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Kyoto University
On solutions of quasi-linear partial differential equations

$-\text{div} \mathbf{A}(x, \nabla u) + \mathbf{B}(x, u) = 0$

§0. Introduction

Recently, a nonlinear potential theory has been developed in [1] for quasi-linear elliptic partial differential equations of second order of the form

$-\text{div} \mathbf{A}(x, \nabla u) = 0,$

where \( \mathbf{A} \) is a mapping of \( R^n \times R^n \) to \( R^n \) \((n \geq 2)\) satisfying a growth condition \( \mathbf{A}(x, h) \cdot h \approx w(x)|h|^p \) \((1 < p < \infty)\) with a "weight" \( w(x) \), which is a nonnegative locally integrable function in \( R^n \). A prototype is the so-called weighted \( p \)-Laplace equations

$-\text{div} (w(X)|\nabla u|^{p-2}\nabla u) = 0,$

This purpose of this paper is to extend some of the results in [1] to the equation

(*)

$-\text{div} \mathbf{A}(x, \nabla u) + \mathbf{B}(x, u) = 0,$

where \( \mathbf{B}(x, t) \) is a mapping of \( R^n \times R \) to \( R \), which is non-decreasing in \( t \). A prototype equation may be given by

$-\text{div}(w(x)|\nabla u|^{p-2}\nabla u) + w(x)|u|^{p-2}u = 0.$

As a matter of fact, we treat the following three topics: (i) Existence and uniqueness of solutions of Dirichlet problems for equation (*) with Sobolev boundary values, or more generally of obstacle problems (section 3); (ii) Harnack inequality and Hölder continuity for solutions of (*) (section 4); (iii) Regularity at the boundary for solutions of (*) (section 5).

We can discuss (i) in the same way as in [1, Appendix I], using a general result of monotone operators. For (ii) and (iii), the methods in [1] are no longer applicable. We follow the discussion in [2] (for (ii)) and those in [4] (for (iii)), in which the unweighted case, namely the case \( w = 1 \), is treated.

§1. Weighted Sobolev space

We recall the weighted Sobolev spaces \( H^{1,p}(\Omega; \mu) \) which are adopted in [1].

Throughout this paper \( \Omega \) will denote an open subset of \( R^n \) \((n \geq 2)\) and \( 1 < p < \infty \). We denote \( B(x, r) = \{ y \in R^n : |x - y| < r \} \), and \( \lambda B = B(x, \lambda r) \) if \( B = B(x, r) \) and \( \lambda > 0 \).

Let \( w \) be a locally integrable, nonnegative function in \( R^n \). Then a Radon measure \( \mu \) is canonically associated with the weight \( w \):

\[
\mu(E) = \int_E w(x) \text{d}x.
\]
Thus \( d\mu(x) = w(x)dx \), where \( dx \) is the \( n \)-dimensional Lebesgue measure. We say that \( w \) (or \( \mu \)) is \( p \)-admissible if the following four conditions are satisfied:

I. \( 0 < w < \infty \) almost everywhere in \( \mathbb{R}^n \) and the measure \( \mu \) is doubling, i.e., there is a constant \( C_I > 0 \) such that
\[
\mu(2B) \leq C_I \mu(B)
\]
whenever \( B \) is a ball in \( \mathbb{R}^n \).

II. If \( D \) is an open set and \( \varphi_i \in C^\infty_0(D) \) is a sequence of functions such that \( \int_D |\varphi_i|^p d\mu \to 0 \) and \( \int_D |\nabla \varphi_i - v|^p d\mu \to 0(i \to \infty) \), where \( v \) is a vector-valued measurable function in \( L^p(D; \mu; \mathbb{R}^n) \), then \( v = 0 \).

III. (Sobolev inequality) There are constants \( k > 1 \) and \( C_{III} > 0 \) such that
\[
\left( \frac{1}{\mu(B)} \int_B |\varphi|^{kp} d\mu \right)^{1/kp} \leq C_{III} \left( \frac{1}{\mu(B)} \int_B |\nabla \varphi|^p d\mu \right)^{1/p}
\]
whenever \( B = B(x_0, r) \) is a ball in \( \mathbb{R}^n \) and \( \varphi \in C_0^\infty(B) \).

IV. There is a constant \( C_{IV} > 0 \) such that
\[
\int_B |\varphi - \varphi_B|^p d\mu \leq C_{IV} r^p \int_B |\nabla \varphi|^p d\mu
\]
whenever \( B = B(x_0, r) \) is a ball in \( \mathbb{R}^n \) and \( \varphi \in C^\infty(B) \) is bounded. Here
\[
\varphi_B = \frac{1}{\mu(B)} \int_B \varphi d\mu.
\]

From now on, unless otherwise stated, we assume that \( \mu \) is a \( p \)-admissible measure and \( d\mu(x) = w(x)dx \).

In this paper, both condition IV and the following inequality are called the Poincaré inequality.

**Poincaré inequality** ([1, p.9])

If \( \Omega \) is bounded, then
\[
\int_{\Omega} |\varphi|^p d\mu \leq C_{III}^p (\text{diam } \Omega)^p \int_{\Omega} |\nabla \varphi|^p d\mu
\]
for \( \varphi \in C^\infty_0(\Omega) \).

Throughout this paper, let \( c_\mu \) denote constants depending on \( C_I, C_{II}, C_{III}, k \) and \( C_{IV} \).

For a \( \mu \)-measurable function \( f \) defined on an open set \( \Omega \), \( L^p \)-norm of \( f \) is defined by
\[
\|f\|_{p, \Omega} = \left( \int_{\Omega} |f|^p d\mu \right)^{1/p}.
\]

For a function \( \varphi \in C^\infty(\Omega) \) we let
\[
\|\varphi\|_{1,p, \Omega} = \left( \int_{\Omega} |\varphi|^p d\mu \right)^{1/p} + \left( \int_{\Omega} |\nabla \varphi|^p d\mu \right)^{1/p},
\]
where, we recall, $\nabla \varphi = (\partial_1 \varphi, \cdots, \partial_n \varphi)$ is the gradient of $\varphi$. The Sobolev space $H^{1,p}(\Omega; \mu)$ is defined to be the completion of

$$\{ \varphi \in C^\infty(\Omega) : \| \varphi \|_{1,p;\Omega} < \infty \}$$

with respect to norm $\| \cdot \|_{1,p;\Omega}$. In other words, a function $u$ is in $H^{1,p}(\Omega; \mu)$ if and only if $u$ is in $L^p(\Omega; \mu)$ and there is a vector-valued function $v$ in $L^p(\Omega; \mu; \mathbb{R}^n)$ such that for some sequence $\varphi_i \in C^\infty(\Omega)$

$$\int_\Omega |\varphi_i - u|^p d\mu \to 0$$

and

$$\int_\Omega |\nabla \varphi_i - v|^p d\mu \to 0$$
as $i \to \infty$. The function $v$ is called the gradient of $u$ in $H^{1,p}(\Omega; \mu)$ and denoted by $\nabla u$.

The space $H^1_{0}(\Omega; \mu)$ is the closure of $C_0^\infty(\Omega)$ in $H^{1,p}(\Omega; \mu)$. The corresponding local space $H^{1,p}_{loc}(\Omega; \mu)$ is defined in the obvious manner.

§ 2. Quasilinear PDE’s

$A$ is a mapping of $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$ satisfying the following assumptions for some constants $0 < \alpha_1 \leq \alpha_2 < \infty$:

(a1) the mapping $x \mapsto A(x, h)$ is measurable for all $h \in \mathbb{R}^n$ and

the mapping $h \mapsto A(x, h)$ is continuous for a.e. $x \in \mathbb{R}^n$;

for all $h \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$

(a2) $A(x, h) \cdot h \geq \alpha_1 w(x)|h|^p$,

(a3) $|A(x, h)| \leq \alpha_2 w(x)|h|^{p-1}$,

(a4) $(A(x, h_1) - A(x, h_2)) \cdot (h_1 - h_2) > 0$

whenever $h_1, h_2 \in \mathbb{R}^n$, $h_1 \neq h_2$.

$B$ is a mapping of $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}$ satisfying the following assumptions for a constant $0 < \alpha_3 < \infty$:

(b1) the mapping $x \mapsto B(x, t)$ is measurable for all $t \in \mathbb{R}$ and

the mapping $t \mapsto B(x, t)$ is continuous for a.e. $x \in \mathbb{R}^n$;

for all $t \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^n$

(b2) $|B(x, t)| \leq \alpha_3 w(x)(|t|^{p-1} + 1)$,

(b3) $(B(x, t_1) - B(x, t_2))(t_1 - t_2) \geq 0$

whenever $t_1, t_2 \in \mathbb{R}^n$. Using $A$ and $B$ we consider the quasilinear elliptic equation

(2) $-\text{div} A(x, \nabla u) + B(x, u) = 0.$
A function $u \in H_{loc}^{1,p}(\Omega; \mu)$ is a (weak) solution of (2) if
\begin{equation}
\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} B(x, u) \varphi dx = 0
\end{equation}
whenever $\varphi \in C_0^\infty(\Omega)$. A function $u \in H_{loc}^{1,p}(\Omega; \mu)$ is a supersolution of (2) in $\Omega$ if
\[-\text{div}A(x, \nabla u) + B(x, u) \geq 0\]
weakly in $\Omega$, i.e.
\begin{equation}
\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} B(x, u) \varphi dx \geq 0
\end{equation}
whenever $\varphi \in C_0^\infty(\Omega)$ is nonnegative. A function $u \in H_{loc}^{1,p}(\Omega; \mu)$ is a subsolution in $\Omega$ if (4) holds for all nonpositive $\varphi \in C_0^\infty(\Omega)$.

**Lemma 2.1** If $u \in H^{1,p}(\Omega; \mu)$ is a solution (respectively, a supersolution) of (2) in $\Omega$, then
\begin{equation}
\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} B(x, u) \varphi dx = 0 \quad (\text{respectively,} \geq 0)
\end{equation}
for all $\varphi \in H_{0}^{1,p}(\Omega; \mu)$ (respectively, for all nonnegative $\varphi \in H_{0}^{1,p}(\Omega; \mu)$) with compact support.

**Proof**: Let $\Omega'$ be an open set such that $\text{spt}\varphi \subset \Omega' \subset \subset \Omega$. Since $\varphi \in H_{0}^{1,p}(\Omega'; \mu)$, we can choose a sequence of functions $\varphi_i \in C_0^\infty(\Omega')$ such that $\varphi_i \rightarrow \varphi$ in $H_{0}^{1,p}(\Omega'; \mu)$. If $\varphi$ is nonnegative, pick nonnegative functions $\varphi_i$ ([1, Lemma 1.23, p.21]). Then by (a3)
\[
\left| \int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} B(x, u) \varphi dx - \int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi_i dx + \int_{\Omega} B(x, u) \varphi_i dx \right|
\leq \alpha_2 \int_{\Omega'} |\nabla u|^{p-1} |\varphi - \nabla \varphi_i| d\mu + \alpha_3 \int_{\Omega'} (|\varphi - \varphi_i|^p + 1)|\varphi - \varphi_i| d\mu
\leq \alpha_2 \left( \int_{\Omega'} |\nabla u|^{p} d\mu \right)^{(p-1)/p} \left( \int_{\Omega'} |\nabla \varphi - \nabla \varphi_i|^p d\mu \right)^{1/p}
+ 2\alpha_3 \left( \int_{\Omega'} (|\varphi - \varphi_i|^p + 1) d\mu \right)^{(p-1)/p} \left( \int_{\Omega'} |\varphi - \varphi_i|^p d\mu \right)^{1/p}.
\]
Because the last integral tends to zero as $i \rightarrow 0$, we have
\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} B(x, u) \varphi dx = \lim_{i \rightarrow \infty} \left( \int_{\Omega'} A(x, \nabla u) \cdot \nabla \varphi_i dx + \int_{\Omega'} B(x, u) \varphi_i dx \right) = (\geq) 0,
\]
and the lemma follows. $\square$

The proof of Lemma 2.1 implies that (5) holds for all (nonnegative) $\varphi \in H_{0}^{1,p}(\Omega; \mu)$ if $\Omega$ is bounded.

A function $u$ is a solution of (2) if and only if $u$ is a supersolution and a subsolution. Indeed, if $u$ is a supersolution and a subsolution of (2), since the positive part $\varphi^+$ of a test function $\varphi \in C_0^\infty(\Omega)$, belongs $H_{0}^{1,p}(\Omega; \mu)$ and has compact support, $u$ satisfies (3) for $\varphi^+$. Similarly, $u$ satisfies (3) for the negative part of $\varphi$. Hence $u$ is a solution of (2).
§3. The existence of solutions

In this section, the existence of solutions of Dirichlet problems for equation (2) with Sobolev boundary values will be proved, using a general result in the theory of monotone operators.

Let $X$ be a reflexive Banach space with dual $X'$ and let $\langle \cdot, \cdot \rangle$ denote a pairing between $X'$ and $X$. If $K \subset X$ is a closed convex set, then a mapping $\mathcal{S} : K \to X'$ is called monotone if

$$\langle \mathcal{S}u - \mathcal{S}v , u - v \rangle \geq 0$$

for all $u, v$ in $K$. Further, $\mathcal{S}$ is called coercive on $K$ if there exists $\varphi \in K$ such that

$$\frac{\langle \mathcal{S}u_j - \mathcal{S}\varphi , u_j - \varphi \rangle}{\|u_j - \varphi\|} \to \infty$$

whenever $u_j$ is a sequence in $K$ with $\|u_j\| \to \infty$.

We recall the following proposition. ([3, Corollary III.1.8, p.87]).

**Proposition 3.1** Let $K$ be a nonempty closed convex subset of $X$ and let $\mathcal{S} : K \to X'$ be monotone, coercive, and weakly continuous on $K$. Then there exists an element $u$ in $K$ such that

$$\langle \mathcal{S}u , v - u \rangle \geq 0$$

whenever $v \in K$.

Throughout this section, we assume that $\Omega$ is bounded.

Suppose that $\psi$ is any function in $\Omega$ with values in the extended reals $[-\infty, \infty]$, and that $\theta \in H^{1,p}(\Omega; \mu)$. Let

$$\mathcal{K}_{\psi, \theta} = \mathcal{K}_{\psi, \theta}(\Omega) = \{ v \in H^{1,p}(\Omega; \mu) : v \geq \psi \text{ a.e in } \Omega, v - \theta \in H^{1,p}_0(\Omega; \mu) \}.$$

Set $X = L^p(\Omega; \mu; \mathbb{R}^n) \times L^p(\Omega; \mu; \mathbb{R})$ and $K = \{ (\nabla v, v) : v \in \mathcal{K}_{\psi, \theta}(\Omega) \}$.

**Lemma 3.2** $K$ is a closed convex set in $X$.

**Proof:** $K$ is clearly convex. To show the closedness, let $(\nabla v_i, v_i) \in K$ be a sequence converging to $(f, u)$ in $X$. By $\nabla v_i \to f$ in $L^p(\Omega; \mu; \mathbb{R}^n)$ and $v_i \to u$ in $L^p(\Omega; \mu; \mathbb{R})$, $v_i$ is a bounded sequence in $H^{1,p}(\Omega; \mu)$. Since $\mathcal{K}_{\psi, \theta}$ is a convex and closed subset of $H^{1,p}(\Omega; \mu)$, there is a function $v \in \mathcal{K}_{\psi, \theta}$ such that $v = u$ and $\nabla v = f$ ([1, Theorem 1.31, p.25]). Thus $(f, u) \in K$. The lemma is proved. $\square$

Let $\langle \cdot, \cdot \rangle$ be the pairing between $X$ and $X'$,

$$\langle (f, u), (g, v) \rangle = \int_{\Omega} f \cdot g d\mu + \int_{\Omega} uv d\mu,$$

where $(f, u)$ is in $X$ and $(g, v)$ in $X' = L^{p/(p-1)}(\Omega; \mathbb{R}^n) \times L^{p/(p-1)}(\Omega; \mu; \mathbb{R})$.

A mapping $\mathcal{S} : K \to X'$ is well defined by the formula

$$\langle \mathcal{S}(\nabla v, f), (g, u) \rangle = \int_{\Omega} A(x, \nabla v(x)) \cdot f(x) dx + \int_{\Omega} B(x, v(x)) u(x) dx$$

where

$$\langle (f, u), (g, v) \rangle = \int_{\Omega} f \cdot g d\mu + \int_{\Omega} uv d\mu.$$
for $(f, u) \in X$; indeed, by (a3) and (b2),

\[
| \int_{\Omega} A(x, \nabla v) \cdot f dx | \leq \alpha_2 \left( \int_{\Omega} |\nabla v|^p d\mu \right)^{(p-1)/p} \left( \int_{\Omega} |f|^p d\mu \right)^{1/p}
\]

\[
| \int_{\Omega} B(x, v) u dx | \leq 2\alpha_3 \left( \int_{\Omega} (|v| + 1)^p d\mu \right)^{(p-1)/p} \left( \int_{\Omega} |u|^p d\mu \right)^{1/p}
\]

**Lemma 3.3** $\mathfrak{S}$ is monotone, coercive, and weakly continuous on $K$.

Proof: By (a4) and (b3), $\mathfrak{S}$ is monotone.

Next we show that $\mathfrak{S}$ is coercive on $K$. Fix $(\nabla \varphi, \varphi) \in K$. Hereafter, for simplicity, we shall write $\| \cdot \|$ for $\| \cdot \|_{\Omega}$. By (a2), (a3) and (b3)

\[
\langle \mathfrak{S}(\nabla u) - \mathfrak{S}(\nabla \varphi), (\nabla u, u) - (\nabla \varphi, \varphi) \rangle
\]

\[
= \int_{\Omega} (A(x, \nabla u) - A(x, \nabla \varphi)) \cdot (\nabla u - \nabla \varphi) dx + \int_{\Omega} (B(x, u) - B(x, \varphi))(u - \varphi) dx
\]

\[
(6) \geq \alpha_1 (\| \nabla u \|^p + \| \nabla \varphi \|^p) - \alpha_2 (\| \nabla u \|^p \| \nabla \varphi \|^p + \| \nabla u \|^p \| \nabla \varphi \|^p - 1)
\]

\[
\geq \| \nabla u - \nabla \varphi \|^p (\| \nabla \varphi \|^p - 1) - \alpha_2 \| \nabla \varphi \|^p - 1 (\| \nabla \varphi \| + \| \nabla u - \nabla \varphi \|).
\]

Since $u - \varphi \in H_0^{1,p}(\Omega; \mu)$,

\[
(7) \quad \| u - \varphi \| \leq c \| \nabla u - \nabla \varphi \|.
\]

By (6) and (7), $\mathfrak{S}$ is coercive on $K$.

Finally, to show that $\mathfrak{S}$ is weakly continuous on $K$, let $(\nabla u_i, u_i) \in K$ be a sequence that converges to an element $(\nabla u, u) \in K$ in $X$. For any subsequence $(\nabla u_{i_j}, u_{i_j})$ of $(\nabla u_i, u_i)$, there is a subsequence $(\nabla u_{i_j}', u_{i_j}')$ of $(\nabla u_{i_j}, u_{i_j})$ such that $(\nabla u_{i_j}', u_{i_j}') \to (\nabla u, u)$ a.e. in $\Omega$.

By (a1) and (b1), we have

\[
A(x, \nabla u_{i_j}(x))w^{-1/p}(x) \to A(x, \nabla u(x))w^{-1/p}(x)
\]

\[
B(x, u_{i_j}(x))w^{-1/p}(x) \to B(x, u(x))w^{-1/p}(x)
\]

a.e. in $\Omega$. Since

\[
\int_{\Omega} |A(x, \nabla u_{i_j})w^{-1/p}(x)|^p dx \leq \alpha_2^{p/(p-1)} \int_{\Omega} |\nabla u_{i_j}|^p d\mu
\]

\[
\int_{\Omega} |B(x, u_{i_j})w^{-1/p}(x)|^p dx \leq 2\alpha_3^{p/(p-1)} \int_{\Omega} (|u_{i_j}| + 1)^p d\mu
\]

$L^p/(p-1)(\Omega; dx)$-norms of $A(x, \nabla u_{i_j})w^{-1/p}$ and $B(x, u_{i_j})w^{-1/p}$ are uniformly bounded. Therefore

\[
A(x, \nabla u_{i_j})w^{-1/p} \to A(x, \nabla u)w^{-1/p}
\]

\[
B(x, u_{i_j})w^{-1/p} \to B(x, u)w^{-1/p}
\]

weakly in $L^p/(p-1)(\Omega; dx)$. Since the weak limit is independent of $(\nabla u_{i_j}, u_{i_j})$,

\[
A(x, \nabla u_{i_j})w^{-1/p} \to A(x, \nabla u)w^{-1/p}
\]

\[
B(x, u_{i_j})w^{-1/p} \to B(x, u)w^{-1/p}.
\]
weakly in $L^{p/(p-1)}(\Omega; dx)$. Hence we have for all $(f, g) \in X$ that

$$\langle \mathfrak{A}(\nabla u_i, u_i), (f, g) \rangle = \int_\Omega A(x, \nabla u_i) \cdot f dx + \int_\Omega B(x, u_i) g dx$$

$$= \int_\Omega A(x, \nabla u_i) w^{-1/p} \cdot f w^{1/p} dx + \int_\Omega B(x, u_i) w^{-1/p} g w^{1/p} dx$$

$$\rightarrow \int_\Omega A(x, \nabla u) w^{-1/p} \cdot f w^{1/p} dx + \int_\Omega B(x, u) w^{-1/p} g w^{1/p} dx$$

$$= \langle \mathfrak{A}(\nabla u, u), (f, g) \rangle.$$ 

Therefore the lemma follows. \qed

Now the following theorem follows form Proposition 3.1, Lemma 3.2 and Lemma 3.3.

**Theorem 3.4** Suppose that $\mathcal{K}_{\psi, \theta}(\Omega) \neq \emptyset$, then there is a function $u$ in $\mathcal{K}_{\psi, \theta}$ such that

$$\int_\Omega A(x, \nabla u) \cdot \nabla(v - u) dx + \int_\Omega B(x, u)(v - u) dx \geq 0$$

whenever $v \in \mathcal{K}_{\psi, \theta}$.

A function $u$ in $\mathcal{K}_{\psi, \theta}(\Omega)$ that satisfies (8) for all $v \in \mathcal{K}_{\psi, \theta}(\Omega)$ is called a solution to the obstacle problem in $\mathcal{K}_{\psi, \theta}(\Omega)$.

As a corollary to this theorem, we have the existence of solutions of Dirichlet problems with Sobolev boundary values.

**Corollary 3.5** Suppose that $\theta \in H^{1,p}(\Omega; \mu)$. Then, there is a function $u \in H^{1,p}(\Omega; \mu)$ with $u - \theta \in H_0^{1,p}(\Omega; \mu)$ such that

$$-\text{div} A(x, \nabla u) + B(x, u) = 0$$

weakly in $\Omega$, that is

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi dx + \int_\Omega B(x, u) \varphi dx = 0$$

whenever $\varphi \in H_0^{1,p}(\Omega; \mu)$.

Proof: Choose $\psi \equiv -\infty$. Let $u$ be the solution to the obstacle problem in $\mathcal{K}_{\psi, \theta}$ and $\varphi \in H_0^{1,p}(\Omega; \mu)$. Since $u + \varphi, u - \varphi \in \mathcal{K}_{\psi, \theta}$, we have

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi dx + \int_\Omega B(x, u) \varphi dx \geq 0$$

and

$$-\int_\Omega A(x, \nabla u) \cdot \nabla \varphi dx - \int_\Omega B(x, u) \varphi dx \geq 0.$$ 

Then

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi dx + \int_\Omega B(x, u) \varphi dx = 0.$$ 

Hence Corollary 3.5 follows. \qed
The uniqueness of solutions of Dirichlet problems for equation (2) and obstacle problems in $\mathcal{K}_{\psi,\theta}$ follows from the following comparison principle Lemma 3.6 and Lemma 3.7 respectively.

**Lemma 3.6** Let $u \in H^1(\Omega; \mu)$ be a supersolution and $v \in H^1(\Omega; \mu)$ a subsolution of (2) in $\Omega$. If $\eta = \min(u - v, 0) \in H_0^1(\Omega; \mu)$, then $u \geq v$ a.e. in $\Omega$.

**Proof:** By (a4) and (b3),

$$
\int_{\Omega} (A(x, \nabla v) - A(x, \nabla u)) \cdot \nabla \eta dx \leq - \int_{\{u < v\}} (A(x, \nabla v) - A(x, \nabla u)) \cdot (\nabla v - \nabla u) dx \leq 0,
$$

$$
\int_{\Omega} (B(x, v) - B(x, u)) \eta dx \leq - \int_{\{v \leq u\}} (B(x, v) - B(x, u))(v - u) dx \leq 0.
$$

From this we have

$$
0 \leq \int_{\Omega} A(x, \nabla v) \cdot \nabla \eta dx + \int_{\Omega} B(x, v) \eta dx - \left( \int_{\Omega} A(x, \nabla u) \cdot \nabla \eta dx + \int_{\Omega} B(x, u) \eta dx \right) \leq 0.
$$

and, hence

$$
\int_{\Omega} (A(x, \nabla v) - A(x, \nabla u)) \cdot \nabla \eta dx = 0
$$

and

$$
\int_{\Omega} (B(x, v) - B(x, u)) \eta dx = 0.
$$

Therefore $\nabla \eta = 0$ a.e. in $\Omega$. Because $\eta \in H_0^1(\Omega; \mu)$, $\eta = 0$ a.e. in $\Omega$ ([1, Lemma 1.17, p.18]). The lemma follows. \qed

**Lemma 3.7** Suppose that $u$ is a solution to the obstacle problem in $\mathcal{K}_{\psi,\theta}(\Omega)$. If $v \in H^1(\Omega; \mu)$ is a supersolution of (2) in $\Omega$ such that $\min(u, v) \in \mathcal{K}_{\psi,\theta}(\Omega)$, then $v \geq u$ a.e. in $\Omega$.

**Proof:** Since $u - \min(u, v) \in H_0^1(\Omega; \mu)$ and is nonnegative, the lemma is proved in the same manner as in the proof of Lemma 3.6. \qed

**§4. The local behavior of solutions**

In this section, we study the local behavior of solutions of (2).

The next theorem can be shown in the same manner as [2, Theorem 1].

**Theorem 4.1** Each solution of (2) in $\Omega$ is locally bounded.

We obtain, using the Moser iteration technique, the following Harnack inequality.

Let $B(R)$ denote an open ball of radius $R$.

**Theorem 4.2** Let $u$ be a nonnegative solution of equation (2) in $\Omega$. Given $R_0 > 0$ there is a constant $c > 0$ such that

$$
\text{ess sup}_{B(R)} u \leq c \text{ ess inf}_{B(R)} (u + R)
$$

whenever $B(R)$ is a ball in $\Omega$ such that $3B(R) \subset \Omega$ and $R \leq R_0$. Here $c$ depends only on $n, p, \alpha_1, \alpha_2, \alpha_3, c_\mu$ and $R_0$. 

We require some lemmas to prove Theorem 4.2.

**Lemma 4.3** ([2, Lemma 2, p.252]) Let $a$ be a positive exponent, and let $a_i$, $b_i$ ($i = 1, \cdots, N$), be two sets of $N$ real numbers such that $0 < a_i < \infty$ and $0 \leq b_i < a$. Suppose that $z$ is a positive number satisfying

\[ z^a \leq \sum a_i z^{b_i}. \]

Then

\[ z \leq c \sum (a_i)^{\gamma_i} \]

where $c$ depends only on $N$, $a$, and $b_i$, and where $\gamma_i = (a - b_i)^{-1}$.

**Lemma 4.4** (John-Nirenberg lemma) ([1, Appendix II]) Suppose that $v$ is a locally $\mu$-integrable function in $\Omega$ with

\[ \sup_{\mu(B)} \frac{1}{B} \left| \int_{B} v - v_B \right| d\mu \leq c_0, \]

where

\[ v_B = \frac{1}{\mu(B)} \int_{B} v d\mu \]

and the supremum is taken over all balls $B \subset \subset \Omega$. Then there are positive constants $c_1$ and $c_2$ depending on $c_0$, $n$, and $\alpha_\mu$ such that

\[ \sup_{\mu(B)} \frac{1}{B} \left| \int_{B} e^{c_1|v-v_B|} \right| d\mu \leq c_2, \]

where the supremum is taken over all balls $B \subset \subset \Omega$.

Let $u$ be a nonnegative solution of equation (2) in $\Omega$ and $B = B(R)$ is a ball in $\Omega$. We set $\overline{u} = u + R$. Thus, by Theorem 4.1, if $\eta \in C_0^\infty(B)$ is nonnegative, then $\varphi(x) = \eta^p \overline{u}^{1-p} \in H_0^{1,p}(B; \mu)$ for any real value of $\beta$. Moreover,

\[ |B(x, u)| \leq 2 \alpha_3 w \max(1, 1/R^{p-1}) \overline{u}^{p-1}. \]

We set $\alpha'_3 = 2 \alpha_3 \max(1, 1/R^{p-1})$.

Next lemma guarantees that $v = \log \overline{u}$ satisfies the hypothesis of John-Nirenberg lemma.

**Lemma 4.5** Suppose that $u$ is a nonnegative solution of equation (2) in $\Omega$ and $B = B(R)$ is a ball in $\Omega$ such $3B \subset \Omega$. Then there is a constant $c > 0$ such that

\[ \int_{B_1} |v - v_{B_1}| d\mu \leq c\mu(B_1) \quad (v = \log \overline{u}), \]

whenever $B_1$ is a ball with $B_1 \subset 2B$. Here $c$ depends on $p$, $\alpha_1$, $\alpha_2$, $\alpha'_3 R^p$ and $\alpha_\mu$.

Proof: Setting $\varphi = \eta^p \overline{u}^{1-p}$, we have

\[ 0 = \int_{3B} A(x, \nabla u) \cdot \nabla \varphi dx + \int_{3B} B(x, u) \varphi dx \]
\[
\int_{3B} A(x, \nabla u) \cdot \{p(\eta/\overline{u})^{p-1}\nabla \eta + (1-p)(\eta/\overline{u})^{p-1}\nabla \eta\}dx + \int_{3B} B(x, u)\eta^{p} \overline{u}^{p-1}d\mu \\
\leq -\alpha_1(p-1) \int_{3B} (\eta/\overline{u})^{p} |\nabla u|^{p}d\mu + \alpha_2 p \int_{3B} |\nabla \eta| |\eta \nabla v|^{p-1}d\mu \\
+ \alpha_3' \int_{3B} \eta^{p-1} |\eta|^{p} \overline{u}^{p-1}d\mu
\]
\[
= -\alpha_1(p-1) \int_{3B} \eta \nabla v|^{p}d\mu + \alpha_2 p \int_{3B} |\nabla \eta| \eta \nabla v|^{p-1}d\mu + \alpha_3' \int_{3B} \eta^{p} d\mu,
\]
where \(v = \log \overline{u}\). Hence

\[
(9) \quad \alpha_1(p-1) \eta |\nabla v|^{p} \leq \alpha_2 p \int_{3B} |\nabla \eta| |\eta \nabla v|^{p-1}d\mu + \alpha_3' \int_{3B} \eta^{p} d\mu.
\]

Let \(B_1 \subset 2B\) be any open ball of radius \(h\). Let \(\eta\) be so chosen that \(\eta = 1\) in \(B_1\), \(0 \leq \eta \leq 1\) in \(3B \setminus B_1\), the support of \(\eta\) is contained in \((3/2)B_1\), and \(|\nabla \eta| \leq 3/h\). Then by Hölder’s inequality we obtain

\[
\int_{3B} |\nabla \eta| |\eta \nabla v|^{p-1}d\mu \leq \left( \int_{(3/2)B_1} |\nabla \eta|^{p}d\mu \right)^{1/p} \left( \int_{(3/2)B_1} |\eta \nabla v|^{p}d\mu \right)^{(p-1)/p}
\]
\[
\leq \frac{3}{h} \mu((3/2)B_1)^{1/p} \eta |\nabla v|^{p-1}_{p,3B}.
\]

By the above inequalities and (9) we have

\[
\alpha_1(p-1) \eta |\nabla v|^{p} \leq \frac{3\alpha_2 p}{h} \mu((3/2)B_1)^{1/p} |\nabla v|^{p-1}_{p,3B} + \frac{\alpha_3'(3R)^{p}}{h^{p}} \mu((3/2)B_1).
\]

Application of Lemma 4.3 yields,

\[
|\nabla v|_{p,B_1} \leq c h^{-1} \mu((3/2)B_1)^{1/p},
\]

where \(\eta = 1\) in \(B_1\) have been used. Finally by the the doubling property, Hölder’s inequality and Poincaré inequality we have

\[
\int_{B_1} |v - v_{B_1}| d\mu \leq c \mu((3/2)B_1)^{(p-1)/p} h \left( \int_{B_1} |\nabla v|^{p}d\mu \right)^{1/p} \leq c \mu(B_1) \quad (v = \log \overline{u}),
\]

where \(c = c(p, \alpha_1, \alpha_2, \alpha_3, R, \mu)\). \(\square\)

The following estimates will be used when we apply to the Moser iteration technique.

**Lemma 4.6** Suppose that \(u\) is a nonnegative solution of equation (2) in \(\Omega\) and \(B = B(R)\) is a ball in \(\Omega\). For \(\beta \neq 0, p-1\), let \(q\) satisfying \(pq = p + \beta - 1\) and \(v = \overline{u}^{q}\). Then there is a constant \(c > 0\) such that

(i) if \(\beta > 0\),

\[
\|\eta v\|_{k, B} \leq c(\mu(B))^{(1-k)/kp} R(1 + \beta^{-1})(1 + q)^{p}(\|v \nabla \eta\|_{p,B} + \|\eta v\|_{p,B}),
\]
(ii) if \(1 - p < \beta < 0\),

\[
\|\eta v\|_{k_p,B} \leq c\{\mu(B)\}^{(1-k)/kp} R(1 - \beta^{-1}) (\|v \nabla \eta\|_{p,B} + \|\eta v\|_{p,B}),
\]

(iii) if \(\beta < 1 - p\),

\[
\|\eta v\|_{k_p,B} \leq c\{\mu(B)\}^{(1-k)/kp} R(1 + |q|)^p (\|v \nabla \eta\|_{p,B} + \|\eta v\|_{p,B}),
\]

where \(c\) depends only on \(p, \alpha_1, \alpha_2, c_\mu\) and \(\alpha'_3 R^{p-1}\).

Proof: We prove only (i), the proofs of (ii) and (iii) being similar. For \(\varphi = \eta^p \bar{u}^\beta\), we have

\[
0 = \int_B A(x, \nabla u) \cdot \nabla \varphi \, dx + \int_B B(x, u) \varphi \, dx
\]

\[
= \int_B A(x, \nabla u) \cdot (p\eta^{p-1} \bar{u}^\beta \nabla \eta + \beta \eta^{p\beta-1} \nabla u) \, dx + \int_B B(x, u) \eta^p \bar{u}^\beta \, dx
\]

\[
\geq \alpha_1 \beta \int_B \eta^{p\beta-1} |\nabla u| \eta^{p-1} \nabla \eta \, d\mu - \alpha_2 \int_B |\nabla u|^p \eta^{p-1} \eta^p \bar{u}^\beta \, d\mu - \alpha_3' \int_B \eta^p \bar{u}^\beta \, d\mu.
\]

Since \(pq = p + \beta - 1\) and \(v = \bar{u}^q\),

\[
\frac{\alpha_1 \beta}{q^p} \|\eta \nabla v\|_p^p \leq \frac{p \alpha_2}{q^{p-1}} \int_B |v \nabla \eta| \|\eta \nabla v\|_p^{p-1} \, d\mu + \alpha_3' \int_B (\eta v)^p \, d\mu.
\]

Here for simplicity we have written \(\| \cdot \|_p\) for \(\| \cdot \|_{p,B}\).

By Hölder's inequality,

\[
\int_B |v \nabla \eta| \|\eta \nabla v\|_p^{p-1} \, d\mu \leq \|v \nabla \eta\|_p \|\eta \nabla v\|_p^{p-1},
\]

\[
\int_B (\eta v)^p \, d\mu = \|\eta v\|_p \left( \int_B (\eta v)^p \, d\mu \right)^{(p-1)/p}
\]

\[
\leq \|\eta v\|_p \left( \int_B (\eta v)^{kp} \, d\mu \right)^{1/k} \left( \int_B |\nabla \eta|^{(k-1)/k} \, d\mu \right)^{(p-1)/p}
\]

\[
= \mu(B)^{(k-1)(p-1)/(kp)} \|\eta v\|_p \|\eta \nabla v\|_{kp}^{p-1}
\]

\[
\leq c_\mu R^{p-1} \|\eta v\|_p (\|v \nabla \eta\|_p^{p-1} + \|\eta \nabla v\|_{p-1}^{p-1}),
\]

where we have used Sobolev inequality. By the above inequalities, if we set

\[
z = \frac{\|\eta \nabla v\|_p}{\|v \nabla \eta\|_p}, \quad \zeta = \frac{\|\eta v\|_p}{\|v \nabla \eta\|_p},
\]

then (10) can be written as

\[
\beta z^p \leq c \{qz^{p-1} + q^p \zeta (1 + z^{p-1})\},
\]

where \(c = c(p, \alpha_1, \alpha_2, \alpha'_3 R^{p-1}, c_\mu)\). Application of Lemma 4.3 yields

\[
z \leq c(1 + \beta^{-1})(1 + q)^p (1 + \zeta),
\]

that is,

\[
\|\eta \nabla v\|_p \leq c(1 + \beta^{-1})(1 + q)^p (\|v \nabla \eta\|_p + \|\eta v\|_p).
\]
Finally using Sobolev inequality again, from (11) we obtain the desired estimate. \hfill \square

Proof of Theorem 4.2: Set $v = \log \overline{u}$. By Lemma 4.4 and Lemma 4.5, there are positive constants $r_0$ and $c_0$ such that

$$
\left( \int_{B_1} e^{r_0 v} d\mu \right) \left( \int_{B_1} e^{-r_0 v} d\mu \right) = \left( \int_{B_1} e^{r_0 (v - v_{B_1})} d\mu \right) \left( \int_{B_1} e^{r_0 (v_{B_1} - v)} d\mu \right) \leq \left( \int_{B_1} e^{r_0 |v - v_{B_1}|} d\mu \right) \leq c_0^2 \{\mu(B_1)\}^2.
$$

Because $B_1$ is any ball contained in $2B$,

$$
\left( \int_{2B} e^{r_0 v} d\mu \right) \left( \int_{2B} e^{-r_0 v} d\mu \right) \leq c_0^2 \{\mu(2B)\}^2.
$$

Hence

$$
(12) \quad \left( \int_{2B} \overline{u}^{r_0} d\mu \right)^{1/r_0} \leq c \{\mu(B)\}^{(1-k)/kpR(h-h')^{-1}(1+|\beta|^{-1})(1+r)^p} \left( \int_{2B} \overline{u}^{-r_0} d\mu \right)^{-1/r_0}.
$$

Next, let $0 < h' < h \leq 3R$. Let the function $\eta \in C_0^\infty(B(h))$ be so chosen that $\eta = 1$ in $B(h')$, $0 \leq \eta \leq 1$ in $B(h)$ and $|\nabla \eta| \leq 3(h-h')^{-1}$. Then Lemma 4.6 yields

(i) if $\beta > 0$,

$$
(13) \quad ||\overline{u}||_{kp,B(h')} \leq c \{\mu(B)\}^{(1-k)/kpR(1+q)^p(h-h')^{-1}(1+\beta^{-1})||\overline{u}^q||_{p,B(h)}},
$$

(ii) if $1 - p < \beta < 0$,

$$
(14) \quad ||\overline{u}||_{kp,B(h')} \leq c \{\mu(B)\}^{(1-k)/kpR(h-h')^{-1}(1-\beta^{-1})||\overline{u}^q||_{p,B(h)}},
$$

(iii) if $\beta < 1 - p$,

$$
(15) \quad ||\overline{u}||_{kp,B(h')} \leq c \{\mu(B)\}^{(1-k)/kpR(h-h')^{-1}(1+|q|)^p||\overline{u}^q||_{p,B(h)}},
$$

where $c$ depends only on $p, \alpha_1, \alpha_2, c_\mu$ and $\alpha'_2 R^{-p-1}$.

Putting $r = pq = p + \beta - 1$ in (13) and (14), combining the result in a single inequality, we obtain

$$
(16) \quad \left( \int_{B(h')} \overline{u}^{kr} d\mu \right)^{1/kr} \leq \left\{ c \{\mu(3B)\}^{(1-k)/kpR(h-h')^{-1}(1+|\beta|^{-1})(1+r)^p} \right\}^{p/r} \times \left( \int_{B(h)} \overline{u}^r d\mu \right)^{1/r},
$$

for all $0 < r \neq p - 1$. Let $r_\nu = k^\nu r'_0 \quad \nu = 0, 1, 2, \ldots$,

and $h_\nu = R(1 + 2^{-\nu})$, $h'_\nu = h_{\nu+1}$, where $r'_0 \leq r_0$ is so chosen that $r_\nu \neq p - 1$ for any $\nu = 0, 1, 2, \ldots$. Thus

$$
|\beta| = |r - (p - 1)| \geq c > 0,
$$

whenever $r = r_\nu$, where $c$ depends only on $p, k, r_0$. The term $(1 + |\beta|^{-1})$ in (16) can thus be absorbed into the general constant $c$. Hence from(16) we have that

$$
\left( \int_{B(h'_\nu)} \overline{u}^{r_{\nu+1}} d\mu \right)^{1/r_{\nu+1}} \leq \left\{ c \{\mu(3B)\}^{(1-k)/kp2^{\nu+1}(1+r_\nu)^p} \right\}^{p/r_\nu} \left( \int_{B(h_\nu)} \overline{u}^{r_\nu} d\mu \right)^{1/r_\nu}.
$$
\[
= c^{1/k_r} \{ \mu(3B) \}^{(1-k)/kr} 2^{p/r_0} \{ (1 + r_0' k^{r_0})^p r_0' \}^{1/k_r} \int_{B(h_0)} \bar{u}^{r_0} d\mu \right)^1/r_0
\leq c^{1/k_r} c_2^{1/k_r} \{ \mu(3B) \}^{(1-k)/kr} \left( \int_{B(h_0)} \bar{u}^{r_0} d\mu \right)^1/r_0.
\]

By iterating, it follows that

\[(17) \quad \text{ess sup}_B \bar{u} \leq c \{ \mu(3B) \}^{-1/r_0} \left( \int_{2B} \bar{u}^{r_0} d\mu \right)^1/r_0.\]

Setting \( s = pq \) in (15), since \( s \) and \( q \) are negative, we obtain

\[
\left( \int_{B(h')} \bar{u}^{ks} d\mu \right)^{1/ks} \geq \left\{ c \{ \mu(3B) \}^{(1-k)/kp} R (h - h')^{-1} (1 + |s|)^p \right\}^{p/ks} \left( \int_{B(h)} \bar{u}^{s} d\mu \right)^{1/s}.
\]

Let \( s_\nu = -k^{\nu} r_0 \), \( h_\nu = R (1 + 2^{-\nu}) \) and \( h_\nu' = h_\nu + 1 \). Then

\[
\left( \int_{B(h_\nu')} \bar{u}^{s_{\nu+1}} d\mu \right)^{1/s_{\nu+1}} \geq c^{1/k_r} c_2^{1/k_r} \{ \mu(3B) \}^{(1-k)/kr} \left( \int_{B(h_\nu)} \bar{u}^{s_{\nu}} d\mu \right)^{1/s_{\nu}}.
\]

By iterating, we obtain

\[(18) \quad \text{ess inf}_B \bar{u} \geq c^{-1} \{ \mu(3B) \}^{1/ro} \left( \int_{2B} \bar{u}^{-ro} d\mu \right)^{-1/ro}.\]

Finally, by (12), (17), (18), and a simple application of Hölder's inequality, we have

\[
\text{ess sup}_B \bar{u} \leq c \{ \mu(3B) \}^{-1/r_0} \left( \int_{2B} \bar{u}^{r_0} d\mu \right)^{1/r_0} \leq c \{ \mu(3B) \}^{-1/ro} \left( \int_{2B} \bar{u}^{ro} d\mu \right)^{1/ro} \leq c \text{ ess inf}_B \bar{u}.
\]

Since \( \bar{u} = u + R \), this concludes the proof of Theorem 4.2. □

We apply Theorem 4.4 to show that any solutions of (2) has Hölder continuous representative.

**Theorem 4.7** Let \( u \) be a solution of (2) in \( \Omega \) and \( x_0 \) be any point of \( \Omega \). If \( 0 < R < \infty \) is such that \( B(x_0, R) \subset \Omega \) and if \( |u| \leq L \ a.e \ in \ B(x_0, R) \), then there are constants \( c \) and \( 0 < \lambda < 1 \) such that

\[
\text{ess sup}_{B(x_0, \rho)} u - \text{ess inf}_{B(x_0, \rho)} u \leq c \left( \frac{\rho}{R} \right) \lambda,
\]

whenever \( 0 < \rho < R \). Here \( c \) and \( \lambda \) depend only on \( n, p, \alpha, \alpha_2, \alpha_3, c_\mu, R \) and \( L \).

**Proof:** We write \( B(r) = B(x_0, r) \) and

\[
M(r) = \text{ess sup}_{B(r)} u, \quad m(r) = \text{ess inf}_{B(r)} u.
\]

Then \( M(r) \) and \( m(r) \) are well defined for \( 0 < r \leq R \), and

\[
\bar{u} = M(r) - u, \quad \bar{u} = u - m(r).
\]
are non-negative in $B(r)$. Obviously $\bar{u}$ is a solution of
\[-\text{div} \bar{A}(x, \nabla \bar{u}) + \bar{B}(x, \bar{u}) = 0\]
where $\bar{A}(x, \bar{h}) = -A(x, -\bar{h})$ and $\bar{B}(x, \bar{t}) = -B(x, M(r) - \bar{t})$. Thus
\[|\bar{B}(x, \bar{t})| \leq \alpha'_3 w(x)(|\bar{t}|^{p_1} + 1),\]
where $\alpha'_3$ is a constant depending only on $\alpha_3$, $p$ and $L$. By applying Harnack inequality to $\bar{u}$, we have
\[(19) \quad M(r) - m(r/3) = \text{ess \ sup}_{B(r/3)} \bar{u} \leq c(\text{ess \ inf}_{B(r/3)} \bar{u} + r) = c\{M(r) - M(r/3) + r\}.\]
Similarly we have
\[(20) \quad M(r/3) - m(r) = \text{ess \ sup}_{B(r/3)} \bar{u} \leq c(\text{ess \ inf}_{B(r/3)} \bar{u} + r) = c\{m(r/3) - m(r) + r\}.\]
Here $c > 1$ depends on $n, p, \alpha_1, \alpha_2, \alpha_3, c_{\mu}, R$ and $L$. By (19) and (20),
\[(21) \quad M(r/3) - m(r/3) \leq \frac{c-1}{c+1}\{M(r) - m(r)\} + \frac{2c}{c+1} r.\]
Thus setting
\[\theta = \frac{c-1}{c+1}, \quad \tau = \frac{2cR}{c-1}\]
and
\[\omega = M(r) - m(r),\]
(21) can be written as
\[\omega(r/3) \leq \theta\{\omega(r) + \tau(r/R)\}.\]
Since $\omega(r)$ is an increasing function, for any number $s \geq 3$ we have also
\[\omega(r/s) \leq \theta\{\omega(r) + \tau(r/R)\}, \quad 0 < r \leq R.\]
By iterating, we obtain
\[(22) \quad \omega(R/s^\nu) \leq \theta^\nu\{\omega(R) + \tau\{1 + (\theta s)^{-1} + \cdots + (\theta s)^{-\nu+1}\}\},\]
for $\nu = 1, 2, 3, \cdots$. Let $s$ be so chosen that $\theta s = 3$. Then (22) implies
\[(23) \quad \omega(R/s^\nu) \leq \theta^\nu\{\omega(R) + 2\tau\}.\]
For any $\rho$ such that $0 < \rho \leq R/s$ choose $\nu$ such that $R/s^{\nu+1} < \rho \leq R/s^\nu$. Then from (23) we have
\[(24) \quad \omega(\rho) \leq \omega(R/s^\nu) \leq \theta^\nu(\omega(R) + 2\tau).\]
If we set $\gamma = -\log_3 \theta$, then we have $\theta = s^{-\lambda}$ where $\lambda = \gamma/(\gamma+1) > 0$. Thus
\[\theta^\nu = \left(\frac{R}{s^{\nu+1}}\frac{s}{R}\right)^{\lambda s} \leq c\left(\frac{\rho}{R}\right)^{\lambda s}.\]
Hence, since $\omega(R) + 2\tau \leq c(L + R)$, (22) implies
\[
\omega(\rho) \leq c(L + R) \left( \frac{\rho}{R} \right)^{\lambda}, \quad (\rho < R),
\]
as desired. \[\square\]

§5. A regularity at the boundary for solutions
In this section, we are concerned with the continuity of solutions at the boundary. First, we recall the definition of the $(p, \mu)$-capacity which is adopted in [1]. Suppose that $K$ is a compact subset of $\Omega$. Let
\[
W(K, \Omega) = \{ u \in C_{0}^{\infty}(\Omega) : u \geq 1 \text{ on } K \}
\]
and define
\[
\text{cap}_{p,\mu}(K, \Omega) = \inf_{u \in W(K, \Omega)} \int_{\Omega} |\nabla u|^{p} d\mu.
\]
Further, if $U \subset \Omega$ is open, set
\[
\text{cap}_{p,\mu}(U, \Omega) = \sup_{K \subset U \text{ compact}} \text{cap}_{p,\mu}(K, \Omega),
\]
and, finally, for an arbitrary set $E \subset \Omega$
\[
\text{cap}_{p,\mu}(E, \Omega) = \inf_{E \subset \Omega} \sup_{U \text{ open}} \text{cap}_{p,\mu}(U, \Omega).
\]
The number $\text{cap}_{p,\mu}(E, \Omega) \in [0, \infty]$ is called the $(p, \mu)$-capacity of the condenser $(E, \Omega)$. If $u \in H_{loc}^{1,p}(\Omega; \mu), x_{0} \in \partial \Omega,$ and $l \in \mathbb{R}$ we say that
\[
(25) \quad u(x_{0}) \leq l \quad \text{weakly}
\]
if for every $k > l$ there is an $r > 0$ such that $\eta(u - k)^{+} \in H_{0}^{1,p}(\Omega; \mu)$ whenever $\eta \in C_{0}^{\infty}(B(x_{0}, r))$. The condition
\[
(26) \quad u(x_{0}) \geq l \quad \text{weakly}
\]
is defined analogously and $u(x_{0}) = l$ weakly if both (25) and (26) hold. Observe that if $f$ is a continuous function on $\mathbb{R}^{n} \setminus \Omega$, $f \in H_{loc}^{1,p}(\mathbb{R}^{n}; \mu)$, and $u - f \in H_{0}^{1,p}(\Omega; \mu)$, then $u(x) = f(x)$ weakly for every $x \in \partial \Omega$.

**Lemma 5.1** Suppose that $u \in H_{loc}^{1,p}(\Omega; \mu)$ is a subsolution of (2) in $\Omega$, that $u \leq L$ a.e. in $\Omega$, and that $u(x_{0}) \leq l$ weakly for $x_{0} \in \partial \Omega$. For $k > l$, let
\[
u_{k} = \begin{cases} (u - k)^{+} & \text{on } \Omega \\ 0 & \text{otherwise} \end{cases}
\]
and define
\[
M(r) = \text{ess sup}_{B(x_{0}, r)} \nu_{k}.
\]
Choose $r_{0} > 0$ so small that $\eta \nu_{k} \in H_{0}^{1,p}(\Omega; \mu)$ whenever $\eta \in C_{0}^{\infty}(B(x_{0}, r_{0}))$. 

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Then there is a constant $c$ depending only on $n, p, l, r_0, \alpha_1, \alpha_2, \alpha_3, c_\mu$ and $L$ such that

$$
\int_{B(x_0,r/2)} |\nabla(v^{-})|^p d\mu \leq c(M(r) + r)(M(r) - M(r/2) + r)^{p-1} \mu(B(x_0, r))r^{-p}
$$

where $0 < r \leq r_0/2$, $v^{-} = M(r) + r - u_k$ and $\eta \in C_0^\infty(B(x_0, r/2))$ with $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq 5/r$.

Before proving Lemma 5.1, we will state its implication.

**Theorem 5.2** Let $u \in H_{loc}^{1,p}(\Omega; \mu)$ be a subsolution of (2) which is bounded above on $\Omega$, $x_0 \in \partial\Omega$, and $u(x_0) \leq l$ weakly. If

$$
\int_0^1 \left( \frac{\text{cap}_{p,\mu}(B(x_0, t) \setminus \Omega, B(x_0, 2t))}{\text{cap}_{p,\mu}(B(x_0, t), B(x_0, 2t))} \right)^{1/(p-1)} dt = \infty,
$$

then

$$
\text{ess} \lim_{x \to x_0} \sup_{x \to x_0} u(x) \leq l.
$$

**Proof:** Since, for any $k > l$, it follows immediately from Theorem 5.1, the definition of $(p, \mu)$-capacity and [1, Lemma 2.14] that

$$
(M(r) + r)^{1/(p-1)} \left( \frac{\text{cap}_{p,\mu}(B(x_0, r/4) \cap \{u_k = 0\}, B(x_0, r/2))}{\text{cap}_{p,\mu}(B(x_0, r/4), B(x_0, r/2))} \right)^{1/(p-1)} 
\leq c(M(r) - M(r/2) + r),
$$

the theorem is proved in the same manner as in the proof of [4, Theorem 2.2]. □

If $u$ is a supersolution of (2), then $-u$ is a subsolution of

$$
-\text{div} \vec{A}(x, \nabla v) + \vec{B}(x, v) = 0,
$$

where $\vec{A}(x, h) = -\vec{A}(x, -h)$ and $\vec{B}(x, t) = -\vec{B}(x, -t)$. Consequently, Theorem 5.2 has the obvious counterpart for supersolutions of (2). These results yield

**Theorem 5.3** Let $u \in H_{loc}^{1,p}(\Omega; \mu)$ be a bounded solution of (2), that $x_0 \in \partial\Omega$, and that $u(x_0) = l$ weakly. If (27) holds, then

$$
\lim_{x \to x_0} u(x) = l.
$$

**Proof of Lemma 5.1:** Fix $r > 0$ so that $0 < r \leq r_0/2$, let $\eta \in C_0^\infty(B(x_0, r/2))$ with $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq 5/r$. Set

$$
I(r) = (M(r) + r)(M(r) - M(r/2) + r)^{p-1} \mu(B(x_0, r))r^{-p}.
$$
Since
\[ \int |\nabla(\eta v^{-1})|^p d\mu \leq c \left( \int \eta^p |\nabla u_k|^p d\mu + \int v^{-p} |\nabla \eta|^p d\mu \right), \]
we will show that
\[ \int \eta^p |\nabla u_k|^p d\mu \leq c I(r) \quad \text{and} \quad \int v^{-p} |\nabla \eta|^p d\mu \leq c I(r), \]
by using following two estimates.

**Estimate 1**  
For \((1-p)/p < \alpha \neq 0\)
\[
(c m(\alpha))^{-1} \int_{B(x_0, r)} |\nabla(\omega v^\alpha)|^p d\mu \leq \int_{B(x_0, r)} v^\alpha \{(\omega v)^p + |\nabla \omega|^p\} d\mu,
\]
whenever \(\omega \in C_0^\infty(B(x_0, r))\) with \(0 \leq \omega \leq 1\), where \(c\) is a constant depending on \(p, \alpha_1, \alpha_2, \alpha_3, l, r_0, L\), and
\[
0 < m(\alpha) < 1 + \alpha^p \quad \text{if} \quad \alpha > 0,
\]
\[
m(\alpha) > 0 \quad \text{and} \quad \text{a decreasing function of} \quad \alpha \quad \text{if} \quad (1-p)/p < \alpha < 0.
\]

**Estimate 2**  
For \(0 < \sigma < p-1\),
\[
\mu(B(x_0, r))^{-1} \|v^{-\sigma k}\|_{1, B(x_0, r/2)} \leq c(M(r) - M(r/2) + r)^{\sigma k},
\]
where \(c\) is a constant depending on \(p, n, \alpha_1, \alpha_2, \alpha_3, l, r_0, L\) and \(\sigma\).

Let us suppose that Estimate 1 and Estimate 2 are true. Fix \(\alpha < 0\) such that \(1 < (1+\alpha)p < k\), then putting \(B = B(x_0, r/2)\), we have
\[
\int_B \eta^{p-1} |\nabla u_k|^{p-1} |\nabla \eta| d\mu = \int_B (\eta v^{1+\alpha} |\nabla u_k|)^{p-1} (v^{-1+\alpha})^{p-1} |\nabla \eta| d\mu
\]
(28)
\[
= c \int_B (\eta |\nabla v|^\alpha)^{p-1} (v^{-1+\alpha})^{p-1} |\nabla \eta| d\mu
\]
\[
\leq c \left( \int_B (\eta |\nabla v|^\alpha)^p d\mu \right)^{(p-1)/p} \left( \int_B (v^{-1+\alpha})^{p-1} |\nabla \eta|^p d\mu \right)^{1/p}
\]
\[
\leq c \left\{ \left( \int_B |\nabla(\eta v^\alpha)|^p d\mu \right)^{1/p} + \left( \int_B |v^\alpha \nabla \eta|^p d\mu \right)^{1/p} \right\}^{p-1}
\]
\[
\times \left( \int_B (v^{-1+\alpha})^{p-1} |\nabla \eta|^p d\mu \right)^{1/p}
\]
\[
\leq c \left( r^{-p} \int_B v^\alpha d\mu \right)^{(p-1)/p} \left( \int_B (v^{-1+\alpha})^{p-1} |\nabla \eta|^p d\mu \right)^{1/p}
\]
\[
\leq c \left\{ (M(r) - M(r/2) + r)^{-\alpha p} \mu(B(x_0, r)) r^{-p} \right\}^{(p-1)/p}
\]
\[
\times \left\{ (M(r) - M(r/2) + r)^{(1+\alpha)p-1} \mu(B(x_0, r)) r^{-p} \right\}^{1/p}
\]
\[
= c (M(r) - M(r/2) + r)^{(p-1)} \mu(B(x_0, r)) r^{-p},
\]
in the last inequality we have used Estimate 2 with \(\sigma = -\alpha p/k\) and \(\sigma = (1+\alpha)(p-1)p/k\) respectively. Also since \(\eta \leq 1\),
\[
\int_B \eta^p d\mu \leq \mu(B(x_0, r)) \leq c I(r).
\]
Hence, by (28) and (29),

\[ (30) \quad \int_B \eta^p |\nabla u_k|^p d\mu \leq c \left( \int_B \eta^p d\mu + M(r) \int_B \eta^{p-1} |\nabla u_k|^{p-1} |\nabla \eta| d\mu \right) \leq c I(r). \]

Here the first inequality has been obtained by using the facts that \( \varphi = \eta^p u_k \in H_0^{1,p}(\Omega; \mu) \), \( \varphi \) is nonnegative, \( u \) is a subsolution and the structure of \( A \) and \( B \). From Estimate 2 with \( \sigma = (p - 1)/k \) again

\[ (31) \quad \int_B |v^{-1} \nabla \eta|^p d\mu \leq c r^{-p} (M(r) + r) \int_B v^{-p+1} d\mu \leq c I(r). \]

Therefore we obtain from (30) and (31)

\[ \int_B |\nabla (\eta v^{-1})|^p d\mu \leq c I(r). \]

Finally, we will prove Estimate 1 and Estimate 2. For \( \beta > 0 \), let

\[ \psi = v^\beta - (M(r) + r)^{-\beta} \]

and

\[ \varphi = \omega^p \psi, \]

where \( \omega \in C_0^\infty(B(x_0, r)) \). Then \( \varphi \in H_0^{1,p}(\Omega; \mu) \). Since \( \varphi = 0 \) on \( \{u_k = 0\} \) and \( \varphi \geq 0 \) on \( \Omega \),

\[ \int \beta \omega^p v^{\beta+1} A(x, \nabla u) \cdot \nabla u_k dX + \int p \omega^{p-1} \psi A(x, \nabla u) \cdot \nabla \omega dx + \int B(x, u) \varphi dX \leq 0, \]

where the integrals are taken over \( B(x_0, r) \cap \{u_k > 0\} \). Hereafter we will suppress explicit indication of this domain of integration.

Using (a2), (a3) and (b2) we have

\[ \alpha_1 \beta \int \omega^p v^{\beta+1} |\nabla u_k|^p d\mu \leq p \alpha_2 \int \omega^{p-1} \psi |\nabla u_k|^{p-1} |\nabla \omega| d\mu + \alpha_3 \int \omega^p |\nabla (u|^{p-1} + 1) d\mu. \]

Since \( \psi \leq v^\beta \), \( v^{-1} \leq M(r_0) + r_0 \) and \( l \leq u \leq L \), we obtain

\[ (32) \quad c^{-1} \beta \int \omega^p v^{\beta+1} |\nabla u_k|^p d\mu \leq \int \omega^{p-1} \psi |\nabla u_k|^{p-1} |\nabla \omega| d\mu + \int \omega^p v^{\beta+1} d\mu, \]

where \( c \) depends on \( p, \alpha_1, \alpha_2, \alpha_3, r_0, L \). Application of Young's inequality yields that

\[ \int \omega^{p-1} \psi |\nabla u_k|^{p-1} |\nabla \omega| d\mu \leq \varepsilon^{p/(p-1)} (p-1)^{p-1} \int \omega^p v^{\beta+1} |\nabla u_k|^p d\mu \]

\[ + \varepsilon^{-p} \int v^{\beta-p+1} |\nabla \omega|^p d\mu, \]

for any \( \varepsilon > 0 \). By the above inequality and (32), with an appropriate choice for \( \varepsilon \), we have

\[ (33) \quad c^{-1} \beta \int \omega^p v^{\beta+1} |\nabla u_k|^p d\mu \leq \int \omega^p v^{\beta+1} d\mu + \beta^{-p} \int v^{\beta-p+1} |\nabla \omega|^p d\mu. \]
By letting $\beta = p\alpha + p - 1$ with $0 < \beta \neq p - 1$, , we obtain Estimate 1.

Next we prove Estimate 2. In (33) letting $\beta = p - 1$, 
\[
\int \omega^p|\nabla(\log v)|^p \leq c\{(p - 1)^{-1} \int \omega^p v^p d\mu + (p - 1)^{-p} \int |\nabla \omega|^p d\mu\}.
\]
Since, by using $v \leq 1/r$ and Sobolev inequality,
\[
\int \omega^p v^p d\mu \leq r^{-p} \mu(B(x_0, r))^{(k-1)/k} \left( \int \omega^{pk} d\mu \right)^{1/k} \leq c \int |\nabla \omega|^p d\mu,
\]
we have
\[
\int \omega^p|\nabla(\log v)|^p \leq c \int |\nabla \omega|^p d\mu
\]
whenever $0 \leq \omega \in C^\infty_0(B(x_0, r))$. Using Lemma 4.4 (John-Nirenberg lemma) in the same manner as in the proof of Lemma 4.5 and Theorem 4.2, it follows that there are positive constants $c$ and $\sigma_0$ such that

\[
\int_{B(x_0, s)} v^{-\sigma} d\mu \int_{B(x_0, s)} v^\sigma d\mu \leq c \left\{ \mu(B(x_0, s)) \right\}^2,
\]
whenever $\sigma \leq \sigma_0$ and $0 < s \leq 3r/4$.

Let $0 < s < t \leq r$ and let a function $\omega \in C^\infty_0(B(x_0, t))$ be chosen such that $0 \leq \omega \leq 1$, $\omega = 1$ on $B(x_0, s)$ and $|\nabla \omega| \leq 2(t - s)^{-1}$. Then $(\omega v)^p \leq v^p \leq r^{-p} \leq 2(t - s)^{-p}$. Hence, from Sobolev inequality and Estimate 1,
\[
\left( \int_{B(x_0, t)} |v^\alpha|^{kp} d\mu \right)^{1/k} \leq c \left\{ \mu(B(x_0, r)) \right\}^{(1-k)/kp} (t-s)^{-p} \int_{B(x_0, t)} v^{p\alpha} d\mu,
\]
whenever $0 < s < t \leq r$ and $(1-p)p^{-1} < \alpha \neq 0$.

Let $r_j = r(2^{-1} + 2^{-j-2})$ for $j = 0, 1, \ldots$. Then since $m(\alpha k^j) \leq c (k^p)^j$ for $0 < \alpha_0 \leq \sigma_0^{-1}$, (35) yields that
\[
\left( \int_{B(x_0, r_{j+1})} |v^\alpha|^{kp} d\mu \right)^{1/k} \leq c (k^p)^j \left\{ \mu(B(x_0, r)) \right\}^{(1-k)/kp} \int_{B(x_0, r_j)} v^{p\alpha k^j} d\mu,
\]
and hence
\[
\|v^{p\alpha k^j}\|_{L^{p,k^j}(B(x_0, r_{j+1}))} \leq c \left\{ \mu(B(x_0, r)) \right\}^{(1-k)/kp} \int_{B(x_0, r_j)} v^{p\alpha k^j} d\mu,
\]
for $j = 0, 1, \ldots$. Hereafter, for simplicity, we shall write $\| \cdot \|_{p,r}$ for $\| \cdot \|_{p,B(x_0,r)}$. By iterating, we have
\[
(M(r) - M(r/2) + r)^{-p\alpha_0} \leq c \left\{ \mu(B(x_0, r)) \right\}^{-1} \|v^{p\alpha_0}\|_{L^{1,3r/4}} ,
\]
whenever $0 < p\alpha_0 \leq \sigma_0$. From (34) and (36), we obtain that
\[
\mu(B(x_0, r))^{-1} \|v^{-p\alpha_0}\|_{L^{1,3r/4}} \leq c (M(r) - M(r/2) + r)^{p\alpha_0}
\]
whenever $0 < p\alpha_0 \leq \sigma_0$.

Return to (35) with $1 - p < p\alpha < 0$. Let $0 < \sigma < p - 1$ and let $j_0$ be a positive integer such that $p - 1 \leq \sigma_0 k^{j_0}$. Put $\sigma_1 = \sigma k^{-j_0}$. Since $0 < \sigma_1 k^j \leq \sigma < p - 1$ for $0 \leq j \leq j_0$, $m(-\sigma_1 k^j p^{-1}) \leq m(-\sigma p^{-1})$ for $0 \leq j \leq j_0$. 

\[\int \omega^p|\nabla(\log v)|^p \leq c \int |\nabla \omega|^p d\mu\]
whenever $0 \leq \omega \in C^\infty_0(B(x_0, r))$. Using Lemma 4.4 (John-Nirenberg lemma) in the same manner as in the proof of Lemma 4.5 and Theorem 4.2, it follows that there are positive constants $c$ and $\sigma_0$ such that

\[
\int_{B(x_0, s)} v^{-\sigma} d\mu \int_{B(x_0, s)} v^\sigma d\mu \leq c \left\{ \mu(B(x_0, s)) \right\}^2,
\]
whenever $\sigma \leq \sigma_0$ and $0 < s \leq 3r/4$.

Let $0 < s < t \leq r$ and let a function $\omega \in C^\infty_0(B(x_0, t))$ be chosen such that $0 \leq \omega \leq 1$, $\omega = 1$ on $B(x_0, s)$ and $|\nabla \omega| \leq 2(t - s)^{-1}$. Then $(\omega v)^p \leq v^p \leq r^{-p} \leq 2(t - s)^{-p}$. Hence, from Sobolev inequality and Estimate 1,
\[
\left( \int_{B(x_0, t)} |v^\alpha|^{kp} d\mu \right)^{1/k} \leq c \left\{ \mu(B(x_0, r)) \right\}^{(1-k)/kp} (t-s)^{-p} \int_{B(x_0, t)} v^{p\alpha} d\mu,
\]
whenever $0 < s < t \leq r$ and $(1-p)p^{-1} < \alpha \neq 0$.

Let $r_j = r(2^{-1} + 2^{-j-2})$ for $j = 0, 1, \ldots$. Then since $m(\alpha_0 k^j) \leq c (k^p)^j$ for $0 < \alpha_0 \leq \sigma_0^{-1}$, (35) yields that
\[
\left( \int_{B(x_0, r_{j+1})} |v^\alpha|^{kp} d\mu \right)^{1/k} \leq c (k^p)^j \left\{ \mu(B(x_0, r)) \right\}^{(1-k)/kp} \int_{B(x_0, r_j)} v^{p\alpha k^j} d\mu,
\]
and hence
\[
\|v^{p\alpha k^j}\|_{L^{p,k^j}(B(x_0, r_{j+1}))} \leq c \left\{ \mu(B(x_0, r)) \right\}^{(1-k)/kp} \int_{B(x_0, r_j)} v^{p\alpha k^j} d\mu,
\]
for $j = 0, 1, \ldots$. Hereafter, for simplicity, we shall write $\| \cdot \|_{p,r}$ for $\| \cdot \|_{p,B(x_0,r)}$. By iterating, we have
\[
(M(r) - M(r/2) + r)^{-p\alpha_0} \leq c \left\{ \mu(B(x_0, r)) \right\}^{-1} \|v^{p\alpha_0}\|_{L^{1,3r/4}} ,
\]
whenever $0 < p\alpha_0 \leq \sigma_0$. From (34) and (36), we obtain that
\[
\mu(B(x_0, r))^{-1} \|v^{-p\alpha_0}\|_{L^{1,3r/4}} \leq c (M(r) - M(r/2) + r)^{p\alpha_0}
\]
whenever $0 < p\alpha_0 \leq \sigma_0$.

Return to (35) with $1 - p < p\alpha < 0$. Let $0 < \sigma < p - 1$ and let $j_0$ is a positive integer such that $p - 1 \leq \sigma_0 k^{j_0}$. Put $\sigma_1 = \sigma k^{-j_0}$. Since $0 < \sigma_1 k^j \leq \sigma < p - 1$ for $0 \leq j \leq j_0$, $m(-\sigma_1 k^j p^{-1}) \leq m(-\sigma p^{-1})$ for $0 \leq j \leq j_0$. 

\[\int \omega^p|\nabla(\log v)|^p \leq c \int |\nabla \omega|^p d\mu\]
Let $r_j = (r/4)\{3 - j/(j_0 + 1)\}$ for $0 \leq j \leq j_0 + 1$. Then (35) yields that

$$\|v^{-\sigma_1}\|_{kj+1,r_{j}+1} \leq \left[ c \, m(-\sigma p^{-1})\{\mu(B(x_0, r))\}^{(1-k)/k}\{4(j_0 + 1)\}^p \right]^{1-j} \|v^{-\sigma_1}\|_{kj,r_{j}}.$$ 

By iterating for $0 \leq j \leq j_0$, we have

$$\mu(B(x_0, r))^{-1}\|v^{-\sigma_1}\|_{k^{j+1},r_{j+1}} \leq \left[ c \, m(-\sigma p^{-1})\{4(j_0 + 1)\}^p \right]^{k^{j+1}-1} \times \left[ \{\mu(B(x_0, r))\}^{-1}\|v^{-\sigma_1}\|_{1,3r/4} \right]^{k^{j+1}}.$$ 

Since $0 < \sigma_1 < \sigma_0$, from (37) we obtain Estimate 2. Hence Lemma 5.1 follows. \square

References


