]
\]
Now we define the sequence $\{\mathcal{K}_t^\alpha(g)\}_{t=1}^\infty$ by (2.2) - (2.3), that is,
\[
\mathcal{K}_t^\alpha(g) = Q_\alpha^{-(t-1)}(\mathcal{K}_\alpha(g)) \quad \text{for } t \geq 1.
\] (3.5)

**Assumption B2.** For all $\alpha \in [0, 1]$, it holds that
\[
C(S) = \bigcup_{t=1}^\infty \mathcal{K}_t^\alpha(g).
\]

Using the sequence $\{\mathcal{K}_t^\alpha(g)\}_{t=1}^\infty$ given in (3.5), we define $\hat{\sigma}_\alpha, \hat{\mathcal{F}}$, $\hat{\sigma}$ and $\hat{\sigma}(\cdot, \cdot)_\alpha$, respectively, by (2.4), (2.5), (2.6) and (2.7). Then, from Theorems 2.1 and 2.2, $\hat{\sigma}$ is a Markov FST on $\hat{\mathcal{F}}$.

The following theorem will be proved by applying the idea of the one-step look ahead (OLA) policy[3, 8, 14] for stochastic stopping problems.

**Theorem 3.1.** Under Assumptions B1 and B2, $\hat{\sigma}$ is optimal in $\hat{\mathcal{F}}$.

**Proof.** Firstly, consider the deterministic stopping problem which maximizes $g(\varphi(\tilde{s}, t)\alpha)$ over $t \geq 1$. As $g$ is additive, $g(\varphi(\tilde{s}, t)\alpha) = \Sigma_{l=1}^{t-1} g(R_\alpha(\tilde{s}_l, \alpha))$. Therefore $g(\varphi(\tilde{s}, l)\alpha) \geq g(\varphi(\tilde{s}, l+1)\alpha)$ if and only if $\tilde{s}_{l+1} \in K_\alpha(g)$. By the assumption B1, $\tilde{s}_{l+1} \in K_\alpha(g)$ implies $g(\varphi(\tilde{s}, l)\alpha) \geq g(\varphi(\tilde{s}, l)\alpha)$ for all $l \geq 1$. Thus, since $\sigma(\tilde{s}) = \sigma(\cdot, \cdot)_\alpha$ by Theorem 2.1, we can show
\[
g(\varphi(\tilde{s}, \tilde{\sigma}(\tilde{s}, \cdot))_{\alpha}) \geq g(\varphi(\tilde{s}, \tilde{\sigma}(\tilde{s}, \cdot))_{\alpha})
\]
for all $\tilde{\sigma} \in \Sigma(\tilde{\mathcal{F}}')$ and $\alpha \in [0, 1]$. This implies that $G(\tilde{s}, \tilde{\sigma}) \geq G(\tilde{s}, \tilde{\sigma})$ for all $\tilde{\sigma} \in \Sigma(\tilde{\mathcal{F}}')$ by using (3.3). This complete the proof. $q.e.d.$

4. A numerical example

An example is given to illustrate the previous results of fuzzy stopping problem in this section.

Let $S := [0, 1]$. The fuzzy relations $\tilde{q}$ and $\tilde{r}$ are given by
\[
\tilde{q}(x, y) = \begin{cases} 
1 & \text{if } y = \beta x \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
\tilde{r}(x, z) = \begin{cases} 
1 & \text{if } z = x - \lambda \\
0 & \text{otherwise}
\end{cases}
\]
where $\lambda > 0$ is an observation cost and $0 < \beta < 1$ for $x, y, z \in [0, 1]$ and $z \in \mathbb{R}$. Then, $Q_\alpha$ and $R_\alpha$ defined by (1.5) and (3.2) are independent of $\alpha$ and are calculated as follows:
\[
Q_\alpha([a, b]) = \beta[a, b] \quad \text{and} \quad R_\alpha([a, b]) = [a - \lambda, b - \lambda]
\]
for $0 \leq a \leq b \leq 1$.

Let $g([a, b]) := (a + 2b)/3$ for $0 \leq a \leq b \leq 1$, which is additive. Then, $K_\alpha(g)$ is given as
\[
K_\alpha(g) = \{[a, b] \in C(S) \mid a + 2b \leq 0 \},
\]
So $K_\alpha(g) = Q_\alpha^{-1}(K_\alpha(g)) = \{[a, b] \in C(S) \mid a + 2b \leq 3\lambda \beta^{1-t} \}$. Since $K_\alpha(g)$ is independent of $\alpha$, we see that $Q_\alpha(K_\alpha(g)) = \{\beta[a, b] \mid [a, b] \in K_\alpha(g)\} \subset K_\alpha(g)$ and $\bigcup_{t=1}^\infty K_t^\alpha(g) = C(S)$. Thus Assumptions B1 and B2 in Section 3 are satisfied in this example.

Let the initial fuzzy state be
\[
\tilde{s}(x) := (1 - \mid 8x - 4 \mid) \vee 0 \quad \text{for } x \in [0, 1].
\]
For the stopping time $\hat{\sigma}(\tilde{s})$ given in (2.4), we easily obtain that $\tilde{s}_\alpha = [(3 + \alpha)/8, (5 - \alpha)/8]$ and $\hat{\sigma}(\tilde{s}_\alpha) = \min\{t \geq 1 \mid 13 - \alpha \leq 24\lambda \beta^{1-t}\}$. Thus, as $\hat{\sigma}(\tilde{s}_\alpha)$ is non-increasing in $\alpha \in [0, 1]$, we have $\tilde{s} \in \hat{\mathcal{F}}$. 
Since \( \hat{\sigma}_{\alpha}(\hat{\tilde{s}}_{\alpha}) \in K^{t}(g) \) means \( 13 - \alpha \leq 24\lambda \beta^{1-t} \), then
\[
\hat{\sigma}(\tilde{s}, t) = 1 \wedge (13 - 24\lambda \beta^{1-t} \vee 0).
\]

The numerical value of \( \hat{\sigma} \) is given in Table 1.

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<th>3</th>
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<th>8</th>
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**References**


