Title
On a property of fuzzy stopping times
(Optimization Theory in
Discrete and Continuous Mathematical Sciences)

Author(s)
YOSHIDA, Y.; YASUDA, M.; NAKAGAMI, J.; KURANO, M.

Citation
数理解析研究所講究録 (1997), 1015: 176-182

Issue Date
1997-11

URL
http://hdl.handle.net/2433/61601

Type
Departmental Bulletin Paper

Textversion
publisher
Kyoto University
On a property of fuzzy stopping times

Y. YOSHDIA, M. YASUDA, J. NAKAGAMI and M. KURANO

Abstract
This note is concerned with a fuzzy stopping time for a dynamic fuzzy system. A new class of fuzzy stopping times is introduced and constructed by subsets of $\alpha$-cut for fuzzy states. The results are applied to the optimization of a corresponding problem with an additive weighting function.

Keywords: Fuzzy stopping times; Markov property; $\alpha$-cuts of fuzzy sets; optimality.

1 Introduction and notations
The stopping time with fuzziness, which is called a fuzzy stopping time, is considered by our previous paper [11] in which optimization of a corresponding fuzzy problem is pursued by the constructive method.

In this note, we introduce a new class of fuzzy stopping times defined by subsets of the $\alpha$-cuts of fuzzy states and we apply it to a fuzzy stopping problem with additive weighting functions as the scalarization of the fuzzy total rewards. As related works, refer to [1, 5, 6, 7, 15].

In the remainder of this section, a fuzzy stopping time for a fuzzy dynamic system is defined explicitly. A new class of fuzzy stopping time is introduced in Section 2 and its construction is discussed. These results are applied to the 'optimization' of a corresponding fuzzy stopping problem in Section 3. In Section 4, a example is given to illustrate the results.

Let $E$, $E_1$, $E_2$ be convex compact subsets of some Banach space. Throughout the paper, we will denote a fuzzy set and a fuzzy relation by their membership functions. For the theory of fuzzy sets, refer to Zadeh [16] and Novák [12]. A fuzzy set $\tilde{u} : E \mapsto [0, 1]$ is called convex if

$$\tilde{u}(\lambda x + (1 - \lambda)y) \geq \tilde{u}(x) \land \tilde{u}(y), \quad x, y \in E, \lambda \in [0, 1],$$

where $a \land b := \min\{a, b\}$ for real numbers $a, b$ (c.f. Chen-wei Xu [2]). Also, a fuzzy relation $\tilde{h} : E_1 \times E_2 \mapsto [0, 1]$ is called convex if

$$\tilde{h}(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \geq \tilde{h}(x_1, y_1) \land \tilde{h}(x_2, y_2)$$

for $x_1, x_2 \in E_1, y_1, y_2 \in E_2$ and $\lambda \in [0, 1]$. A fuzzy set $\tilde{u} \in C(E)$ is called upper semicontinuous and have compact supports and the normality condition : $\sup_{x \in E} \tilde{u}(x) = 1$. The $\alpha$-cut ($\alpha \in [0, 1]$) of the fuzzy set $\tilde{u}$ is defined by

$$\tilde{u}_\alpha := \{x \in E \mid \tilde{u}(x) \geq \alpha\} (\alpha > 0) \quad \text{and} \quad \tilde{u}_0 := \text{cl}\{x \in E \mid \tilde{u}(x) > 0\},$$

where cl denotes the closure of a set. We denote by $\mathcal{C}(E)$ the collection of all compact convex subsets of $E$. Clearly, $\tilde{u} \in \mathcal{F}(E)$ means $\tilde{u}_\alpha \in \mathcal{C}(E)$ for all $\alpha \in [0, 1]$.

Let $\mathbb{R}$ be the set of all real numbers. We see, from the definition, that $\mathcal{C}(\mathbb{R})$ is the set of all bounded closed intervals in $\mathbb{R}$. The elements of $\mathcal{F}(\mathbb{R})$ are called fuzzy numbers. The addition and the scalar multiplication on $\mathcal{F}(\mathbb{R})$ are defined as follows (see Puri and Ralescu [13]): For $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbb{R})$ and $\lambda \geq 0$,

$$(\tilde{m} + \tilde{n})(x) := \sup_{x_1, x_2 \in \mathbb{R} : x_1 + x_2 = x} \{\tilde{m}(x_1) \land \tilde{n}(x_2)\} \quad (x \in \mathbb{R}) \quad (1.1)$$

and

$$(\lambda \tilde{m})(x) := \begin{cases} \tilde{m}(x/\lambda) & \text{if } \lambda > 0 \\ 1_{\{0\}}(x) & \text{if } \lambda = 0 \end{cases} \quad (x \in \mathbb{R}). \quad (1.2)$$

And hence

$$(\tilde{m} + \tilde{n})_\alpha = \tilde{m}_\alpha + \tilde{n}_\alpha \quad \text{and} \quad (\lambda \tilde{m})_\alpha = \lambda \tilde{m}_\alpha \quad (\alpha \in [0, 1]),$$

where $\tilde{m}_\alpha := \{x \in \mathbb{R} : \tilde{m}(x) \geq \alpha\}$ and $1_{\{0\}}(x)$ is the indicator function of the singleton set $\{0\}$.
where $A + B := \{x + y \mid x \in A, y \in B \}$, $\lambda A := \{\lambda x \mid x \in A \}$, $A + \emptyset = \emptyset + A := A$ and $\lambda \emptyset := \emptyset$ for any non-empty closed intervals $A, B \in C(\mathbb{R})$. We use the following lemma.

**Lemma 1.1** (Chen-wei Xu [2]).

(i) For any $\tilde{m}, \tilde{n} \in \mathcal{F}(\mathbb{R})$ and $\lambda \geq 0$, it holds that $\tilde{m} + \lambda \tilde{n} \in \mathcal{F}(\mathbb{R})$.

(ii) Let $\tilde{u} \in \mathcal{F}(E_1)$ and $\tilde{p} \in \mathcal{F}(E_1 \times E_2)$. Then $\sup_{x \in E_1} \{\tilde{u}(x) \wedge \tilde{p}(x, \cdot)\} \in \mathcal{F}(E_2)$.

We consider the dynamic fuzzy system ([9]), which is denoted by the elements $(S, \tilde{q})$ as follows.

**Definition 1.**

(i) The state space $S$ is a convex compact subset of some Banach space. In general, the system is fuzzy, so that the state of the system is called a fuzzy state and is denoted by an element of $\mathcal{F}(S)$.

(ii) The law of motion for the system is denoted by time-invariant fuzzy relations $\tilde{q} : S \times S \mapsto [0, 1]$, and assume that $\tilde{q} \in \mathcal{F}(S \times S)$.

If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}(S)$, the state is moved to a new fuzzy state $Q(\tilde{s})$ after unit time, where $Q : \mathcal{F}(S) \mapsto \mathcal{F}(S)$ is defined by

$$Q(\tilde{s})(y) := \sup_{x \in S} \{\tilde{s}(x) \wedge \tilde{q}(x, y)\} \quad (y \in S). \quad (1.3)$$

Note that the map $Q$ is well-defined by Lemma 1.1.

For the dynamic fuzzy system $(S, \tilde{q})$ with a given initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$, we can define a sequence of fuzzy states $\{\tilde{s}_t\}_{t=1}^\infty$ by

$$\tilde{s}_1 := \tilde{s} \quad \text{and} \quad \tilde{s}_{t+1} := Q(\tilde{s}_t) \quad (t \geq 1). \quad (1.4)$$

A fuzzy stopping time for this sequence $\{\tilde{s}_t\}_{t=1}^\infty$ is defined in the next section. In order to define a fuzzy stopping time, we need the following preliminaries.

Associated with the fuzzy relation $\tilde{q}$, the corresponding maps $Q_\alpha : C(S) \mapsto C(S) (\alpha \in [0, 1])$ are defined as follows: For $D \in C(S)$,

$$Q_\alpha(D) := \{y \in S \mid \tilde{q}(x, y) \geq \alpha \text{ for some } x \in D\} \quad \text{for } \alpha > 0,$$

$$\text{cl}\{y \in S \mid \tilde{q}(x, y) > 0 \text{ for some } x \in D\} \quad \text{for } \alpha = 0. \quad (1.5)$$

From the assumption on $\tilde{q}$, the maps $Q_\alpha$ is well-defined. The iterates $Q_\alpha^t (t \geq 0)$ are defined by setting $Q_\alpha^0 := I$(identity) and iteratively,

$$Q_\alpha^{t+1} := Q_\alpha Q_\alpha^t \quad (t \geq 0).$$

In the following lemma, which is easily verified by the idea in the proof of Kurano et al. [9, Lemma 1], the $\alpha$-cuts of $Q(\tilde{s})$ defined by (1.3) is specified using the maps $Q_\alpha$.

**Lemma 1.2** ([9, 10]). For any $\alpha \in [0, 1]$ and $\tilde{s} \in \mathcal{F}(S)$, we have:

(i) $Q(\tilde{s})_\alpha = Q_\alpha(\tilde{s}_\alpha)$;

(ii) $\tilde{s}_{t, \alpha} = Q_\alpha^{t-1}(\tilde{s}_\alpha) \quad (t \geq 1)$,

where $\tilde{s}_{t, \alpha} := (\tilde{s}_t)_\alpha$ and $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.4) with $\tilde{s}_1 = \tilde{s}$.

## 2 Fuzzy stopping times

In this section, we define a fuzzy stopping time to be discussed here. And a new class of fuzzy stopping times is introduced, which is constructed thorough subsets of $\alpha$-cuts of fuzzy states.

For the sake of simplicity, denote $\mathcal{F} := \mathcal{F}(S)$. Let $N = \{1, 2, \cdots\}$ and $\mathcal{F}'$ a subset of $\mathcal{F}$.

**Definition 2** (cf.[11]). A fuzzy stopping time(FST) on $\mathcal{F}'$ is a fuzzy relation $\tilde{\sigma} : \mathcal{F}' \times N \mapsto [0, 1]$ such that, for each fuzzy state $\tilde{s} \in \mathcal{F}'$, $\tilde{\sigma}(\tilde{s}, t)$ is non-increasing in $t$ and there exists a natural number $t(\tilde{s}) \in \mathbb{N}$ with $\tilde{\sigma}(\tilde{s}, t) = 0$ for all $t \geq t(\tilde{s})$. 

We note here that 0 represents 'stop' and 1 represents 'continue' in the grade of membership (cf. [11]). An FST $\tilde{\sigma}(\tilde{s}, \cdot)$ means the degree of 'continue' at time $t$ starting at a fuzzy state $\tilde{s} \in F'$. The set of all FSTs on $F'$ is denoted by $\Sigma(F')$. Assuming $Q(F') \subset F'$, an FST $\tilde{\sigma} \in \Sigma(F')$ is called Markov if there exist a mapping $\delta : F' \rightarrow [0, 1]$ satisfying

(i) $\delta(Q(\tilde{s})) \leq \delta(\tilde{s})$, and
(ii) $\tilde{\sigma}(\tilde{s}, t) = \delta(\tilde{s}_t)$ for all $\tilde{s} \in F'$ and $t \geq 1$,

where $\{\tilde{s}_t\}_{t=1}^\infty$ is defined by (1.4) with $\tilde{s}_1 = \tilde{s}$.

The above $\delta$ is called a support of $\tilde{\sigma}$. We consider ourselves with the construction of Markov FSTs. For this purpose, we assume the following condition holds.

**Condition A1.** For each $\alpha \in [0, 1]$, there exists a non-empty subset $K_\alpha$ of $C(S)$ satisfying

$$ Q_\alpha(K_\alpha) \subset K_\alpha. \tag{2.1} $$

Using this subset $K_\alpha$, we define a sequence of subsets $\{K_\alpha^t\}_{t=1}^\infty$ inductively by

$$ K_\alpha^1 := K_\alpha \tag{2.2} $$

and for each $t \geq 2$,

$$ K_\alpha^t := \{c \in C(S) \mid Q_\alpha(c) \in K_\alpha^{t-1}\}. \tag{2.3} $$

Clearly, $K_\alpha^t = Q_\alpha^{-1}(K_\alpha^{t-1}) = Q_\alpha^{(t-1)}(K_\alpha)$. Also, it holds from (2.1) that $K_\alpha^t \subset K_\alpha^{t+1}$ ($t \geq 1$).

To simplify our discussion, we assume the following condition holds henceforth.

**Condition A2.** For all $\alpha \in [0, 1]$, it holds that

$$ C(S) = \bigcup_{t=1}^\infty K_\alpha^t. $$

For $c \in C(S)$ and $\alpha \in [0, 1]$, define $\tilde{\sigma}_\alpha(c)$ by

$$ \tilde{\sigma}_\alpha(c) := \min \{t \mid c \in K_\alpha^t\}. \tag{2.4} $$

That is, it is the first entry time of $c \in C(S)$ with the grade $\alpha$. We define a restricted class $\hat{F} \subset F$ by

$$ \hat{F} := \{\tilde{s} \in F \mid \tilde{\sigma}_\alpha(\tilde{s}_\alpha) \text{ is non-increasing in } \alpha \in [0, 1]\}. \tag{2.5} $$

Using the class $\{\tilde{\sigma}_\alpha(\tilde{s}_\alpha) \mid \alpha \in [0, 1]\}$, for the restricted element $\tilde{s} \in \hat{F}$, let us construct

$$ \hat{\sigma}(\tilde{s}, t) := \sup_{\alpha \in [0, 1]} \{\alpha \land 1_{D_\alpha}(t)\} \quad (t \geq 1), \tag{2.6} $$

where $1_{D_\alpha}$ is the indicator of a set $D_\alpha = \{t \in N \mid \tilde{\sigma}_\alpha(\tilde{s}_\alpha) > t\}$. This is the usual technique of constructing a corresponding fuzzy number from the class of level sets. Now let

$$ \hat{\sigma}(\tilde{s}, \cdot)_\alpha := \min \{t \in N \mid \hat{\sigma}(\tilde{s}, t) < \alpha\}. \tag{2.7} $$

Then we obtain the following theorem.

**Theorem 2.1.**

(i) $\hat{\sigma}(\tilde{s}, \cdot)_\alpha = \tilde{\sigma}_\alpha(\tilde{s}_\alpha)$, $\tilde{s} \in \hat{F}$, $\alpha \in [0, 1]$;

(ii) $\hat{\sigma}$ is an FST on $\hat{F}$.  

Proof. By (2.6) and (2.7), we have that $\hat{\sigma}(\tilde{s}, \cdot)_t \leq t$ is equivalent to $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \leq t$ for all $t \geq 1$. This fact shows (i). From Condition A2, there exists $t^* \in \mathbb{N}$ with $\tilde{s}_0 \in K^t_{t^*}_0$. So, $\hat{\sigma}_\alpha(\tilde{s}_\alpha) \leq \tilde{s}_0(\tilde{s}_0) \leq t^*$ for all $\alpha \in [0, 1]$, which shows by (2.5) that $\hat{\sigma}(\tilde{s}, t) = 0$ for all $t \geq t^*$. Since $\hat{\sigma}(\tilde{s}, t + 1) \leq \hat{\sigma}(\tilde{s}, t)$ holds clearly for $t \geq 1$ from the definition (2.6), we also obtain (ii). \textit{q.e.d.}

In order to show the Markov property of $\hat{\sigma}$, we need the following lemma.

Lemma 2.1. Let $\tilde{s} \in \hat{\mathcal{F}}$. Then

(i) $\hat{\sigma}(\tilde{s}, t) = \alpha$ if and only if, for any $\epsilon > 0$,

\[ \tilde{s}_\alpha + \epsilon \in K^t_{\alpha+\epsilon} \quad \text{and} \quad \tilde{s}_\alpha - \epsilon \notin K^t_{\alpha-\epsilon}; \]

(ii) $\tilde{s}_t \in \hat{\mathcal{F}}$ ($t \geq 1$).

Proof. By (2.6), $\hat{\sigma}(\tilde{s}, t) = \sup\{\alpha \mid \hat{\sigma}_\alpha(\tilde{s}_\alpha) > t\}$. So, (i) follows from (2.4). From Lemma 1.2(ii), for $l \geq 1$, $\hat{\sigma}_\alpha((\tilde{s}_t)_\alpha) = \hat{\sigma}_\alpha(\tilde{s}_t) = \hat{\sigma}_\alpha(Q^{l-1}_{a}^{-1}(\tilde{s}_\alpha))$. So, by (2.3) and (2.4),

\[
\hat{\sigma}_\alpha((\tilde{s}_t)_\alpha) = \min\{t \geq 1 \mid Q^{l-1}_{a}^{-1}(\tilde{s}_\alpha) \in K^t_{a}\} = \min\{t \geq 1 \mid \tilde{s}_\alpha \in K^{tl-1}_{a}\} = \max\{\hat{\sigma}_\alpha(\tilde{s}_\alpha) - (l - 1), 1\},
\]

and it is non-increasing in $\alpha \in [0, 1]$ since $\tilde{s} \in \hat{\mathcal{F}}$. Therefore we obtain (ii). \textit{q.e.d.}

Theorem 2.2. Let $\tilde{s} \in \hat{\mathcal{F}}$. Then, $\hat{\sigma}$ is a Markov FST with $\tilde{s}$.

Proof. Let $\{\tilde{s}_t\}_{t=1}^\infty$ be defined by (1.4) with $\tilde{s}_1 = \tilde{s}$. First, we prove

\[
\hat{\sigma}(\tilde{s}, t + r) = \hat{\sigma}(\tilde{s}, t) \wedge \hat{\sigma}(\tilde{s}_{t+1}, r) \quad \text{for} \quad t, r \in \mathbb{N}. \tag{2.8}
\]

Note that $\hat{\sigma}(\tilde{s}_{t+1}, r)$ is well-defined from Lemma 2.1(ii). Let $\alpha = \hat{\sigma}(\tilde{s}, t + r)$. From Lemma 2.1(i), we have

\[ \tilde{s}_{\alpha + \epsilon} \in K^t_{\alpha+\epsilon} \quad \text{and} \quad \tilde{s}_{\alpha - \epsilon} \notin K^t_{\alpha-\epsilon} \quad \text{for any} \quad \epsilon > 0. \]

Noting $Q^t_{a}(K^t_{0}) = K^t_{a-t}$ ($1 \leq t < l$) and Lemma 1.2(ii), we obtain

\[ \tilde{s}_{t+1,\alpha+\epsilon} = Q^t_{a+\epsilon}(\tilde{s}_{\alpha+\epsilon}) \in Q^t_{a+\epsilon}(K^t_{a+\epsilon}) = K^t_{a+\epsilon} \tag{2.9} \]

and

\[ \tilde{s}_{t+1,\alpha-\epsilon} = Q^t_{a-\epsilon}(\tilde{s}_{\alpha-\epsilon}) \notin Q^t_{a-\epsilon}(K^t_{a-\epsilon}) = K^t_{a-\epsilon}. \tag{2.10} \]

Therefore, we get $\hat{\sigma}(\tilde{s}_{t+1}, r) = \alpha$ from Lemma 2.1(i). Namely, $\hat{\sigma}(\tilde{s}, t + r) = \hat{\sigma}(\tilde{s}_{t+1}, r)$. Since $\hat{\sigma}(\tilde{s}, t + r) \leq \hat{\sigma}(\tilde{s}, t)$ from Theorem 2.1(ii), we obtain $\hat{\sigma}(\tilde{s}, t) \wedge \hat{\sigma}(\tilde{s}_{t+1}, r) = \alpha$, and so (2.8) holds.

Next, we put $\delta(\tilde{s}) = \hat{\sigma}(\tilde{s}, 1)$ for $\tilde{s} \in \hat{\mathcal{F}}$. From (2.8), we get

\[
\hat{\sigma}(\tilde{s}, t) = \hat{\sigma}(\tilde{s}, 1) \wedge \hat{\sigma}(\tilde{s}_2, t - 1).
\]

\[
= \hat{\sigma}(\tilde{s}, 1) \wedge \hat{\sigma}(\tilde{s}_2, 1) \wedge \hat{\sigma}(\tilde{s}_3, t - 2)
\]

\[
= \cdots = \bigwedge_{t=1}^\infty \hat{\sigma}(\tilde{s}_t)
\]

\[
\hat{\sigma}(\tilde{s}, 1) \wedge \hat{\sigma}(\tilde{s}_{t+1}, r) = \alpha \quad \text{for any} \quad \epsilon > 0.
\]

Since we also have $\delta(Q(\tilde{s})) \leq \delta(\tilde{s})$ from Theorem 2.1(ii), $\hat{\sigma}$ is a Markov FST with $\tilde{s}$. \textit{q.e.d.}
3 Applications to fuzzy stopping problem

In this section, applying the results in the previous section, we obtain the optimal FST for a fuzzy dynamic system with fuzzy rewards ([10]) when the weighting function is additive.

Firstly, we will formulate the stopping problem to be considered here. Let $\tilde{r} : S \times \mathbb{R} \mapsto [0, 1]$ be a fuzzy relation satisfying $\tilde{r} \in \mathcal{F}(S \times \mathbb{R})$. If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}$, the following fuzzy reward is earned:

$$R(\tilde{s})(z) := \sup_{x \in S} \{ \tilde{s}(x) \vee \tilde{r}(x, z) \}, \quad z \in \mathbb{R}.$$ 

Then we can define a sequence of fuzzy rewards $\{ R(\tilde{s}_t) \}_{t=1}^{\infty}$, where $\{ \tilde{s}_t \}_{t=1}^{\infty}$ is defined in (1.4) with the initial fuzzy state $\tilde{s}_1 = \tilde{s}$. Let

$$\varphi(\tilde{s}, t) := \sum_{l=1}^{t-1} R(\tilde{s}_l) \quad \text{for} \ t \in \mathbb{N}. \quad (3.1)$$

We need the following lemma, which is proved in [9].

**Lemma 3.1** ([9, 10]). For $t \in \mathbb{N}$ and $\alpha \geq 0$,

$$\varphi(\tilde{s}, t) = \sum_{l=1}^{t-1} R_{\alpha}(\tilde{s}_l, \alpha)$$

holds, where

$$R_{\alpha}(\tilde{s}_l, \alpha) := \left\{ \begin{array}{ll} \{ z \in \mathbb{R} \mid \tilde{r}(x, z) \geq \alpha \text{ for some } z \in \tilde{s}_l, \alpha \} & \text{for } \alpha > 0, \\
\{ c \{ z \in \mathbb{R} \mid \tilde{r}(x, z) > \alpha \text{ for some } z \in \tilde{s}_l, \alpha \} & \text{for } \alpha = 0. \end{array} \right. \quad (3.2)$$

Let $g : C(\mathbb{R}) \mapsto \mathbb{R}$ be any additive map with $g(\phi) = 0$, that is,

$$g(c' + c'') = g(c') + g(c'') \quad \text{for } c', c'' \in C(S).$$

Adapting this $g$ for a weighting function (see [4]), when an FST $\hat{\sigma} \in \Sigma(\hat{\mathcal{F}})$ and an initial fuzzy state $\tilde{s} \in \hat{\mathcal{F}}$ are used, the scalarization of the total fuzzy reward is given by

$$G(\tilde{s}, \hat{\sigma}) = \int_{0}^{1} g(\varphi(\tilde{s}, \hat{\sigma}, t)) \, d\alpha \quad = \int_{0}^{1} g\left( \sum_{t=1}^{\infty} R_{\alpha}(\tilde{s}_t, \alpha) \right) \, d\alpha, \quad (3.3)$$

where $\sum_{t=1}^{\infty} R_{\alpha}(\tilde{s}_t, \alpha) = \phi$ and $\hat{\sigma}_\alpha$ means $\hat{\sigma}(\tilde{s}_t, \alpha) = \min\{ t \in \mathbb{N} \mid \hat{\sigma}(\tilde{s}, t) < \alpha \}$ for simplicity. Since $\varphi(\tilde{s}, \hat{\sigma}, t) \in C(\mathbb{R})$ and the map $\alpha \mapsto g(\varphi(\tilde{s}, \hat{\sigma}, t))$ is left-continuous in $\alpha \in (0, 1]$, therefore the right-hand integral of (3.3) is well-defined. For a given $\mathcal{F}' \subset \mathcal{F}$, our objective is to maximize (3.3) over all FSTs $\hat{\sigma} \in \Sigma(\mathcal{F}')$ for each initial fuzzy state $\tilde{s} \in \mathcal{F}'$.

**Definition 3.** An FST $\hat{\sigma}^*$ with $\tilde{s} \in \mathcal{F}'$ is called an $\tilde{s}$-optimal if

$$G(\tilde{s}, \hat{\sigma}) \leq G(\tilde{s}, \hat{\sigma}^*) \quad \text{for all } \hat{\sigma} \in \Sigma(\mathcal{F}').$$

If $\hat{\sigma}^*$ is $\tilde{s}$-optimal for all $\tilde{s} \in \mathcal{F}'$, $\hat{\sigma}^*$ is called optimal in $\mathcal{F}'$.

Now we will seek a $\tilde{s}$-optimal or an optimal FST by using the results in the previous sections. For each $\alpha \in [0, 1]$, let

$$\mathcal{K}_\alpha(g) := \{ c \in C(S) \mid g(R_{\alpha}(c)) \leq 0 \}. \quad (3.4)$$

Here we need the following Assumptions B1 and B2, which are assumed to hold henceforth.

**Assumption B1** (Closedness).

$$Q_\alpha(\mathcal{K}_\alpha(g)) \subset \mathcal{K}_\alpha(g) \quad \text{for all } \alpha \in [0, 1]$$
Now we define the sequence \( \{ K^t_{\alpha}(g) \}_{t=1}^{\infty} \) by (2.2) - (2.3), that is,
\[
K^t_{\alpha}(g) = Q^{-t(t-1)}_{\alpha}(K_{\alpha}(g)) \quad \text{for} \quad t \geq 1. \tag{3.5}
\]

**Assumption B2.** For all \( \alpha \in [0, 1] \), it holds that
\[
C(S) = \bigcup_{t=1}^{\infty} K^t_{\alpha}(g).
\]

Using the sequence \( \{ K^t_{\alpha}(g) \}_{t=1}^{\infty} \) given in (3.5), we define \( \tilde{\sigma}_{\alpha}, \hat{\sigma}, \tilde{\sigma}(\cdot, \cdot)_{\alpha}, \) respectively, by (2.4), (2.5), (2.6) and (2.7). Then, from Theorems 2.1 and 2.2, \( \tilde{\sigma} \) is a Markov FST on \( \hat{\mathcal{F}} \).

The following theorem will be proved by applying the idea of the one-step look ahead (OLA) policy([3, 8, 14]) for stochastic stopping problems.

**Theorem 3.1.** Under Assumptions B1 and B2, \( \tilde{\sigma} \) is optimal in \( \hat{\mathcal{F}} \).

**Proof.** Firstly, consider the deterministic stopping problem which maximizes \( g(\hat{\varphi}(\hat{s}, t)_{\alpha}) \) over \( t \geq 1 \). As \( g \) is additive, \( g(\hat{\varphi}(\hat{s}, t)_{\alpha}) = \Sigma^{t-1}_{l=1} g(R_{\alpha}(\hat{s}_{l}, \alpha)) \). Therefore \( g(\varphi(\hat{s}, t)_{\alpha}) \geq g(\varphi(\hat{s}, t+1)_{\alpha}) \) if and only if \( \tilde{s}_{l, \alpha} \in K_{\alpha}(g) \). By the assumption B1, \( \tilde{s}_{l, \alpha} \in K_{\alpha}(g) \) implies \( g(\varphi(\hat{s}, t)_{\alpha}) \geq g(\varphi(\hat{s}, t)_{\alpha}) \) for all \( l \geq t \). Thus, since \( \tilde{s}_{\alpha} = (3 + \alpha)/8, (5 - \alpha)/8 \) and \( \tilde{\sigma}_{\alpha} = \sigma(\tilde{s}, \cdot)_{\alpha} \) by Theorem 2.1, we can show
\[
g(\varphi(\hat{s}, \tilde{\sigma}(\hat{s}, \cdot)_{\alpha})) \geq g(\varphi(\hat{s}, \tilde{\sigma}(\hat{s}, \cdot)_{\alpha}))
\]
for all \( \tilde{\sigma} \in \Sigma(F') \) and \( \alpha \in [0, 1] \). This implies that \( G(\hat{s}, \tilde{\sigma}) \geq G(\tilde{s}, \tilde{\sigma}) \) for all \( \tilde{\sigma} \in \Sigma(F') \) by using (3.3). This complete the proof. q.e.d.

4 A numerical example

An example is given to illustrate the previous results of fuzzy stopping problems in this section.

Let \( S := [0, 1] \). The fuzzy relations \( \tilde{q} \) and \( \tilde{r} \) are given by
\[
\tilde{q}(x, y) = \begin{cases} 
1 & \text{if } y = \beta x \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
\tilde{r}(x, z) = \begin{cases} 
1 & \text{if } z = x - \lambda \\
0 & \text{otherwise},
\end{cases}
\]
where \( \lambda > 0 \) is an observation cost and \( 0 < \beta < 1 \) for \( x, y, z \in [0, 1] \) and \( z \in \mathbb{R} \). Then, \( Q_{\alpha} \) and \( R_{\alpha} \) defined by (1.5) and (3.2) are independent of \( \alpha \) and are calculated as follows:
\[
Q_{\alpha}([a, b]) = \beta[a, b] \quad \text{and} \quad R_{\alpha}([a, b]) = [a - \lambda, b - \lambda]
\]
for \( 0 \leq a \leq b \leq 1 \).

Let \( g([a, b]) := (a + 2b)/3 \) for \( 0 \leq a \leq b \leq 1 \), which is additive. Then, \( K_{\alpha}(g) \) is given as
\[
K_{\alpha}(g) = \{ [a, b] \in C(S) \mid a + 2b \leq 0 \}
\]
So \( K^t_{\alpha}(g) = Q^{-t(t-1)}_{\alpha}(K_{\alpha}(g)) = \{ [a, b] \in C(S) \mid a + 2b \leq 3(\beta-1) \} \). Since \( K^t_{\alpha}(g) \) is independent of \( \alpha \), we see that \( Q_{\alpha}(K_{\alpha}(g)) = \beta[a, b] \mid [a, b] \in K_{\alpha}(g) \} \subset K_{\alpha}(g) \) and \( \bigcup_{t=1}^{\infty} K^t_{\alpha}(g) = C(S) \). Thus Assumptions B1 and B2 in Section 3 are satisfied in this example.

Let the initial fuzzy state be
\[
\tilde{s}(x) := (1 - |8x - 4|) \vee 0 \quad \text{for } x \in [0, 1].
\]
For the stopping time \( \tilde{\sigma}_{\alpha}(\tilde{s}_{\alpha}) \) given in (2.4), we easily obtain that \( \tilde{s}_{\alpha} = [(3 + \alpha)/8, (5 - \alpha)/8] \) and \( \tilde{\sigma}_{\alpha}(\tilde{s}_{\alpha}) = \min\{t \geq 1 \mid 13 - \alpha \leq 24\lambda\beta^{t-1} \} \). Thus, as \( \tilde{\sigma}_{\alpha}(\tilde{s}_{\alpha}) \) is non-increasing in \( \alpha \in [0, 1] \), we have \( \tilde{s} \in \hat{\mathcal{F}} \).
Since $\hat{\sigma}_\alpha(\hat{\tilde{s}}_\alpha) \in K^t(g)$ means $13 - \alpha \leq 24\lambda \beta^{1-t}$, then

$$\hat{\sigma}(\tilde{s},t) = 1 \wedge ((13 - 24\lambda)\beta^{1-t} \vee 0).$$

The numerical value of $\hat{\sigma}$ is given in Table 1.

| Table 1. An $\tilde{s}$-optimal FST ($\lambda = 0.48$, $\beta = 0.98$). |
|------------------------|--|---|---|---|---|---|---|---|
| $t$                  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
| $\hat{\sigma}(\tilde{s},t)$ | 1 | 1 | 1 | .7603 | .5108 | .2552 | .00 | .00 | ... |

References


