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Finite Hilbert Networks

1 Introduction

Let $N = \{X, Y, K\}$ be a finite connected graph which has no self-loop. Namely $X$ is a finite set of nodes, $Y$ is a finite set of arcs and $K$ is the node-arc incidence matrix.

Let $\mathcal{H}$ be a real Hilbert space with an inner product $(\cdot, \cdot)$ and the norm $\| \cdot \|$. Denote by $L(X; \mathcal{H})$ the set of all functions $u$ on $X$ such that $u(x) \in \mathcal{H}$. We call an element of $L(X, \mathcal{H})$ a $\mathcal{H}$-potential. The meaning of the notation $L(Y; \mathcal{H})$ is similar. Let $r \in L(Y; \mathcal{L}(\mathcal{H}))$. For each $y \in Y$, we have $r(y) \in \mathcal{L}(\mathcal{H})$ and there exists $\rho(y) > 0$ such that

$$(r(y)h, h) \geq \rho(y)\|h\|^2 \quad \text{for all} \quad h \in \mathcal{H}.$$ 

Here $r(y)h$ means the image of $h$ under $r(y)$, i.e., $r(y)(h)$. In this paper, we use this convention unless no confusion occurs from the context. Denote by $r(y)^{-1}$ the inverse operator of $r(y)$. Since $r(y) \in \mathcal{L}(\mathcal{H})$, there exists $\rho^*(y) > 0$ such that

$$(r(y)^{-1}h, h) \geq \rho^*(y)\|h\|^2 \quad \text{for all} \quad h \in \mathcal{H}.$$ 

By [1], we see that there exists a unique square root $r(y)^{1/2} \in \mathcal{L}(\mathcal{H})$ of $r(y)$ for each $y \in Y$, i.e.,

$$[r(y)^{1/2}]^2 = r(y).$$

Definition 1.1 Let $e$ be a fixed element of $\mathcal{H}$ such that $\|e\| = 1$.

Lemma 1.1 For every $y \in Y$, the following relations hold:

1) $|(r(y)w(y), e)|^2 \leq (r(y)w(y), w(y))(r(y)e, e)$.

2) $(r(y)^{-1}e, e)r(y)e, e) \geq 1$.

Proof. By Schwarz's inequality, we have

$$(r(y)w(y), e)^2 = |(r(y)^{1/2}w(y), r(y)^{1/2}e)|^2 \leq \|r(y)^{1/2}w(y)\|^2\|r(y)^{1/2}e\|^2$$

$$= (r(y)w(y), w(y))(r(y)e, e).$$

2) follows from (1) by taking $w(y) := r(y)^{-1}e$. □
Definition 1.2 For $u \in L(X; \mathcal{H})$, let $\delta u$ be the potential drop of $u$ and let $du$ be the discrete derivative of $u$:
\[
\delta u(y) := \sum_{x \in X} K(x, y)u(x)
\]
\[
du(y) := -r(y)^{-1}(\delta u(y)) = -\Gamma(y)^{-1}\delta u(y).
\]
The Dirichlet sum of $u$ is defined by
\[
D(u) := \sum_{y \in \mathrm{Y}} (r(y)du(y), du(y)) = \sum_{y \in \mathrm{Y}} (r(y)^{-1}\delta u(y), \delta u(y)).
\]

Definition 1.3 For $w \in L(Y; \mathcal{H})$, let $\partial w(x)$ be the divergence of $w$ and let $H(w)$ be the energy of $w$:
\[
\partial w(x) := \sum_{y \in \mathrm{Y}} K(x, y)w(y)
\]
\[
H(w) := \sum_{y \in \mathrm{Y}} (r(y)w(y), w(y)).
\]
Notice that $D(u) = H(du)$.

Lemma 1.2 Let $u \in L(X; \mathcal{H})$ and $w \in L(Y; \mathcal{H})$. Then
\[
\sum_{y \in \mathrm{Y}} (w(y), \delta u(y)) \leq H(w)^{1/2}D(u)^{1/2}.
\]

Proof. We have by Schwarz's inequality
\[
\sum_{y \in \mathrm{Y}} (w(y), \delta u(y)) = \sum_{y \in \mathrm{Y}} (r(y)^{1/2}w(y), r(y)^{-1/2}\delta u(y))
\]
\[
\leq \sum_{y \in \mathrm{Y}} \|r(y)^{1/2}w(y)\|\|r(y)^{-1/2}\delta u(y)\|
\]
\[
\leq \left[\sum_{y \in \mathrm{Y}} \|r(y)^{1/2}w(y)\|^2\right]^{1/2}\left[\sum_{y \in \mathrm{Y}} \|r(y)^{-1/2}\delta u(y)\|^2\right]^{1/2}
\]
\[
= H(w)^{1/2}D(u)^{1/2}. \quad \square
\]
To emphasize the analogy to [2], we put
\[
D(N; \mathcal{H}; a) := \{u \in L(X; \mathcal{H}); u(a) = 0\}.
\]
Note that $D(u) < \infty$ for every $L(X; \mathcal{H})$, since $G$ is a finite graph. We see that $D(u)^{1/2}$ is a norm on $D(N; \mathcal{H}; a)$ by the following lemma:

Lemma 1.3 Let $a \in X$. For any $x \in X$, there exists a constant $M_x$ which satisfies:
\[
\|u(x)\| \leq M_x D(u)^{1/2}
\]
for all $u \in L(X; \mathcal{H})$ with $u(a) = 0$. 

Proof. There exists a path $P$ from $a$ to $x$. Let $C_X(P)$ and $C_Y(P)$ be the sets of nodes and arcs on $P$ respectively (cf. [2]), i.e.,

$$C_X(P) := \{x_0, x_1, \cdots, x_n\} \quad (x_0 = a, x_n = x)$$

$$C_Y(P) := \{y_1, y_2, \cdots, y_n\}, e(y_i) = \{x_{i-1}, x_i\} \quad (i = 1, 2, \cdots, n).$$

Let $u \in L(X; \mathcal{H})$ and $u(a) = 0$. We have

$$D(u) \geq \sum_{y \in P} (r(y)^{-1} \delta u(y), \delta u(y))$$

$$= \sum_{i=1}^{n} (r(y_i)^{-1} \delta u(y_i), \delta u(y_i))$$

$$\geq \sum_{i=1}^{n} \rho^*(y_i) \|u(x_i) - u(x_{i-1})\|^2$$

$$\geq \sum_{i=1}^{n} \rho^*(y_i) \|u(x_i)\|^2 - \|u(x_{i-1})\|^2,$$

so that, for $i = 1, 2, \cdots$

$$\|u(x_i)\| - \|u(x_{i-1})\| \leq D(u)^{1/2}[\rho^*(y_i)]^{-1/2}.$$

Since $u(a) = 0$, we have

$$\|u(x)\| = \sum_{i=1}^{n} \|u(x_i)\| - \|u(x_{i-1})\| \leq M_x D(u)^{1/2}$$

with

$$M_x := \sum_{i=1}^{n} [\rho^*(y_i)]^{-1/2}.$$

This completes the proof. \square

Since $G$ is a finite graph, the following fact is obvious:

**Proposition 1.1** $D(N; \mathcal{H}, a)$ is a Hilbert space with respect to the inner product:

$$D(u_1, u_2) := \sum_{y \in V} (r(y)^{-1} \delta u_1(y), \delta u_2(y)).$$

$L(Y; \mathcal{H})$ is a Hilbert space with respect to the inner product:

$$H(w_1, w_2) := \sum_{y \in Y} (r(y)w_1(y), w_2(y)).$$. 
2 \( \mathcal{H} \)-flows

Definition 2.1 Let \( a \) and \( b \) be distinct nodes. We say that \( w \in L(Y;\mathcal{H}) \) is a \( \mathcal{H} \)-flow from \( a \) to \( b \) if
\[
\partial w(x) = 0 \quad \text{for all} \quad x \in X, \quad x \neq a, b.
\]
Denote by \( F(a,b;\mathcal{H}) \) the set of all \( \mathcal{H} \)-flows from \( a \) to \( b \).

Notice that
\[
\partial w(a) + \partial w(b) = 0,
\]
since \( G \) is a finite graph.

Definition 2.2 For \( w \in F(a, b;\mathcal{H}) \), we define two real valued functions:
\[
I_e(w) := (\partial w(b), e) = -(\partial w(a), e),
\]
\[
I(w) := \|\partial w(a)\| = \|\partial w(b)\|.
\]

3 Extremum problems

Let us consider several extremum problems related to \( \mathcal{H} \)-potentials and \( \mathcal{H} \)-flows:
\[
d(a,b;\mathcal{H},e) := \inf\{D(u);u \in L(X;\mathcal{H}), u(a) = 0, u(b) = e\}
\]
\[
d_e(a,b;\mathcal{H}) := \inf\{D(u);u \in L(X;\mathcal{H}), (u(a), e) = 0, (u(b), e) = 1\}
\]
\[
d(a,b;\mathcal{H}) := \inf\{D(u);u \in L(X;\mathcal{H}), u(a) = 0, \|u(b)\| = 1\}
\]
\[
d_e^*(a,b;\mathcal{H}) := \inf\{H(w);w \in F(a,b;\mathcal{H}), K w(b) = e\}
\]
\[
d_e^*(a,b;\mathcal{H}) := \inf\{H(w);w \in F(a,b;\mathcal{H}), I_e(w) = 1\}
\]
\[
d^*(a,b;\mathcal{H}) := \inf\{H(w);w \in F(a,b;\mathcal{H}), I(w) = 1\}
\]

Clearly
\[
d_e(a,b;\mathcal{H}) \leq d(a,b;\mathcal{H},e), \quad d(a,b;\mathcal{H}) \leq d(a,b;\mathcal{H},e),
\]
\[
d_e^*(a,b;\mathcal{H}) \leq d_e^*(a,b;\mathcal{H}), \quad d^*(a,b;\mathcal{H}) \leq d_e^*(a,b;\mathcal{H}).
\]

Lemma 3.1 Let \( u \) be a feasible solution for \( d(a,b;\mathcal{H},e) \) and \( w \) be a feasible solution for \( d_e^*(a,b;\mathcal{H}) \). Then \( 1 \leq H(w)^{1/2}D(u)^{1/2} \).

Proof. By definition and Lemma 1.2
\[
1 = I_e(w) = (Kw(b), e) = \sum_{x \in X}(Kw(x), u(x))
\]
\[
= \sum_{y \in Y}(w(y), \delta u(y))
\]
\[
\leq H(w)^{1/2}D(u)^{1/2}. \quad \square
\]

Similarly we can prove
Lemma 3.2 Let $u$ be a feasible solution for $d_e(a, b; \mathcal{H})$ and $w$ be a feasible solution for $d^*(a, b; \mathcal{H}, e)$. Then $1 \leq H(w)^{1/2}D(u)^{1/2}$.

By the above observation, we obtain

Theorem 3.1 The following relations hold:

1. $1 \leq d(a, b; \mathcal{H}, e)d^*(a, b; \mathcal{H})$,
2. $1 \leq d_e(a, b; \mathcal{H})d^*(a, b; \mathcal{H}, e)$.

Lemma 3.3 There exists a unique optimal solution for $d(a, b; \mathcal{H}, e)$.

Proof. Let $\{u_n\}$ be a minimizing sequence for $d(a, b; \mathcal{H}, e)$, i.e., $\{u_n\} \subset L(X; \mathcal{H})$, $u_n(a) = 0$, $u_n(b) = e$ and $D(u_n) \to d(a, b; \mathcal{H}, e)$ as $n \to \infty$. Since $(u_n + u_m)/2$ is a feasible solution for $d(a, b; \mathcal{H}, e)$, we have

$$d(a, b; \mathcal{H}, e) \leq D((u_n + u_m)/2)$$

as $m, n \to \infty$. Therefore $D(u_n - u_m) \to 0$ as $n, m \to \infty$. It follows from Lemma 1.3 that $\{u_n(x)\}$ is a Cauchy sequence in $\mathcal{H}$ for each $x \in X$. Therefore $\{u_n(x)\}$ converges strongly to $\tilde{u}(x) \in \mathcal{H}$ for each $x \in X$. We see easily that $\tilde{u}(a) = 0$, $\tilde{u}(b) = e$ and $d(a, b; \mathcal{H}, e) = D(\tilde{u})$. Namely $\tilde{u}$ is an optimal solution. We omit the proof of the uniqueness of the optimal solution.

Now we study some properties of the optimal solution $\tilde{u}$ of $d(a, b; \mathcal{H}, e)$.

Lemma 3.4 Let $\tilde{w}(y) := d\tilde{u}(y)$. Then $\tilde{w} \in F(a, b; \mathcal{H})$ and $I_e(\tilde{w}) = D(\tilde{u})$.

Proof. Let $f \in L(X; \mathcal{H})$ satisfy $f(a) = 0$ and $f(b) = 0$. Then for any $t \in \mathbb{R}$, $\tilde{u} + tf$ is a feasible solution for $d(a, b; \mathcal{H}, e)$, we have

$$D(\tilde{u}) \leq D(\tilde{u} + tf)$$

By the standard variational argument, we have

$$D(\tilde{u}, f) = 0.$$

On the other hand, we have

$$D(\tilde{u}, f) = \sum_{y \in Y}(\tilde{w}(y), \sum_{z \in X} K(z, y)f(z))$$

$$= \sum_{z \in X} \sum_{y \in Y}(K(z, y)\tilde{w}(y), f(z))$$

$$= \sum_{z \in X}(\partial\tilde{w}(z), f(z)).$$
Denote by $\epsilon_x$ the characteristic function of $\{x\}$, i.e., $\epsilon_x(x) = 1$ and $\epsilon_x(z) = 0$ for $z \neq x$. Let $x \neq a, b$. For any $h \in \mathcal{H}$, we may take $\epsilon_x h$ for $f$, and hence
\[(\partial \tilde{w}(x), h) = 0.\]
Therefore $\partial \tilde{w}(x) = 0$ for $x \neq a, b$.

Let $\tilde{\omega} \in L(X; \mathcal{H})$ such that $\tilde{\omega}(y) = e$ for all $y \in X$. By taking $\tilde{\omega} - \tilde{u} - \epsilon_a e$ for $f$, we obtain
\[D(\tilde{u}, \tilde{\omega} - \tilde{u} - \epsilon_a e) = 0,\]
so that $I_e(\tilde{\omega}) = D(\tilde{u})$.

\textbf{Theorem 3.2} \hspace{1em} $d(a, b; \mathcal{H}, e) d^*_e(a, b; \mathcal{H}) = 1$.

\textbf{Proof.} It suffices to show that $d(a, b; \mathcal{H}, e) d^*_e(a, b; \mathcal{H}) \leq 1$. Let $\tilde{u}$ be the optimal solution for $d(a, b; \mathcal{H}, e)$ and put $\tilde{w}(y) := d\tilde{u}(y)$. Then we see by the above observation that $\tilde{w}(y)/D(\tilde{u})$ is a feasible solution for $d^*_e(a, b; \mathcal{H})$, so that
\[d^*_e(a, b; \mathcal{H}) \leq H(\tilde{w}(y)/D(\tilde{u})) = D(\tilde{u})/D(\tilde{u})^2 = 1/D(\tilde{u}) = 1/d(a, b; \mathcal{H}, e). \]

\textbf{Lemma 3.5} \hspace{1em} There exists a unique optimal solution $\tilde{w}$ for $d^*(a, b; \mathcal{H}, e)$.

\textbf{Proof.} There exists a minimizing sequence $\{w_n\}$ for $d^*(a, b; \mathcal{H}, e)$. Since $(w_n + w_m)/2$ is a feasible solution for $d^*(a, b; \mathcal{H}, e)$, we have
\[d^*(a, b; \mathcal{H}) \leq H((w_n + w_m)/2) \leq H((w_n + w_m)/2) + H((w_n - w_m)/2) = [H(w_n) + H(w_m)]/2 \to d^*(a, b; \mathcal{H}, e)\]
as $m, n \to \infty$. Therefore $\{w_n\}$ is a Cauchy sequence in the Hilbert space $L(Y; \mathcal{H})$ and converges to $\tilde{w} \in L(Y; \mathcal{H})$. Then we see easily that $\tilde{w}$ is an optimal solution for $d^*(a, b; \mathcal{H}, e)$. We omit the proof of the uniqueness of the optimal solution.

\textbf{Definition 3.1} \hspace{1em} We say that $\omega \in L(Y; \mathcal{H})$ is a cycle if $\partial \omega(x) = 0$ for all $x \in X$. Denote by $C(Y; \mathcal{H})$ the set of cycles on $N$.

By the standard variational argument, we have
Lemma 3.6 Let \( \tilde{w} \) be the optimal solution of \( d^*(a, b; \mathcal{H}, e) \). For any cycle \( \omega \in C(Y; \mathcal{H}) \),
\[
H(\tilde{w}, \omega) := \sum_{y \in Y} (r(y)\tilde{w}(y), \omega(y)) = 0.
\]

Definition 3.2 Let \( P_{a,x} \) the set of all paths from \( a \) to \( x (x \neq a) \).

Theorem 3.3 \( d_e(a, b; \mathcal{H}) d^*(a, b; \mathcal{H}, e) = 1 \).

Proof. Let \( \tilde{w} \) be the optimal solution of \( d^*(a, b; \mathcal{H}, e) \). Let \( h \in \mathcal{H} \) and let \( P_1, P_2 \in P_{a,x} \).

Then
\[
\omega(y) = (p_1(y) - p_2(y))h \in C(Y; \mathcal{H}),
\]
where \( p_1 \) and \( p_2 \) are path indices of \( P_1 \) and \( P_2 \) respectively. By Lemma 3.6, we have
\[
H(\tilde{w}, p_1 h) = H(\tilde{w}, p_2 h).
\]
We set \( \tilde{u}(a) = 0 \). For \( x \neq a \) and a path index \( p_x \) of a path \( P \subset P_{a,x} \), the function \( \tilde{u} \in L(X) \) defined by \( \tilde{u}(a) = 0 \) and
\[
\tilde{u}(x) := \sum_{y \in Y} p_x(y)\tilde{w}(y)
\]
is well-defined by the above observation. Then we have \( \delta \tilde{u}(y) = -\tilde{w}(y) \). In case \( P \in P_{a,b} \), \( \tilde{w} - pe \) is a feasible solution for \( d^*(a, b; \mathcal{H}, e) \), so that \( H(\tilde{w}, \tilde{w} - pe) = 0 \) or
\[
H(\tilde{w}) = H(\tilde{w}, pe) = (\tilde{u}(b), e).
\]
Now \( \tilde{u}/\beta \) is a feasible solution for \( d_e(a, b; \mathcal{H}) \) and
\[
d_e(a, b; \mathcal{H}) \leq D(\tilde{u}) = H(\tilde{w})/H(\tilde{w})^2 = 1/H(\tilde{w}) = 1/d^*(a, b; \mathcal{H}, e). \quad \square
\]

4 Extremal length

Let \( a \) and \( b \) be distinct two nodes. The extremal length \( EL(a, b; \mathcal{L}(\mathcal{H})) \) is defined by the inverse of the value of the extremum problem \( (EL) \):

Minimize \( H(w) \) subject to
\[
w \in L(Y; \mathcal{H}),
\]
\[
\sum_{y \in P} ||r(y)w(y)|| \geq 1 \quad \text{for all} \ P \in P_{a,b}.
\]

The extremal length \( EL_e(a, b; \mathcal{L}(\mathcal{H})) \) is defined by the inverse of the value of the extremum problem \( (EL_e) \):

Minimize \( H(w) \) subject to
\[
w \in L(Y; \mathcal{H}),
\]
\[
\sum_{y \in P} |(r(y)w(y), e)| \geq 1 \quad \text{for all} \ P \in P_{a,b}.
\]
Since \( |(r(y)w(y), e)| \leq ||r(y)w(y)|| ||e|| = ||r(y)w(y)|| \), we have
\[
(4.1) \quad EL(a, b; \mathcal{L}(\mathcal{H})) \geq EL_e(a, b; \mathcal{L}(\mathcal{H})).
\]
Lemma 4.1 $d_e(a, b; \mathcal{H}) \geq EL_e(a, b; \mathcal{H})^{-1}$.

Proof. Let $u$ be any feasible solution for $d_e(a, b; \mathcal{H})$. Then

$$w(y) := r(y)^{-1} \delta u(y) \in \mathcal{H}$$

for each $y \in Y$. As in the proof of Lemma 1.3, for $P \in \mathcal{P}_{a,b}$ let

$$C_X(P) := \{x_0, x_1, \ldots, x_n\} \ (x_0 = a, x_n = b)$$

$$C_Y(P) := \{y_1, y_2, \ldots, y_n\}, e(y_i) = \{x_{i-1}, x_i\} \ (i = 1, 2, \ldots, n).$$

Then we have

$$\sum_{y \in P} |(r(y)w(y), e)| = \sum_{i=1}^{n} |(r(y_i)w(y_i), e)|$$

$$= \sum_{i=1}^{n} |(\delta u(y_i), e)|$$

$$\geq \sum_{i=1}^{n} |(u(x_i) - u(x_{i-1}), e)|$$

$$\geq (u(b), e) - (u(a), e) = 1.$$ 

Therefore

$$EL_e(a, b; \mathcal{H})^{-1} \leq H(w) = D(u),$$

so that $EL_e(a, b; \mathcal{H})^{-1} \leq d_e(a, b; \mathcal{H})$. \qed

Lemma 4.2 Let $w$ be a feasible solution for the problem $(EL_e)$. Then

$$d_e(a, b; \mathcal{H}) \leq \sum_{y \in Y} (r(y)w(y), w(y))(r(y)e, e)(r(y)^{-1}e, e).$$

Proof. Put $V(y) := |(r(y)w(y), e)|$. Then

$$\sum_{y \in P} V(y) \geq 1 \text{ for all } P \in \mathcal{P}_{a,b}.$$ 

By the duality between the max-potential problem and the min-work problem, there exists $\beta \in L(X; \mathbb{R})$ such that $\beta(a) = 0$, $\beta(b) = 1$ and $|\delta \beta(y)| \leq V(y)$ on $Y$. Let $u(x) := \beta(x)e$. Then $u \in L(X; \mathcal{H})$, $u(a) = 0$ and $u(b) = e$, so that by Lemma 1.1 (1)

$$d_e(a, b; \mathcal{H}) \leq D(u) = \sum_{y \in Y} (r(y)^{-1}\delta u(y), \delta u(y))$$

$$= \sum_{y \in Y} (\delta \beta(y))^2 (r(y)^{-1}e, e)$$

$$\leq \sum_{y \in Y} V(y)^2 (r(y)^{-1}e, e)$$

$$\leq \sum_{y \in Y} (r(y)w(y), w(y))(r(y)e, e)(r(y)^{-1}e, e) \ \square$$
Theorem 4.1 Let \( M(r) := \sup\{(r(y)e, e)(r(y)^{-1}e, e); y \in Y\} \). Then
\[
EL_e(a, b; \mathcal{H})^{-1} \leq d_e(a, b; \mathcal{H}) \leq M(r) EL_e(a, b; \mathcal{H})^{-1}.
\]

Corollary 4.1 Assume that \((r(y)e, e)(r(y)^{-1}e, e) = 1\) for all \( y \in Y \). Then \( d_e(a, b; \mathcal{H}) = EL_e(a, b; \mathcal{H})^{-1} \).

Remark 1. Let \( I \) be the identity map of \( \mathcal{H} \) and let \( \gamma \in L(Y; \mathbb{R}) \) be positive. Then \( r(y) = \gamma(y)I \) is positive and invertible. Clearly, we have \((r(y)e, e) = \gamma(y)\) and \((r(y)^{-1}e, e) = 1/\gamma(y)\), so that the condition in the above theorem holds in this case.

We show by an example that the equality \( d_e(a, b; \mathcal{H}) = EL_e(a, b; \mathcal{H})^{-1} \) does not hold in general.

Example Let \( X = \{x_0, x_1, x_2\}, Y = \{y_1, y_2\} \)
\[
K(x_i, y_i) = 1, \ K(x_i, y_{i-1}) = -1 (i = 1, 2)
\]
and \( K(x, y) = 0 \) for any other pair. Then \( \{X, Y, K\} \) is a finite graph. Take \( \mathcal{H} \) as \( \mathbb{R}^2 \) and define \( r(y) \) by
\[
r(y) := \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad r(y_2) := \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}.
\]
Then
\[
r(y_1)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/s \end{pmatrix}, \quad r(y_2)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/t \end{pmatrix}.
\]
Let \( a = x_0, b = x_2 \) in the above setting and let \( e = (e_1, e_2)^T \in \mathbb{R}^2 \). Let \( u \in L(X, \mathbb{R}^2) \) be a feasible solution for \( d_e(a, b; \mathbb{R}^2) \) and set \( u(x_1) = (\alpha, \beta)^T \). Then
\[
D(u) = \alpha^2 + \beta^2/s + (\alpha - e_1)^2 + (\beta - e_2)^2/t.
\]
It is easily seen that
\[
d_e(a, b; \mathbb{R}^2) = \frac{e_1^2}{2} + \frac{e_2^2}{s + t}
\]
and \( \tilde{u}(x_1) := (e_1/2, e_2 s/(s + t))^T \) is the optimal solution. For \( w \in L(Y, \mathbb{R}^2) \), set \( w(y_1) = (p_1, q_1)^T, w(y_2) = (p_2, q_2)^T \). Then
\[
H(w) = p_1^2 + sq_1^2 + p_2^2 + t q_2^2.
\]
Clearly, \( P_{a,b} \) is a singleton. The feasibility of \( w \in L(Y, \mathbb{R}^2) \) for the problem \( (EL_u) \) implies
\[
(p_1 + p_2)e_1 + (sq_1 + bq_2)e_2 \geq 1.
\]
Therefore we have
\[ EL_e(a, b; \mathbb{R}^2)^{-1} = \frac{1}{2e_1^2 + (s + t)e_2^2} \]
and the optimal solution is given by
\[ p_1 = p_2 = \frac{e_1 \lambda}{2}, \quad q_1 = q_2 = \frac{e_2 \lambda}{2} \]
with \( \lambda := \frac{2}{2e_1^2 + (s + t)e_2^2} \).

We have
\[ d_e(a, b; \mathbb{R}^2) - EL_e(a, b; \mathbb{R}^2)^{-1} = \frac{e_1^2 e_2^2 (c-2)^2}{2c(2e_1^2 + ce_2^2)} \geq 0 \]
with \( c = s + t \).

Thus the equality holds only if \( c = 2 \) or \( e_1 = 0 \) or \( e_2 = 0 \).

## 5 Extremal width

Let \( a \) and \( b \) be distinct two nodes. Denote by \( Q_{a,b} \) the set of all cuts between \( a \) and \( b \) (cf. [2]). For \( Q \in Q_{a,b} \), there exist two disjoint subsets \( Q(a) \) and \( Q(b) \) of \( X \) such that \( a \in Q(a), \ b \in Q(b), \ X = Q(a) \cup Q(b) \) and \( Q = Q(a) \ominus Q(b) \). The index function \( u_Q \in L(X; \mathcal{H}) \) of \( Q \) is defined by
\[ u := \epsilon_{Q(A)}e = \sum_{z \in Q} \epsilon_{z}e. \]

The characteristic function \( s_Q \) of \( Q \) is defined by
\[ s_Q := \delta u_Q e = \sum_{z \in Q} \delta \epsilon_{z}e. \]

Notice that \(|\delta \epsilon_{Q(A)}(y)| = 1\) if \( y \in Q \) and \( \delta \epsilon_{Q(A)}(y) = 0 \) otherwise. Observe that \( \|s_Q(y)\| = 1 \) if \( y \in Q \) and \( \|s_Q(y)\| = 0 \) otherwise.

The extremal width \( EW(a, b; \mathcal{H}) \) is defined by the inverse of the value of the extremum problem \((EW)\):

**Minimize** \( H(w) \) subject to
\[ w \in L(Y; \mathcal{H}), \]
\[ \sum_{y \in Q} \|w(y)\| \geq 1 \quad \text{for all} \quad Q \in Q_{a,b}. \]

The extremal width \( EW_e(a, b; \mathcal{H}) \) is defined by the inverse of the value of the extremum problem \((EW_e)\):

**Minimize** \( H(w) \) subject to
\[ w \in L(Y; \mathcal{H}), \]
\[ \sum_{y \in Q} |(w(y), e)| \geq 1 \quad \text{for all } Q \in \mathcal{Q}_{a,b}. \]

Since \(|(w(y), e)| \leq ||w(y)|| ||e|| = ||w(y)||\), we have

(5.1). \[ EW(a, b; \mathcal{H}) \geq EW_e(a, b; \mathcal{H}) \]

**Lemma 5.1** \( d^*_{e}(a, b; \mathcal{H}) \geq EW_e(a, b; \mathcal{H})^{-1}. \)

**Proof.** Let \( w \) be any feasible solution for \( d^*_{e}(a, b; \mathcal{H}) \) and let \( Q \in \mathcal{Q}_{a,b} \) Then

\[
1 = I_{e}(w) = - (\partial w(a), e) = - \sum_{x \in X} (\partial w(x), \epsilon_Q(x)e) = - \sum_{y \in Y} (w(y), \delta_{\epsilon_Q(y)}e) \leq \sum_{y \in Q} |(w(y), e)|
\]

Therefore \( EW_e(a, b; \mathcal{H})^{-1} \leq H(w) \), so that \( EW_e(a, b; \mathcal{H})^{-1} \leq d^*_{e}(a, b; \mathcal{H}) \).

**Lemma 5.2** Let \( w \) be a feasible solution for the problem \( (EW_e) \). Then

\[
d^*_{e}(a, b; \mathcal{H}) \leq \sum_{y \in Y} (r(y)w(y), w(y))(r(y)e, e)(r(y)^{-1}e, e).
\]

**Proof.** Put \( V(y) := |(w(y), e)|. \) Then

\[
\sum_{y \in Q} V(y) \geq 1 \quad \text{for all } Q \in \mathcal{Q}_{a,b}.
\]

By the duality between the max-flow problem and the min-cut problem, there exists \( \varphi \in L(Y; \mathbb{R}) \) such that \(|\varphi(y)| \leq V(y) \) on \( Y \),

\[
\partial \varphi(x) = 0 \quad \text{for } x \in X \setminus \{a, b\} \quad \text{and} \quad - \partial \varphi(a) = - \partial \varphi(b) = 1.
\]

Let \( w(y) := \varphi(y)e. \) Then \( w \in F(a, b; \mathcal{H}) \) and \( I_{e}(w) = 1. \) Therefore

\[
d^*_{e}(a, b; \mathcal{H}) \leq H(w) = \sum_{y \in Y} (r(y)\varphi(y)e, \varphi(y)e) = \sum_{y \in Y} |\varphi(y)|^2 (r(y)e, e) \leq \sum_{y \in Y} |(w(y), e)|^2 (r(y)e, e) \leq \sum_{y \in Y} (r(y)w(y), w(y))(r(y)^{-1}e, e)(r(y)e, e). \]

**Theorem 5.1** Let \( M(r) := \sup\{(r(y)e, e)(r(y)^{-1}e, e); y \in Y\} \). Then

\[
EW_e(a, b; \mathcal{H})^{-1} \leq d^*_{e}(a, b; \mathcal{H}) \leq M(r) EW_e(a, b; \mathcal{H})^{-1}.
\]
Corollary 5.1 Assume that $(r(y)e, e)(r(y)^{-1}e, e) = 1$ for all $y \in Y$. Then $d_e^*(a, b; \mathcal{H}) = EW_e(a, b; \mathcal{H})^{-1}$.

We recall the example in Section 4 and calculate $EW_e(a, b; \mathcal{H})^{-1}$ and $d_e^*(a, b; \mathcal{H})$ in this case. If $w \in F(a, b; \mathbb{R}^2)$, then $w(y_1) = w(y_2) = (p, q)^T$ and

$$H(w) = 2p^2 + (s + t)q^2, \quad I_e(w) = pe_1 + qe_2.$$ 

By a simple calculus, we see easily that

$$d_e^*(a, b; \mathbb{R}^2) = \frac{1}{e_1^2/2 + e_2^2/(s + t)}.$$

On the other hand, if $w$ is feasible for $EW_e(a, b; \mathbb{R}^2)^{-1}$, then we have

$$(*) \quad p_1e_1 + q_1e_1 \leq 1, \quad p_2e_1 + q_2e_1 \leq 1$$

with $w(y_1) = (p_1, q_1)^T$, $w(y_2) = (p_2, q_2)$. Minimizing $H(w)$ subject to the condition $(*)$, we have

$$EW_e(a, b; \mathbb{R}^2)^{-1} = \frac{s}{se_1^2 + e_2^2} + \frac{t}{te_1^2 + e_2^2}.$$ 

Therefore

$$d_e^*(a, b; \mathbb{R}^2) - EW_e(a, b; \mathbb{R}^2)^{-1} = \frac{(s - t)^2e_1^2e_2^2}{[(s + t)e_1^2 + 2e_2^2(te_1^2 + e_2^2)(se_1^2 + e_2^2)]} \geq 0$$

and the equality holds if $s = t$ or $e_1 = 0$ or $e_2 = 0$.

References

