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Finite Hilbert Networks

Maretsugu Yamasaki

1 Introduction

Let $N = \{X, Y, K\}$ be a finite connected graph which has no self-loop. Namely $X$ is a finite set of nodes, $Y$ is a finite set of arcs and $K$ is the node-arc incidence matrix.

Let $\mathcal{H}$ be a real Hilbert space with an inner product $(\cdot, \cdot)$ and the norm $\| \cdot \|$. Denote by $L(X; \mathcal{H})$ the set of all functions $u$ on $X$ such that $u(x) \in \mathcal{H}$. We call an element of $L(X, \mathcal{H})$ a $\mathcal{H}$-potential. The meaning of the notation $L(Y; \mathcal{H})$ is similar. Let $r \in L(Y; \mathcal{L}(\mathcal{H}))$. For each $y \in Y$, we have $r(y) \in \mathcal{L}(\mathcal{H})$ and there exists $\rho(y) > 0$ such that

$$(r(y)h, h) \geq \rho(y)\|h\|^2 \quad \text{for all } h \in \mathcal{H}.$$ 

Here $r(y)h$ means the image of $h$ under $r(y)$, i.e., $r(y)(h)$. In this paper, we use this convention unless no confusion occurs from the context. Denote by $r(y)^{-1}$ the inverse operator of $r(y)$. Since $r(y) \in \mathcal{L}(\mathcal{H})$, there exists $\rho^*(y) > 0$ such that

$$(r(y)^{-1}h, h) \geq \rho^*(y)\|h\|^2 \quad \text{for all } h \in \mathcal{H}.$$ 

By [1], we see that there exists a unique square root $r(y)^{1/2} \in \mathcal{L}(\mathcal{H})$ of $r(y)$ for each $y \in Y$, i.e.,

$$\|r(y)^{1/2}\|^2 = r(y).$$

Definition 1.1 Let $e$ be a fixed element of $\mathcal{H}$ such that $\|e\| = 1$.

Lemma 1.1 For every $y \in Y$, the following relations hold:

1) $|(r(y)w(y), e)|^2 \leq (r(y)w(y), w(y))(r(y)e, e)$.

2) $(r(y)^{-1}e, (r(y)e, e)) \geq 1$.

Proof. By Schwarz's inequality, we have

$$|(r(y)w(y), e)|^2 = |(r(y)^{1/2}w(y), r(y)^{1/2}e)|^2 \leq \|r(y)^{1/2}w(y)\|^2\|r(y)^{1/2}e\|^2 = (r(y)w(y), w(y))(r(y)e, e).$$

2) follows from (1) by taking $w(y) := r(y)^{-1}e$. □
Definition 1.2 For $u \in L(X; \mathcal{H})$, let $\delta u$ be the potential drop of $u$ and let $du$ be the discrete derivative of $u$:

$$
\delta u(y) := \sum_{x \in X} K(x, y)u(x)
$$

$$
du(y) := -r(y)^{-1}(\delta u(y)) = -\Gamma(y)^{-1}\delta u(y).
$$

The Dirichlet sum of $u$ is defined by

$$
D(u) := \sum_{y \in Y}(r(y)du(y), du(y)) = \sum_{y \in Y}(r(y)^{-1}\delta u(y), \delta u(y)).
$$

Definition 1.3 For $w \in L(Y; \mathcal{H})$, let $\partial w(x)$ be the divergence of $w$ and let $H(w)$ be the energy of $w$:

$$
\partial w(x) := \sum_{y \in Y} K(x, y)w(y)
$$

$$
H(w) := \sum_{y \in Y}(r(y)w(y), w(y)).
$$

Notice that $D(u) = H(du)$.

Lemma 1.2 Let $u \in L(X; \mathcal{H})$ and $w \in L(Y; \mathcal{H})$. Then

$$
\sum_{y \in Y}(w(y), \delta u(y)) \leq H(w)^{1/2}D(u)^{1/2}.
$$

Proof. We have by Schwarz's inequality

$$
\sum_{y \in Y}(w(y), \delta u(y)) = \sum_{y \in Y}(r(y)^{-1/2}w(y), r(y)^{-1/2}\delta u(y))
$$

$$
\leq \sum_{y \in Y} ||r(y)^{-1/2}w(y)||||r(y)^{-1/2}\delta u(y)||
$$

$$
\leq \left[\sum_{y \in Y} ||r(y)^{-1/2}w(y)||^2\right]^{1/2}\left[\sum_{y \in Y} ||r(y)^{-1/2}\delta u(y)||^2\right]^{1/2}
$$

$$
= H(w)^{1/2}D(u)^{1/2}. \quad \square
$$

To emphasize the analogy to [2], we put

$$
D(N; \mathcal{H}; a) := \{u \in L(X; \mathcal{H}); u(a) = 0\}.
$$

Note that $D(u) < \infty$ for every $L(X; \mathcal{H})$, since $G$ is a finite graph. We see that $D(u)^{1/2}$ is a norm on $D(N; \mathcal{H}; a)$ by the following lemma:

Lemma 1.3 Let $a \in X$. For any $x \in X$, there exists a constant $M_x$ which satisfies:

$$
||u(x)|| \leq M_x D(u)^{1/2}
$$

for all $u \in L(X; \mathcal{H})$ with $u(a) = 0$. 
Proof. There exists a path $P$ from $a$ to $x$. Let $C_X(P)$ and $C_Y(P)$ be the sets of nodes and arcs on $P$ respectively (cf. [2]), i.e.,

$$C_X(P) := \{x_0, x_1, \cdots, x_n\} \ (x_0 = a, x_n = x)$$

$$C_Y(P) := \{y_1, y_2, \cdots, y_n\}, e(y_i) = \{x_{i-1}, x_i\} \ (i = 1, 2, \cdots, n).$$

Let $u \in L(X; \mathcal{H})$ and $u(a) = 0$. We have

$$D(u) \geq \sum_{y \in P} (r(y)^{-1} \delta u(y), \delta u(y))$$

$$= \sum_{i=1}^{n} (r(y_i)^{-1} \delta u(y_i), \delta u(y_i))$$

$$\geq \sum_{i=1}^{n} \rho^*(y_i) \|u(x_i) - u(x_{i-1})\|^2$$

$$\geq \sum_{i=1}^{n} \rho^*(y_i) \|u(x_i)\| - \|u(x_{i-1})\|^2,$$

so that, for $i = 1, 2, \cdots$

$$\|u(x_i)\| - \|u(x_{i-1})\| \leq D(u)^{1/2} [\rho^*(y_i)]^{-1/2}.$$ 

Since $u(a) = 0$, we have

$$\|u(x)\| = \sum_{i=1}^{n} [\|u(x_i)\| - \|u(x_{i-1})\|] \leq M_x D(u)^{1/2}$$

with

$$M_x := \sum_{i=1}^{n} [\rho^*(y_i)]^{-1/2}.$$ 

This completes the proof. □

Since $G$ is a finite graph, the following fact is obvious:

**Proposition 1.1** $D(N; \mathcal{H}, a)$ is a Hilbert space with respect to the inner product:

$$D(u_1, u_2) := \sum_{y \in Y} (r(y)^{-1} \delta u_1(y), \delta u_2(y)).$$

$L(Y; \mathcal{H})$ is a Hilbert space with respect to the inner product:

$$H(w_1, w_2) := \sum_{y \in Y} (r(y)w_1(y), w_2(y)).$$. 

2 \( \mathcal{H} \)-flows

Definition 2.1 Let \( a \) and \( b \) be distinct nodes. We say that \( w \in L(Y; \mathcal{H}) \) is a \( \mathcal{H} \)-flow from \( a \) to \( b \) if

\[
\partial w(x) = 0 \quad \text{for all} \quad x \in X, \quad x \neq a, b.
\]

Denote by \( F(a, b; \mathcal{H}) \) the set of all \( \mathcal{H} \)-flows from \( a \) to \( b \).

Notice that

\[
\partial w(a) + \partial w(b) = 0,
\]

since \( G \) is a finite graph.

Definition 2.2 For \( w \in F(a, b; \mathcal{H}) \), we define two real valued functions:

\[
I_e(w) := (\partial w(b), e) = -(\partial w(a), e),
\]

\[
I(w) := \|\partial w(a)\| = \|\partial w(b)\|.
\]

3 Extremum problems

Let us consider several extremum problems related to \( \mathcal{H} \)-potentials and \( \mathcal{H} \)-flows:

\[
d(a, b; \mathcal{H}, e) := \inf \{D(u); u \in L(X; \mathcal{H}), u(a) = 0, u(b) = e\}
\]

\[
d_e(a, b; \mathcal{H}) := \inf \{D(u); u \in L(X; \mathcal{H}), (u(a), e) = 0, (u(b), e) = 1\}
\]

\[
d(a, b; \mathcal{H}) := \inf \{D(u); u \in L(X; \mathcal{H}), u(a) = 0, \|u(b)\| = 1\}
\]

\[
d^*(a, b; \mathcal{H}, e) := \inf \{H(w); w \in F(a, b; \mathcal{H}), Kw(b) = e\}
\]

\[
d^*_e(a, b; \mathcal{H}) := \inf \{H(w); w \in F(a, b; \mathcal{H}), I_e(w) = 1\}
\]

\[
d^*(a, b; \mathcal{H}) := \inf \{H(w); w \in F(a, b; \mathcal{H}), I(w) = 1\}
\]

Clearly

\[
d_e(a, b; \mathcal{H}) \leq d(a, b; \mathcal{H}, e), \quad d(a, b; \mathcal{H}) \leq d(a, b; \mathcal{H}, e),
\]

\[
d^*_e(a, b; \mathcal{H}) \leq d^*(a, b; \mathcal{H}, e), \quad d^*(a, b; \mathcal{H}) \leq d^*(a, b; \mathcal{H}, e).
\]

Lemma 3.1 Let \( u \) be a feasible solution for \( d(a, b; \mathcal{H}, e) \) and \( w \) be a feasible solution for \( d^*_e(a, b; \mathcal{H}) \). Then \( 1 \leq H(w)^{1/2}D(u)^{1/2} \).

Proof. By definition and Lemma 1.2

\[
1 = I_e(w) = (Kw(b), e) = \sum_{x \in X} (Kw(x), u(x)) = \sum_{y \in Y} (w(y), \delta u(y)) \leq H(w)^{1/2}D(u)^{1/2}.
\]

Similarly we can prove
Lemma 3.2  Let \( u \) be a feasible solution for \( d_e(a, b; \mathcal{H}) \) and \( w \) be a feasible solution for \( d^*(a, b; \mathcal{H}, e) \). Then \( 1 \leq H(w)^{1/2}D(u)^{1/2} \).

By the above observation, we obtain

Theorem 3.1  The following relations hold:
1. \( 1 \leq d(a, b; \mathcal{H}, e)d^*(a, b; \mathcal{H}) \),
2. \( 1 \leq d_e(a, b; \mathcal{H})d^*(a, b; \mathcal{H}, e) \).

Lemma 3.3  There exists a unique optimal solution for \( d(a, b; \mathcal{H}, e) \).

Proof.  Let \( \{u_n\} \) be a minimizing sequence for \( d(a, b; \mathcal{H}, e) \), i.e., \( \{u_n\} \subset L(X; \mathcal{H}) \), \( u_n(a) = 0 \), \( u_n(b) = e \) and \( D(u_n) \to d(a, b; \mathcal{H}, e) \) as \( n \to \infty \). Since \( (u_n + u_m)/2 \) is a feasible solution for \( d(a, b; \mathcal{H}, e) \), we have

\[
\begin{aligned}
d(a, b; \mathcal{H}, e) &\leq D((u_n + u_m)/2) \\
&\leq D((u_n + u_m)/2) + D((u_n - u_m)/2) \\
&= [D(u_n) + D(u_m)]/2 \to d(a, b; \mathcal{H}, e)
\end{aligned}
\]

as \( n, m \to \infty \). Therefore \( D(u_n - u_m) \to 0 \) as \( n, m \to \infty \). It follows from Lemma 1.3 that \( \{u_n(x)\} \) is a Cauchy sequence in \( \mathcal{H} \) for each \( x \in X \). Therefore \( \{u_n(x)\} \) converges strongly to \( \tilde{u}(x) \in \mathcal{H} \) for each \( x \in X \). We see easily that \( \tilde{u}(a) = 0, \tilde{u}(b) = e \) and \( d(a, b; \mathcal{H}, e) = D(\tilde{u}) \). Namely \( \tilde{u} \) is an optimal solution. We omit the proof of the uniqueness of the optimal solution. \( \square \)

Now we study some properties of the optimal solution \( \tilde{u} \) of \( d(a, b; \mathcal{H}, e) \).

Lemma 3.4  Let \( \tilde{w}(y) := d\tilde{u}(y) \). Then \( \tilde{w} \in F(a,b;\mathcal{H}) \) and \( I_e(\tilde{w}) = D(\tilde{u}) \).

Proof.  Let \( f \in L(X; \mathcal{H}) \) satisfy \( f(a) = 0 \) and \( f(b) = 0 \). Then for any \( t \in \mathbb{R} \), \( \tilde{u} + tf \) is a feasible solution for \( d(a, b; \mathcal{H}, e) \), we have

\[
\begin{aligned}
D(\tilde{u}) &\leq D(\tilde{u} + tf) \\
&= D(\tilde{u}) + 2tD(\tilde{u}, f) + t^2D(f).
\end{aligned}
\]

By the standard variational argument, we have

\[
D(\tilde{u}, f) = 0.
\]

On the other hand, we have

\[
\begin{aligned}
D(\tilde{u}, f) &= \sum_{y \in Y} (\tilde{w}(y), \sum_{z \in X} K(z, y)f(z)) \\
&= \sum_{z \in X} \sum_{y \in Y} (K(z, y)\tilde{w}(y), f(z)) \\
&= \sum_{z \in X} (\partial \tilde{w}(z), f(z)).
\end{aligned}
\]
Denote by $\epsilon_x$ the characteristic function of $\{x\}$, i.e., $\epsilon_x(x) = 1$ and $\epsilon_x(z) = 0$ for $z \neq x$. Let $x \neq a, b$. For any $h \in \mathcal{H}$, we may take $\epsilon_x h$ for $f$, and hence

$$(\partial \tilde{w}(x), h) = 0.$$ 

Therefore $\partial \tilde{w}(x) = 0$ for $x \neq a, b$. Let $\tilde{\epsilon} \in L(X; \mathcal{H})$ such that $\tilde{\epsilon}(x) = e$ for all $x \in X$. By taking $\tilde{\epsilon} - \tilde{u} - \epsilon_a e$ for $f$, we obtain

$$D(\tilde{u}, \tilde{\epsilon} - \tilde{u} - \epsilon_a e) = 0,$$

so that

$$D(\tilde{u}) = -D(\tilde{u}, \epsilon_a e) = -(\partial \tilde{w}(a), e).$$

Therefore $I_e(\tilde{w}) = D(\tilde{u})$.

**Theorem 3.2** $d(a, b; \mathcal{H}, e) d^*_e(a, b; \mathcal{H}) = 1$.

**Proof.** It suffices to show that $d(a, b; \mathcal{H}, e) d^*_e(a, b; \mathcal{H}) \leq 1$. Let $\tilde{u}$ be the optimal solution for $d(a, b; \mathcal{H}, e)$ and put $\tilde{w}(y) := d\tilde{u}(y)$. Then we see by the above observation that $\tilde{w}(y)/D(\tilde{u})$ is a feasible solution for $d^*_e(a, b; \mathcal{H})$, so that

$$d^*_e(a, b; \mathcal{H}) \leq H(\tilde{w}(y)/D(\tilde{u})) = D(\tilde{u})/D(\tilde{u})^2 = 1/D(\tilde{u}) = 1/d(a, b; \mathcal{H}, e).$$

**Lemma 3.5** There exists a unique optimal solution $\tilde{w}$ for $d^*(a, b; \mathcal{H}, e)$.

**Proof.** There exists a minimizing sequence $\{w_n\}$ for $d^*(a, b; \mathcal{H}, e)$. Since $(w_n + w_m)/2$ is a feasible solution for $d^*(a, b; \mathcal{H}, e)$, we have

$$d^*(a, b; \mathcal{H}, e) \leq H((w_n + w_m)/2) \leq H((w_n + w_m)/2) + H((w_n - w_m)/2) = [H(w_n) + H(w_m)]/2 \rightarrow d^*(a, b; \mathcal{H}, e)$$

as $m, n \rightarrow \infty$. Therefore $\{w_n\}$ is a Cauchy sequence in the Hilbert space $L(Y; \mathcal{H})$ and converges to $\tilde{w} \in L(Y; \mathcal{H})$. Then we see easily that $\tilde{w}$ is an optimal solution for $d^*(a, b; \mathcal{H}, e)$. We omit the proof of the uniqueness of the optimal solution.

**Definition 3.1** We say that $\omega \in L(Y; \mathcal{H})$ is a cycle if $\partial \omega(x) = 0$ for all $x \in X$. Denote by $C(Y; \mathcal{H})$ the set of cycles on $N$.

By the standard variational argument, we have
Lemma 3.6 Let $\tilde{w}$ be the optimal solution of $d^*(a, b; \mathcal{H}, e)$. For any cycle $\omega \in C(Y; \mathcal{H})$,

$$H(\tilde{w}, \omega) := \sum_{y \in Y} (r(y)\tilde{w}(y), \omega(y)) = 0.$$ 

Definition 3.2 Let $P_{a,x}$ be the set of all paths from $a$ to $x(x \neq a)$.

Theorem 3.3 $d_e(a, b; \mathcal{H})d^*(a, b; \mathcal{H}, e) = 1$.

Proof. Let $\tilde{w}$ be the optimal solution of $d^*(a, b; \mathcal{H}, e)$. Let $h \in \mathcal{H}$ and let $P_1, P_2 \in P_{a,x}$. Then

$$\omega(y) = (p_1(y) - p_2(y))h \in C(Y; \mathcal{H}),$$

where $p_1$ and $p_2$ are path indices of $P_1$ and $P_2$ respectively. By Lemma 3.6, we have $H(\tilde{w}, p_1h) = H(\tilde{w}, p_2h)$. We set $\tilde{u}(a) = 0$. For $x \neq a$ and a path index $p_x$ of a path $P \subset P_{a,x}$, the function $\tilde{u} \in L(X)$ defined by $\tilde{u}(a) = 0$ and

$$\tilde{u}(x) := \sum_{y \in Y} p_x(y)\tilde{w}(y)$$

is well-defined by the above observation. Then we have $\delta\tilde{u}(y) = -\tilde{w}(y)$. In case $P \in P_{a,b}$, $\tilde{w} - pe$ is a feasible solution for $d^*(a, b; \mathcal{H}, e)$, so that $H(\tilde{w}, \tilde{w} - pe) = 0$ or

$$H(\tilde{w}) = H(\tilde{w}, pe) = (\tilde{u}(b), e).$$

Now $\tilde{u}/\beta$ is a feasible solution for $d_e(a, b; \mathcal{H})$ and

$$d_e(a, b; \mathcal{H}) \leq D(\tilde{u}) = H(\tilde{w})/H(\tilde{w})^2 = 1/H(\tilde{w}) = 1/d^*(a, b; \mathcal{H}, e). \quad \Box$$

4 Extremal length

Let $a$ and $b$ be distinct two nodes. The extremal length $EL(a, b; \mathcal{L}(\mathcal{H}))$ is defined by the inverse of the value of the extremum problem $(EL)$:

Minimize $H(w)$ subject to

$$w \in L(Y; \mathcal{H}),$$

$$\sum_{y \in P} ||r(y)w(y)|| \geq 1 \quad \text{for all} P \in P_{a,b}.$$ 

The extremal length $EL_e(a, b; \mathcal{L}(\mathcal{H}))$ is defined by the inverse of the value of the extremum problem $(EL_e)$:

Minimize $H(w)$ subject to

$$w \in L(Y; \mathcal{H}),$$

$$\sum_{y \in P} |(r(y)w(y), e)| \geq 1 \quad \text{for all} P \in P_{a,b}.$$ 

Since $|(r(y)w(y), e)| \leq ||r(y)w(y)|| ||e|| = ||r(y)w(y)||$, we have

$$EL(a, b; \mathcal{L}(\mathcal{H})) \geq EL_e(a, b; \mathcal{L}(\mathcal{H})). \quad (4.1)$$
Lemma 4.1 \( d_e(a, b; \mathcal{H}) \geq EL_e(a, b; \mathcal{H})^{-1} \).

Proof. Let \( u \) be any feasible solution for \( d_e(a, b; \mathcal{H}) \). Then
\[
w(y) := r(y)^{-1}\delta u(y) \in \mathcal{H}
\]
for each \( y \in Y \). As in the proof of Lemma 1.3, for \( P \in \mathcal{P}_{a,b} \) let
\[
C_X(P) := \{x_0, x_1, \ldots, x_n\} \ (x_0 = a, x_n = b) \\
C_Y(P) := \{y_1, y_2, \ldots, y_n\}, e(y_i) = \{x_{i-1}, x_i\} \ (i = 1, 2, \ldots, n).
\]
Then we have
\[
\sum_{y \in P} |(r(y)w(y), e)| = \sum_{i=1}^{n} |(r(y_i)w(y_i), e)| \\
= \sum_{i=1}^{n} |(\delta u(y_i), e)| \\
\geq \sum_{i=1}^{n} |(u(x_i) - u(x_{i-1}), e)| \\
\geq (u(b), e) - (u(a), e) = 1.
\]
Therefore
\[
EL_e(a, b; \mathcal{H})^{-1} \leq H(w) = D(u),
\]
so that \( EL_e(a, b; \mathcal{H})^{-1} \leq d_e(a, b; \mathcal{H}) \). \( \square \)

Lemma 4.2 Let \( w \) be a feasible solution for the problem \( (EL_e) \). Then
\[
d_e(a, b; \mathcal{H}) \leq \sum_{y \in Y} (r(y)w(y), w(y))(r(y)\delta u, \delta u)\leq (r(y)^{-1}e, e).
\]

Proof. Put \( V(y) := |(r(y)w(y), e)|. \) Then
\[
\sum_{y \in P} V(y) \geq 1 \quad \text{for all} \quad P \in \mathcal{P}_{a,b}.
\]
By the duality between the max-potential problem and the min-work problem, there exists \( \beta \in L(X; \mathbb{R}) \) such that \( \beta(a) = 0, \beta(b) = 1 \) and \(|\delta \beta(y)| \leq V(y) \) on \( Y \). Let \( u(x) := \beta(x)e \). Then \( u \in L(X; \mathcal{H}), u(a) = 0 \) and \( u(b) = e \), so that by Lemma 1.1 (1)
\[
d_e(a, b; \mathcal{H}) \leq D(u) = \sum_{y \in Y} (r(y)^{-1}\delta u(y), \delta u(y)) \\
= \sum_{y \in Y} (\delta \beta(y))^2 (r(y)^{-1}e, e) \\
\leq \sum_{y \in Y} V(y)^2 (r(y)^{-1}e, e) \\
\leq \sum_{y \in Y} (r(y)w(y), w(y))(r(y)e, e)(r(y)^{-1}e, e) \quad \square
Theorem 4.1 Let $M(r) := \sup\{(r(y)e, e)(r(y)^{-1}e, e); y \in Y\}$. Then

$$EL_e(a, b; \mathcal{H})^{-1} \leq d_e(a, b; \mathcal{H}) \leq M(r)EL_e(a, b; \mathcal{H})^{-1}.$$ 

Corollary 4.1 Assume that $(r(y)e, e)(r(y)^{-1}e, e) = 1$ for all $y \in Y$. Then $d_e(a, b; \mathcal{H}) = EL_e(a, b; \mathcal{H})^{-1}$.

Remark 1. Let $I$ be the identity map of $\mathcal{H}$ and let $\gamma \in L(Y; \mathbb{R})$ be positive. Then $r(y) = \gamma(y)I$ is positive and invertible. Clearly, we have $(r(y)e, e) = \gamma(y)$ and $(r(y)^{-1}e, e) = 1/\gamma(y)$, so that the condition in the above theorem holds in this case.

We show by an example that the equality $d_e(a, b; \mathcal{H}) = EL_e(a, b; \mathcal{H})^{-1}$ does not hold in general.

Example Let $X = \{x_0, x_1, x_2\}, Y = \{y_1, y_2\}$,

$$K(x_i, y_i) = 1, \ K(x_i-1, y_i) = -1 \ (i = 1, 2)$$

and $K(x, y) = 0$ for any other pair. Then $\{X, Y, K\}$ is a finite graph. Take $\mathcal{H}$ as $\mathbb{R}^2$ and define $r(y)$ by

$$r(y_1) := \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad r(y_2) := \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}.$$ 

Then

$$r(y_1)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/s \end{pmatrix}, \quad r(y_2)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/t \end{pmatrix}.$$ 

Let $a = x_0$, $b = x_2$ in the above setting and let $e = (e_1, e_2)^T \in \mathbb{R}^2$. Let $u \in L(X, \mathbb{R}^2)$ be a feasible solution for $d_e(a, b; \mathbb{R}^2)$ and set $u(x_1) = (\alpha, \beta)^T$. Then

$$D(u) = \alpha^2 + \beta^2/s + (\alpha - e_1)^2 + (\beta - e_2)^2/t.$$ 

It is easily seen that

$$d_e(a, b; \mathbb{R}^2) = \frac{e_1^2}{2} + \frac{e_2^2}{s + t}$$

and $\bar{u}(x_1) := (e_1/2, e_2s/(s + t))^T$ is the optimal solution. For $w \in L(Y, \mathbb{R}^2)$, set $w(y_1) = (p_1, q_1)^T$, $w(y_2) = (p_2, q_2)^T$. Then

$$H(w) = p_1^2 + sq_1^2 + p_2^2 + tq_2^2.$$ 

Clearly, $P_{a,b}$ is a singleton. The feasibility of $w \in L(Y, \mathbb{R}^2)$ for the problem $(EL_e)$ implies

$$(p_1 + p_2)e_1 + (sq_1 + bq_2)e_2 \geq 1.$$
Therefore we have
\[
EL_e(a, b; \mathbb{R}^2)^{-1} = \frac{1}{2e_1^2 + (s + t)e_2^2}
\]
and the optimal solution is given by
\[
p_1 = p_2 = \frac{e_1 \lambda}{2}, \quad q_1 = q_2 = \frac{e_2 \lambda}{2} \quad \text{with} \quad \lambda := \frac{2}{2e_1^2 + (s + t)e_2^2}.
\]
We have
\[
d_e(a, b; \mathbb{R}^2) - EL_e(a, b; \mathbb{R}^2)^{-1} = \frac{e_1^2 e_2^2 (c - 2)^2}{2c(2e_1^2 + ce_2^2)} \geq 0 \quad \text{with} \quad c = s + t.
\]
Thus the equality holds only if \(c = 2\) or \(e_1 = 0\) or \(e_2 = 0\).

5 Extremal width

Let \(a\) and \(b\) be distinct two nodes. Denote by \(Q_{a,b}\) the set of all cuts between \(a\) and \(b\) (cf. [2]). For \(Q \in Q_{a,b}\), there exist two disjoint subsets \(Q(a)\) and \(Q(b)\) of \(X\) such that \(a \in Q(a), \ b \in Q(b), \ X = Q(a) \cup Q(b)\) and \(Q = Q(a) \ominus Q(b)\). The index function \(u_Q \in L(X; \mathcal{H})\) of \(Q\) is defined by
\[
. := \sum \varepsilon_{Q(a)} e = \sum_{z \in Q(a)} \varepsilon_{z} e.
\]
The characteristic function \(s_Q\) of \(Q\) is defined by
\[
s_Q := \delta u_Q e = \sum_{z \in Q(a)} \delta \varepsilon_{z} e.
\]
Notice that \(|\delta \varepsilon_{Q(a)}(y)| = 1\) if \(y \in Q\) and \(\delta \varepsilon_{Q(a)}(y) = 0\) otherwise. Observe that \(|s_Q(y)| = 1\) if \(y \in Q\) and \(|s_Q(y)| = 0\) otherwise.

The extremal width \(EW(a, b; \mathcal{H})\) is defined by the inverse of the value of the extremum problem \((EW)\):
\[
\text{Minimize } H(w) \quad \text{subject to } \quad w \in L(Y; \mathcal{H}), \quad \sum_{y \in Q} \|w(y)\| \geq 1 \quad \text{for all } Q \in Q_{a,b}.
\]
The extremal width \(EW_e(a, b; \mathcal{H})\) is defined by the inverse of the value of the extremum problem \((EW_e)\):
\[
\text{Minimize } H(w) \quad \text{subject to } \quad w \in L(Y; \mathcal{H}),
\]
\[ \sum_{y \in Q} |(w(y), e)| \geq 1 \quad \text{for all } Q \in Q_{a,b}. \]

Since \(|(w(y), e)| \leq \|w(y)\| \|e\| = \|w(y)\|\), we have

(5.1) \quad EW(a, b; \mathcal{H}) \geq EW_e(a, b; \mathcal{H})

**Lemma 5.1** \(d_e^*(a, b; \mathcal{H}) \geq EW_e(a, b; \mathcal{H})^{-1}\).

**Proof.** Let \( w \) be any feasible solution for \( d_e^*(a, b; \mathcal{H}) \) and let \( Q \in Q_{a,b} \). Then

\[
1 = I_e(w) = - (\partial w(a), e) = - \sum_{x \in X} (\partial w(x), \varepsilon_Q(x)e) = - \sum_{y \in Y} (w(y), \delta \varepsilon_Q(y)e) \leq \sum_{y \in Q} |(w(y), e)|
\]

Therefore \( EW_e(a, b; \mathcal{H})^{-1} \leq H(w) \), so that \( EW_e(a, b; \mathcal{H})^{-1} \leq d_e^*(a, b; \mathcal{H}) \). \( \square \)

**Lemma 5.2** Let \( w \) be a feasible solution for the problem \((EW_e)\). Then

\[
d_e^*(a, b; \mathcal{H}) \leq \sum_{y \in Y} (r(y)w(y), w(y))(r(y)e, e)(r(y)^{-1}e, e).
\]

**Proof.** Put \( V(y) := |(w(y), e)| \). Then

\[
\sum_{y \in Q} V(y) \geq 1 \quad \text{for all } Q \in Q_{a,b}.
\]

By the duality between the max-flow problem and the min-cut problem, there exists \( \varphi \in L(Y; \mathbb{R}) \) such that \( |\varphi(y)| \leq V(y) \) on \( Y \),

\[
\partial \varphi(x) = 0 \quad \text{for } x \in X \setminus \{a, b\} \quad \text{and} \quad - \partial \varphi(a) = \partial \varphi(b) = 1.
\]

Let \( w(y) := \varphi(y)e \). Then \( w \in F(a, b; \mathcal{H}) \) and \( I_e(w) = 1 \). Therefore

\[
d_e^*(a, b; \mathcal{H}) \leq H(w) = \sum_{y \in Y} (r(y)\varphi(y)e, \varphi(y)e) = \sum_{y \in Y} |\varphi(y)|^2 (r(y)e, e) \leq \sum_{y \in Y} |(w(y), e)|^2 (r(y)e, e) \leq \sum_{y \in Y} (r(y)w(y), w(y))(r(y)^{-1}e, e)(r(y)e, e). \quad \square
\]

**Theorem 5.1** Let \( M(r) := \sup \{ (r(y)e, e)(r(y)^{-1}e, e); y \in Y \} \). Then

\[
EW_e(a, b; \mathcal{H})^{-1} \leq d_e^*(a, b; \mathcal{H}) \leq M(r) EW_e(a, b; \mathcal{H})^{-1}.
\]
Corollary 5.1 Assume that $\left( r(y) e, e \right)(r(y)^{-1} e, e) = 1$ for all $y \in Y$. Then $d^*_e(a, b; H) = EW_e(a, b; H)^{-1}$.

We recall the example in Section 4 and calculate $EW_e(a, b; H)^{-1}$ and $d^*_e(a, b; H)$ in this case. If $w \in F(a, b; R^2)$, then $w(y_1) = w(y_2) = (p, q)^T$ and

$$H(w) = 2p^2 + (s + t)q^2, \quad I_e(w) = pe_1 + qe_2.$$ 

By a simple calculus, we see easily that

$$d^*_e(a, b; R^2) = \frac{1}{e_1^2/2 + e_2^2/(s+t)}.$$ 

On the other hand, if $w$ is feasible for $EW_e(a, b; R^2)^{-1}$, then we have

(*)

$$p_1e_1 + q_1e_1 \leq 1, \quad p_2e_1 + q_2e_1 \leq 1$$

with $w(y_1) = (p_1, q_1)^T$, $w(y_2) = (p_2, q_2)$. Minimizing $H(w)$ subject to the condition (*), we have

$$EW_e(a, b; R^2)^{-1} = \frac{s}{se_1^2 + e_2^2} + \frac{t}{te_1^2 + e_2^2}.$$ 

Therefore

$$d^*_e(a, b; R^2) - EW_e(a, b; R^2)^{-1} = \frac{(s - t)e_1^2e_2^2}{[(s + t)e_1^2 + 2e_2^2](te_1^2 + e_2^2)(se_1^2 + e_2^2)} \geq 0$$

and the equality holds if $s = t$ or $e_1 = 0$ or $e_2 = 0$.

References

