<table>
<thead>
<tr>
<th>Title</th>
<th>On solutions of quasi-linear partial differential equations - div ( \mathcal{A}(x, \nabla u) + \mathcal{B}(x, y) = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Ono, Takayori</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1997), 1016: 146-165</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-11</td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
On solutions of quasi-linear partial differential equations

\[ -\text{div} \mathcal{A}(x, \nabla u) + \mathcal{B}(x, u) = 0 \]

Introduction

Recently, a nonlinear potential theory has been developed in [1] for quasi-linear elliptic partial differential equations of second order of the form

\[ -\text{div} \mathcal{A}(x, \nabla u) = 0, \]

where \( \mathcal{A} \) is a mapping of \( R^n \times R^n \) to \( R^n \) \((n \geq 2)\) satisfying a growth condition \( \mathcal{A}(x, h) \cdot h \approx w(x)|h|^p \) \((1 < p < \infty)\) with a "weight" \( w(x) \), which is a nonnegative locally integrable function in \( R^n \). A prototype is the so-called weighted \( p \)-Laplace equations

\[ -\text{div}(w(X)|\nabla u|^{p-2}\nabla u) = 0, \]

This purpose of this paper is to extend some of the results in [1] to the equation

\[ (*) \quad -\text{div} \mathcal{A}(x, \nabla u) + \mathcal{B}(x, u) = 0, \]

where \( \mathcal{B}(x, t) \) is a mapping of \( R^n \times R \) to \( R \), which is non-decreasing in \( t \). A prototype equation may be given by

\[ -\text{div}(w(x)|\nabla u|^{p-2}\nabla u) + w(x)|u|^{p-2}u = 0. \]

As a matter of fact, we treat the following three topics: (i) Existence and uniqueness of solutions of Dirichlet problems for equation \((*)\) with Sobolev boundary values, or more generally of obstacle problems (section 3); (ii) Harnack inequality and Hölder continuity for solutions of \((*)\) (section 4); (iii) Regularity at the boundary for solutions of \((*)\) (section 5).

We can discuss (i) in the same way as in [1, Appendix I], using a general result of monotone operators. For (ii) and (iii), the methods in [1] are no longer applicable. We follow the discussion in [2] (for (ii)) and those in [4] (for (iii)), in which the unweighted case, namely the case \( w = 1 \), is treated.

Weighted Sobolev space

We recall the weighted Sobolev spaces \( H^{1,p}(\Omega; \mu) \) which are adopted in [1].

Throughout this paper \( \Omega \) will denote an open subset of \( R^n \) \((n \geq 2)\) and \( 1 < p < \infty \). We denote \( B(x, r) = \{ y \in R^n : |x - y| < r \} \), and \( \lambda B = B(x, \lambda r) \) if \( B = B(x, r) \) and \( \lambda > 0 \).

Let \( w \) be a locally integrable, nonnegative function in \( R^n \). Then a Radon measure \( \mu \) is canonically associated with the weight \( w \) :

\[ (1) \quad \mu(E) = \int_E w(x)dx. \]
Thus $d\mu(x) = w(x)dx$, where $dx$ is the $n$-dimensional Lebesgue measure. We say that $w$ (or $\mu$) is $p$-admissible if the following four conditions are satisfied:

I. $0 < w < \infty$ almost everywhere in $\mathbb{R}^n$ and the measure $\mu$ is doubling, i.e. there is a constant $C_I > 0$ such that
$$\mu(2B) \leq C_I \mu(B)$$
whenever $B$ is a ball in $\mathbb{R}^n$.

II. If $D$ is an open set and $\varphi_i \in C_0^\infty(D)$ is a sequence of functions such that $\int_D |\varphi_i|^p d\mu \to 0$ and $\int_D |\nabla \varphi_i - v|^p d\mu \to 0 (i \to \infty)$, where $v$ is a vector-valued measurable function in $L^p(D; \mu; \mathbb{R}^n)$, then $v = 0$.

III. (Sobolev inequality) There are constants $k > 1$ and $C_{III} > 0$ such that
$$\left( \frac{1}{\mu(B)} \int_B |\varphi|^{kp} d\mu \right)^{1/kp} \leq C_{III}^p \left( \frac{1}{\mu(B)} \int_B |\nabla \varphi|^p d\mu \right)^{1/p}$$
whenever $B = B(x_0, r)$ is a ball in $\mathbb{R}^n$ and $\varphi \in C_0^\infty(B)$.

IV. There is a constant $C_{IV} > 0$ such that
$$\int_B |\varphi - \varphi_B|^p d\mu \leq C_{IV} r^p \int_B |\nabla \varphi|^p d\mu$$
whenever $B = B(x_0, r)$ is a ball in $\mathbb{R}^n$ and $\varphi \in C^\infty(B)$ is bounded. Here
$$\varphi_B = \frac{1}{\mu(B)} \int_B \varphi d\mu.$$

From now on, unless otherwise stated, we assume that $\mu$ is a $p$-admissible measure and $d\mu(x) = w(x)dx$.

In this paper, both condition IV and the following inequality are called the Poincaré inequality.

**Poincaré inequality ([1, p.9])**

If $\Omega$ is bounded, then
$$\int_\Omega |\varphi|^p d\mu \leq C_{III}^p (\text{diam } \Omega)^p \int_\Omega |\nabla \varphi|^p d\mu$$
for $\varphi \in C_0^\infty(\Omega)$.

Throughout this paper, let $c_\mu$ denote constants depending on $C_I$, $C_{II}$, $C_{III}$, $k$ and $C_{IV}$.

For a $\mu$-measurable function $f$ defined on an open set $\Omega$, $L^p$-norm of $f$ is defined by
$$\|f\|_{p, \Omega} = \left( \int_\Omega |f|^p d\mu \right)^{1/p}.$$

For a function $\varphi \in C^\infty(\Omega)$ we let
$$\|\varphi\|_{1, p, \Omega} = \left( \int_\Omega |\varphi|^p d\mu \right)^{1/p} + \left( \int_\Omega |\nabla \varphi|^p d\mu \right)^{1/p},$$
where, we recall, $\nabla \varphi = (\partial_1 \varphi, \cdots, \partial_n \varphi)$ is the gradient of $\varphi$. The Sobolev space $H^{1,p}(\Omega; \mu)$ is defined to be the completion of

$$\{ \varphi \in C^\infty(\Omega) : \| \varphi \|_{1,p,\Omega} < \infty \}$$

with respect to norm $\| \cdot \|_{1,p,\Omega}$. In other words, a function $u$ is in $H^{1,p}(\Omega; \mu)$ if and only if $u$ is in $L^p(\Omega; \mu)$ and there is a vector-valued function $v$ in $L^p(\Omega; \mu; \mathbb{R}^n)$ such that for some sequence $\varphi_i \in C^\infty(\Omega)$

$$\int_{\Omega} |\varphi_i - u|^p d\mu \to 0$$

and

$$\int_{\Omega} |\nabla \varphi_i - v|^p d\mu \to 0$$

as $i \to \infty$. The function $v$ is called the gradient of $u$ in $H^{1,p}(\Omega; \mu)$ and denoted by $\nabla u$.

The space $H^{1,p}_{0}(\Omega; \mu)$ is the closure of $C^\infty_0(\Omega)$ in $H^{1,p}(\Omega; \mu)$. The corresponding local space $H^{1,p}_{loc}(\Omega; \mu)$ is defined in the obvious manner.

§2. Quasilinear PDE's

$A$ is a mapping of $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$ satisfying the following assumptions for some constants $0 < \alpha_1 \leq \alpha_2 < \infty$:

(a1) the mapping $x \mapsto A(x, h)$ is measurable for all $h \in \mathbb{R}^n$ and

the mapping $h \mapsto A(x, h)$ is continuous for a.e. $x \in \mathbb{R}^n$;

for all $h \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$

(a2) $\alpha_1 w(x)|h|^p$,  

(a3) $|A(x, h)| \leq \alpha_2 w(x)|h|^{p-1}$,  

(a4) $(A(x, h_1) - A(x, h_2)) \cdot (h_1 - h_2) > 0$

whenever $h_1, h_2 \in \mathbb{R}^n$, $h_1 \neq h_2$.

$B$ is a mapping of $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}$ satisfying the following assumptions for a constant $0 < \alpha_3 < \infty$:

(b1) the mapping $x \mapsto B(x, t)$ is measurable for all $t \in \mathbb{R}$ and

the mapping $t \mapsto B(x, t)$ is continuous for a.e. $x \in \mathbb{R}^n$;

for all $t \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^n$

(b2) $|B(x, t)| \leq \alpha_3 w(x)(|t|^{p-1} + 1)$,

(b3) $(B(x, t_1) - B(x, t_2))(t_1 - t_2) \geq 0$

whenever $t_1, t_2 \in \mathbb{R}^n$. Using $A$ and $B$ we consider the quasilinear elliptic equation

(2) $-\text{div}A(x, \nabla u) + B(x, u) = 0$. 

148
A function $u \in H_{loc}^{1,p}(\Omega; \mu)$ is a (weak) solution of (2) if

$$
\int_\Omega A(x, \nabla u) \cdot \nabla \varphi dx + \int_\Omega B(x, u) \varphi dx = 0
$$

whenever $\varphi \in C_0^\infty(\Omega)$. A function $u \in H_{loc}^{1,p}(\Omega; \mu)$ is a supersolution of (2) in $\Omega$ if

$$
-\text{div}A(x, \nabla u) + B(x, u) \geq 0
$$

weakly in $\Omega$, i.e.

$$
\int_\Omega A(x, \nabla u) \cdot \nabla \varphi dx + \int_\Omega B(x, u) \varphi dx \geq 0
$$

whenever $\varphi \in C_0^\infty(\Omega)$, and $u$ is a solution of (2) in $\Omega$.

A function $u \in H_{loc}^{1,p}(\Omega; \mu)$ is a subsolution in $\Omega$ if

$$
\int_\Omega A(x, \nabla u) \cdot \nabla \varphi dx + \int_\Omega B(x, u) \varphi dx \geq 0
$$

for all nonnegative $\varphi \in H_0^{1,p}(\Omega; \mu)$.

**Lemma 2.1** If $u \in H^{1,p}(\Omega; \mu)$ is a solution (respectively, a supersolution) of (2) in $\Omega$, then

$$
\int_\Omega A(x, \nabla u) \cdot \nabla \varphi dx + \int_\Omega B(x, u) \varphi dx = 0 \quad (\text{respectively}, \geq 0)
$$

for all $\varphi \in H_0^{1,p}(\Omega; \mu)$ (respectively, for all nonnegative $\varphi \in H_0^{1,p}(\Omega; \mu)$) with compact support.

**Proof:** Let $\Omega'$ be an open set such that $\text{spt} \varphi \subset \Omega' \subset \subset \Omega$. Since $\varphi \in H_0^{1,p}(\Omega'; \mu)$, we can choose a sequence of functions $\varphi_i \in C_0^\infty(\Omega')$ such that $\varphi_i \to \varphi$ in $H_0^{1,p}(\Omega'; \mu)$. If $\varphi$ is nonnegative, pick nonnegative functions $\varphi_i$ (Lemma 1.23, p.21). Then by (a3)

$$
\left| \int_\Omega A(x, \nabla u) \cdot \nabla \varphi_i dx + \int_\Omega B(x, u) \varphi_i dx - \left( \int_\Omega A(x, \nabla u) \cdot \nabla \varphi_i dx + \int_\Omega B(x, u) \varphi_i dx \right) \right|
$$

$$
\leq \alpha_2 \int_{\Omega'} |\nabla u|^p \varphi_i d\mu + \alpha_3 \int_{\Omega'} (|u|^{p-1} + 1) |\varphi - \varphi_i| d\mu
$$

$$
\leq \alpha_2 \left( \int_{\Omega'} |\nabla u|^p d\mu \right)^{(p-1)/p} \left( \int_{\Omega'} |\varphi - \varphi_i|^p d\mu \right)^{1/p}
$$

$$
+ 2\alpha_3 \left( \int_{\Omega'} (|u| + 1)^p d\mu \right)^{(p-1)/p} \left( \int_{\Omega'} |\varphi - \varphi_i|^p d\mu \right)^{1/p}.
$$

Because the last integral tends to zero as $i \to 0$, we have

$$
\int_\Omega A(x, \nabla u) \cdot \nabla \varphi dx + \int_\Omega B(x, u) \varphi dx = \lim_{i \to \infty} \left( \int_{\Omega'} A(x, \nabla u) \cdot \nabla \varphi_i dx + \int_{\Omega'} B(x, u) \varphi_i dx \right) = (\geq) 0,
$$

and the lemma follows. \(\square\)

The proof of Lemma 2.1 implies that (5) holds for all (nonnegative) $\varphi \in H_0^{1,p}(\Omega; \mu)$ if $\Omega$ is bounded.

A function $u$ is a solution of (2) if and only if $u$ is a supersolution and a subsolution. Indeed, if $u$ is a supersolution and a subsolution of (2), since the positive part $\varphi^+$ of a test function $\varphi \in C_0^\infty(\Omega)$, belongs $H_0^{1,p}(\Omega; \mu)$ and has compact support, $u$ satisfies (3) for $\varphi^+$. Similarly, $u$ satisfies (3) for the negative part of $\varphi$. Hence $u$ is a solution of (2).
§3. The existence of solutions

In this section, the existence of solutions of Dirichlet problems for equation (2) with Sobolev boundary values will be proved, using a general result in the theory of monotone operators.

Let $X$ be a reflexive Banach space with dual $X'$ and let $\langle \cdot, \cdot \rangle$ denote a pairing between $X'$ and $X$. If $K \subset X$ is a closed convex set, then a mapping $\mathcal{S} : K \to X'$ is called monotone if

$$\langle \mathcal{S}u - \mathcal{S}v, u - v \rangle \geq 0$$

for all $u, v$ in $K$. Further, $\mathcal{S}$ is called coercive on $K$ if there exists $\varphi \in K$ such that

$$\frac{\langle \mathcal{S}u_j - \mathcal{S}\varphi, u_j - \varphi \rangle}{\|u_j - \varphi\|} \to \infty$$

whenever $u_j$ is a sequence in $K$ with $\|u_j\| \to \infty$.

We recall the following proposition. ([3, Corollary III.1.8, p.87]).

**Proposition 3.1** Let $K$ be a nonempty closed convex subset of $X$ and let $\mathcal{S} : K \to X'$ be monotone, coercive, and weakly continuous on $K$. Then there exists an element $u$ in $K$ such that

$$\langle \mathcal{S}u, v - u \rangle \geq 0$$

whenever $v \in K$.

Throughout this section, we assume that $\Omega$ is bounded.

Suppose that $\psi$ is any function in $\Omega$ with values in the extended reals $[-\infty, \infty]$, and that $\theta \in H^{1,p}(\Omega; \mu)$. Let

$$K_{\psi, \theta} = K_{\psi, \theta}(\Omega) = \{ v \in H^{1,p}(\Omega; \mu) : v \geq \psi \text{ a.e in } \Omega , \ v - \theta \in H^{1,p}_0(\Omega; \mu) \}.$$ 

Set $X = L^p(\Omega; \mu; \mathbb{R}^n) \times L^p(\Omega; \mu; \mathbb{R})$ and $K = \{(\nabla v, v) : v \in K_{\psi, \theta}(\Omega) \}$. 

**Lemma 3.2** $K$ is a closed convex set in $X$.

**Proof**: $K$ is clearly convex. To show the closedness, let $(\nabla v_i, v_i) \in K$ be a sequence converging to $(f, u)$ in $X$. By $\nabla v_i \to f$ in $L^p(\Omega; \mu; \mathbb{R}^n)$ and $v_i \to u$ in $L^p(\Omega; \mu; \mathbb{R})$, $v_i$ is a bounded sequence in $H^{1,p}(\Omega; \mu)$. Since $K_{\psi, \theta}$ is a convex and closed subset of $H^{1,p}(\Omega; \mu)$, there is a function $v \in K_{\psi, \theta}$ such that $v = u$ and $\nabla v = f$ ([1, Theorem 1.31, p.25]). Thus $(f, u) \in K$. The lemma is proved. $\square$

Let $\langle \cdot, \cdot \rangle$ be the pairing between $X$ and $X'$, 

$$\langle (f, u), (g, v) \rangle = \int_\Omega f \cdot gd\mu + \int_\Omega uv d\mu,$$

where $(f, u)$ is in $X$ and $(g, v)$ in $X' = L^{p/(p-1)}(\Omega; \mathbb{R}^n) \times L^{p/(p-1)}(\Omega; \mathbb{R})$.

A mapping $\mathcal{S} : K \to X'$ is well defined by the formula

$$\langle \mathcal{S}(\nabla v, v), (f, u) \rangle = \int_\Omega A(x, \nabla v(x)) \cdot f(x) dx + \int_\Omega B(x, v(x)) u(x) dx$$
for $(f, u) \in X$; indeed, by (a3) and (b2),
\[
|\int_{\Omega} A(x, \nabla v) \cdot f dx| \leq \alpha_2 \left( \int_{\Omega} |\nabla v|^p d\mu \right)^{(p-1)/p} \left( \int_{\Omega} |f|^p d\mu \right)^{1/p} \\
|\int_{\Omega} B(x, v) u dx| \leq 2\alpha_3 \left( \int_{\Omega} (|v|+1)^p d\mu \right)^{(p-1)/p} \left( \int_{\Omega} |u|^p d\mu \right)^{1/p}
\]

Lemma 3.3 $\mathcal{S}$ is monotone, coercive, and weakly continuous on $K$.

Proof: By (a4) and (b3), $\mathcal{S}$ is monotone.

Next we show that $\mathcal{S}$ is coercive on $K$. Fix $(\nabla \varphi, \varphi) \in K$. Hereafter, for simplicity, we shall write $\| \cdot \|$ for $\| \cdot \|_{p,\Omega}$. By (a2), (a3) and (b3)

\[
\langle \mathcal{S}(\nabla u, u) - \mathcal{S}(\nabla \varphi, \varphi), (\nabla u, u) - (\nabla \varphi, \varphi) \rangle \\
= \int_{\Omega} (A(x, \nabla u) - A(x, \nabla \varphi)) \cdot (\nabla u - \nabla \varphi) dx + \int_{\Omega} (B(x, u) - B(x, \varphi))(u - \varphi) dx \\
\geq \alpha_1 (\|\nabla u\|^p + \|\nabla \varphi\|^p) - \alpha_2 (\|\nabla u\|^{p-1}\|\nabla \varphi\| + \|\nabla u\| \|\nabla \varphi\|^{p-1}) \\
\geq \|\nabla u - \nabla \varphi\| \alpha_1 (\|\nabla u\|^p + \|\nabla \varphi\|^p) - \alpha_2 (\|\nabla \varphi\|^{p-1}\|\nabla \varphi\|^p + \|\nabla u - \nabla \varphi\|^p) \\
- \alpha_2 \|\nabla \varphi\|^{p-1}(\|\nabla \varphi\| + \|\nabla u - \nabla \varphi\|).
\]

Since $u - \varphi \in H_0^1(\Omega; \mu)$,

\[
\|u - \varphi\| \leq c\|\nabla u - \nabla \varphi\|.
\]

By (6) and (7), $\mathcal{S}$ is coercive on $K$.

Finally, to show that $\mathcal{S}$ is weakly continuous on $K$, let $(\nabla u_i, u_i) \in K$ be a sequence that converges to an element $(\nabla u, u) \in K$ in $X$. For any subsequence $(\nabla u_{i_j}, u_{i_j})$ of $(\nabla u_i, u_i)$, there is a subsequence $(\nabla u_{i_j}', u_{i_j}')$ of $(\nabla u_{i_j}, u_{i_j})$ such that $(\nabla u_{i_j}', u_{i_j}') \to (\nabla u, u)$ a.e. in $\Omega$. By (a1) and (b1), we have

\[
A(x, \nabla u_{i_j}(x))^{w-1/p}(x) \to A(x, \nabla u(x))^{w-1/p}(x) \\
B(x, u_{i_j}(x))^{w-1/p}(x) \to B(x, u(x))^{w-1/p}(x)
\]
a.e. in $\Omega$. Since

\[
\int_{\Omega} |A(x, \nabla u_i)^{w-1/p}|^{p/(p-1)} dx \leq \alpha_2^{p/(p-1)} \int_{\Omega} |\nabla u_i|^p d\mu \\
\int_{\Omega} |B(x, u_i)^{w-1/p}|^{p/(p-1)} dx \leq 2\alpha_3^{p/(p-1)} \int_{\Omega} (|u_i|+1)^p d\mu,
\]

$L^p/(p-1)(\Omega; dx)$-norms of $A(x, \nabla u_i)^{w-1/p}$ and $B(x, u_i)^{w-1/p}$ are uniformly bounded. Therefore

\[
A(x, \nabla u_{i_j}')^{w-1/p} \to A(x, \nabla u)^{w-1/p} \\
B(x, u_{i_j}')^{w-1/p} \to B(x, u)^{w-1/p}
\]

weakly in $L^p/(p-1)(\Omega; dx)$. Since the weak limit is independent of $(\nabla u_{i_j}, u_{i_j})$, 

\[
A(x, \nabla u_{i_j})^{w-1/p} \to A(x, \nabla u)^{w-1/p} \\
B(x, u_{i_j})^{w-1/p} \to B(x, u)^{w-1/p}.
\]
weakly in $L^{p/(p-1)}(\Omega; dx)$. Hence we have for all $(f, g) \in X$ that

$$\langle \mathcal{S}(\nabla u_i, u_i), (f, g) \rangle = \int_{\Omega} A(x, \nabla u_i) \cdot f dx + \int_{\Omega} B(x, u_i) g dx$$

$$= \int_{\Omega} A(x, \nabla u_i)w^{-1/p} \cdot fw^{1/p} dx + \int_{\Omega} B(x, u_i)w^{-1/p} gw^{1/p} dx$$

$$\rightarrow \int_{\Omega} A(x, \nabla u)w^{-1/p} \cdot fw^{1/p} dx + \int_{\Omega} B(x, u)w^{-1/p} gw^{1/p} dx$$

$$= \langle \mathcal{S}(\nabla u, u), (f, g) \rangle.$$ 

Therefore the lemma follows. \(\square\)

Now the following theorem follows from Proposition 3.1, Lemma 3.2 and Lemma 3.3.

**Theorem 3.4** Suppose that $\mathcal{K}_{\psi, \theta}(\Omega) \neq \emptyset$, then there is a function $u$ in $\mathcal{K}_{\psi, \theta}$ such that

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla(v - u) dx + \int_{\Omega} B(x, u)(v - u) dx \geq 0$$

whenever $v \in \mathcal{K}_{\psi, \theta}$.

A function $u$ in $\mathcal{K}_{\psi, \theta}(\Omega)$ that satisfies (8) for all $v \in \mathcal{K}_{\psi, \theta}(\Omega)$ is called a solution to the obstacle problem in $\mathcal{K}_{\psi, \theta}(\Omega)$.

As a corollary to this theorem, we have the existence of solutions of Dirichlet problems with Sobolev boundary values.

**Corollary 3.5** Suppose that $\theta \in H^{1,p}(\Omega, \mu)$, Then, there is a function $u \in H^{1,p}(\Omega, \mu)$ with $u - \theta \in H_{0}^{1,p}(\Omega, \mu)$ such that

$$-\text{div} A(x, \nabla u) + B(x, u) = 0$$

weakly in $\Omega$, that is

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} B(x, u) \varphi dx = 0$$

whenever $\varphi \in H_{0}^{1,p}(\Omega, \mu)$.

**Proof:** Choose $\psi \equiv -\infty$. Let $u$ be the solution to the obstacle problem in $\mathcal{K}_{\psi, \theta}$ and $\varphi \in H_{0}^{1,p}(\Omega, \mu)$. Since $u + \varphi$, $u - \varphi \in \mathcal{K}_{\psi, \theta}$, we have

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} B(x, u) \varphi dx \geq 0$$

and

$$-\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi dx - \int_{\Omega} B(x, u) \varphi dx \geq 0.$$ 

Then

$$\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi dx + \int_{\Omega} B(x, u) \varphi dx = 0.$$ 

Hence Corollary 3.5 follows. \(\square\)
The uniqueness of solutions of Dirichlet problems for equation (2) and obstacle problems in $\mathcal{K}_{\psi, \theta}$ follows from the following comparison principle Lemma 3.6 and Lemma 3.7 respectively.

**Lemma 3.6** Let $u \in H^{1,p}(\Omega; \mu)$ be a supersolution and $v \in H^{1,p}(\Omega; \mu)$ a subsolution of (2) in $\Omega$. If $\eta = \min(u - v, 0) \in H_{0}^{1,p}(\Omega; \mu)$, then $u \geq v$ a.e. in $\Omega$.

**Proof:** By (a4) and (b3),
\[
\int_{\Omega} (A(x, \nabla v) - A(x, \nabla u)) \cdot \nabla \eta dx \leq - \int_{\{u < v\}} (A(x, \nabla v) - A(x, \nabla u)) \cdot (\nabla v - \nabla u) dx \leq 0,
\]
\[
\int_{\Omega} (B(x, v) - B(x, u)) \eta dx \leq - \int_{\{v < u\}} (B(x, v) - B(x, u))(v - u) dx \leq 0.
\]
From this we have
\[
0 \leq \int_{\Omega} A(x, \nabla v) \cdot \nabla \eta dx + \int_{\Omega} B(x, v) \eta dx - \left( \int_{\Omega} A(x, \nabla u) \cdot \nabla \eta dx + \int_{\Omega} B(x, u) \eta dx \right) \leq 0.
\]
and, hence
\[
\int_{\Omega} (A(x, \nabla v) - A(x, \nabla u)) \cdot \nabla \eta dx = 0
\]
and
\[
\int_{\Omega} (B(x, v) - B(x, u)) \eta dx = 0.
\]
Therefore $\nabla \eta = 0$ a.e. in $\Omega$. Because $\eta \in H_{0}^{1,p}(\Omega; \mu)$, $\eta = 0$ a.e. in $\Omega$ ([1, Lemma 1.17, p.18]). The lemma follows. \(\square\)

**Lemma 3.7** Suppose that $u$ is a solution to the obstacle problem in $\mathcal{K}_{\psi, \theta}(\Omega)$. If $v \in H^{1,p}(\Omega; \mu)$ is a supersolution of (2) in $\Omega$ such that $\min(u, v) \in \mathcal{K}_{\psi, \theta}(\Omega)$, then $v \geq u$ a.e. in $\Omega$.

**Proof:** Since $u - \min(u, v) \in H_{0}^{1,p}(\Omega; \mu)$ and is nonnegative, the lemma is proved in the same manner as in the proof of Lemma 3.6. \(\square\)

§4. The local behavior of solutions

In this section, we study the local behavior of solutions of (2).

The next theorem can be shown in the same manner as [2, Theorem 1].

**Theorem 4.1** Each solution of (2) in $\Omega$ is locally bounded.

We obtain, using the Moser iteration technique, the following Harnack inequality.

Let $B(R)$ denote an open ball of radius $R$.

**Theorem 4.2** Let $u$ be a nonnegative solution of equation (2) in $\Omega$. Given $R_0 > 0$ there is a constant $c > 0$ such that

\[
\text{ess sup}_{B(R)} u \leq c \text{ ess inf}_{B(R)} (u + R)
\]

whenever $B(R)$ is a ball in $\Omega$ such that $3B(R) \subset \Omega$ and $R \leq R_0$. Here $c$ depends only on $n$, $p$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $c_\mu$ and $R_0$. 

We require some lemmas to prove Theorem 4.2.

**Lemma 4.3** ([2, Lemma 2, p.252]) Let $a$ be a positive exponent, and let $a_i$, $b_i$ ($i = 1, \cdots, N$), be two sets of $N$ real numbers such that $0 < a_i < \infty$ and $0 \leq b_i < a$. Suppose that $z$ is a positive number satisfying

$$z^a \leq \sum a_i z^{b_i}.$$  

Then

$$z \leq c \sum (a_i)^{\gamma_i}$$

where $c$ depends only on $N$, $a$, and $b_i$, and where $\gamma_i = (a - b_i)^{-1}$.

**Lemma 4.4** (John-Nirenberg lemma) ([1, Appendix II]) Suppose that $v$ is a locally $\mu$-integrable function in $\Omega$ with

$$\sup_{\mu(B)} \frac{1}{\mu(B)} \int_B |v - v_B|d\mu \leq c_o,$$

where

$$v_B = \frac{1}{\mu(B)} \int_B v d\mu$$

and the supremum is taken over all balls $B \subset \subset \Omega$. Then there are positive constants $c_1$ and $c_2$ depending on $c_0$, $n$, and $c_\mu$ such that

$$\sup_{\mu(B)} \frac{1}{\mu(B)} \int_B e^{c_1 |v - v_B|}d\mu \leq c_2,$$

where the supremum is taken over all balls $B \subset \subset \Omega$.

Let $u$ be a nonnegative solution of equation (2) in $\Omega$ and $B = B(R)$ is a ball in $\Omega$. We set $\bar{u} = u + R$. Thus, by Theorem 4.1, if $\eta \in C_0^\infty(B)$ is nonnegative, then $\varphi(x) = \eta^p \bar{u}^\beta \in H_0^{1,p}(B; \mu)$ for any real value of $\beta$. Moreover,

$$|B(x, u)| \leq 2\alpha_3 w \max(1, 1/R^{p-1}) \bar{u}^{p-1}.$$  

We set $\alpha'_3 = 2\alpha_3 \max(1, 1/R^{p-1})$.

Next lemma guarantees that $v = \log \bar{u}$ satisfies the hypothesis of John-Nirenberg lemma.

**Lemma 4.5** Suppose that $u$ is a nonnegative solution of equation (2) in $\Omega$ and $B = B(R)$ is a ball in $\Omega$ such $3B \subset \Omega$. Then there is a constant $c > 0$ such that

$$\int_{B_1} |v - v_{B_1}|d\mu \leq c\mu(B_1) \quad (v = \log \bar{u}),$$

whenever $B_1$ is a ball with $B_1 \subset 2B$. Here $c$ depends on $p$, $\alpha_1$, $\alpha_2$, $\alpha'_3 R^p$ and $c_\mu$.

Proof: Setting $\varphi = \eta^p \bar{u}^{1-p}$, we have

$$0 = \int_{3B} A(x, \nabla u) \cdot \nabla \varphi dx + \int_{3B} B(x, u) \varphi dx$$
\[ = \int_{3B} A(x, \nabla u) \cdot \{p(\eta/\overline{u})^{p-1} \nabla \eta + (1-p)(\eta/\overline{u})^{p} \nabla u\} dx + \int_{3B} B(x, u) \eta^{p} \overline{u}^{1-p} dx \]
\[ \leq -\alpha_1 (p-1) \int_{3B} (\eta/\overline{u})^{p} \nabla \eta d\mu + \alpha_2 p \int_{3B} (\eta/\overline{u})^{p-1} |\nabla \eta||\nabla u|^{p-1} d\mu \]
\[ + \alpha_3' \int_{3B} \eta^{p-1} \overline{u}^{1}|\nabla \eta|^{p-1} d\mu \]
\[ = -\alpha_1 (p-1) \int_{3B} (|\eta\nabla v|)^{p} d\mu + \alpha_2 p \int_{3B} |\nabla \eta||\eta\nabla v|^{p-1} d\mu + \alpha_3' \int_{3B} \eta^{p} d\mu, \]

where \( v = \log \overline{u} \). Hence

\[(9) \quad \alpha_1 (p-1) ||\eta \nabla v||_{p,3B} \leq \alpha_2 p \int_{3B} |\nabla \eta||\eta \nabla v|^{p-1} d\mu + \alpha_3' \int_{3B} \eta^{p} d\mu. \]

Let \( B_1 \subset 2B \) be any open ball of radius \( h \). Let \( \eta \) be so chosen that \( \eta = 1 \) in \( B_1 \), \( 0 \leq \eta \leq 1 \) in \( 3B \setminus B_1 \), the support of \( \eta \) is contained in \( (3/2)B_1 \), and \( |\nabla \eta| \leq 3/h \). Then by H"older's inequality we obtain

\[ \int_{3B} |\nabla \eta||\eta \nabla v|^{p-1} d\mu \leq \left( \int_{(3/2)B_1} |\nabla \eta|^{p} d\mu \right)^{1/p} \left( \int_{(3/2)B_1} |\eta \nabla v|^{p} d\mu \right)^{(p-1)/p} \]
\[ \leq \frac{3}{h} \left\{ \mu((3/2)B_1) \right\}^{1/p} ||\eta \nabla v||_{p,3B}, \]
\[ \int_{3B} \eta^{p} d\mu \leq \mu((3/2)B_1). \]

By the above inequalities and (9) we have

\[ \alpha_1 (p-1) ||\eta \nabla v||_{p,3B} \leq \frac{3\alpha_2 p}{h} \left\{ \mu((3/2)B_1) \right\}^{1/p} ||\eta \nabla v||_{p,3B} + \frac{\alpha_3'(3R)^p}{hp} \mu((3/2)B_1). \]

Application of Lemma 4.3 yields,

\[ ||\nabla v||_{p,B_1} \leq ch^{-1} \mu((3/2)B_1)^{1/p}, \]

where \( \eta = 1 \) in \( B_1 \) have been used. Finally by the doubling property, H"older's inequality and Poincaré inequality we have

\[ \int_{B_1} |v - v_{B_1}| d\mu \leq c \{\mu((3/2)B_1)\}^{(p-1)/p} h \left( \int_{B_1} |\nabla v|^{p} d\mu \right)^{1/p} \leq c\mu(B_1) \quad (v = \log \overline{u}), \]

where \( c = c(p, \alpha_1, \alpha_2, \alpha_3'R^{p}, \mu) \).

\[ \square \]

The following estimates will be used when we apply to the Moser iteration technique.

**Lemma 4.6** Suppose that \( u \) is a nonnegative solution of equation (2) in \( \Omega \) and \( B = B(R) \) is a ball in \( \Omega \). For \( \beta \neq 0 \), \( p-1 \), let \( q \) satisfying \( pq = p + \beta - 1 \) and \( v = \overline{u}^{q} \). Then there is a constant \( c > 0 \) such that

(i) if \( \beta > 0 \),

\[ ||\eta v||_{p,B_1} \leq c \{\mu(B)\}^{(1-k)/kp} R \left( 1 + \beta^{-1} \right) (1 + q)^p \left( ||v \nabla \eta||_{p,B} + ||v ||_{p,B} \right), \]
(ii) if $1 - p < \beta < 0$,
\[
\|\eta v\|_{kp, B} \leq c\{\mu(B)\}^{1/(1-k)} (1 - \beta^{-1}) (\|v\nabla\eta\|_{p, B} + \|\eta v\|_{p, B}),
\]

(iii) if $\beta < 1 - p$,
\[
\|\eta v\|_{kp, B} \leq c\{\mu(B)\}^{1/(1-k)} R^{1/(1-k)} (\|v\nabla\eta\|_{p, B} + \|\eta v\|_{p, B}),
\]

where $c$ depends only on $p$, $\alpha_1$, $\alpha_2$, $c_\mu$ and $\alpha_3' R^{p-1}$.

Proof: We prove only (i), the proofs of (ii) and (iii) being similar. For $\varphi = \eta^p \overline{u}^\beta$, we have
\[
0 = \int_B A(x, \nabla u) \cdot \nabla \varphi \, dx + \int_B B(x, u) \varphi \, dx
\]
\[
= \int_B A(x, \nabla u) \cdot (p\eta^{p-1} \overline{u}^\beta \nabla \eta + \beta \eta^{p\beta-1} \overline{u} \nabla u) \, dx + \int_B B(x, u) p\overline{u}^\beta \, dx
\]
\[
\geq \alpha_1 \beta \int_B \eta^{p\beta} |\nabla u|^{p} \, d\mu - p\alpha_2 \int_B |\nabla u|^{p-1} |\nabla \eta| \overline{u} \, d\mu - \eta^{p} \mu \alpha_3 / \int_B \eta^{p} \overline{u}^\beta \, d\mu.
\]

Since $pq = p + \beta - 1$ and $v = \overline{u}^q$,
\[
\frac{\alpha_1 \beta}{q^p} \|\eta \nabla v\|^p \leq \frac{p\alpha_2}{q(p-1)} \int_B |v \nabla \eta| |\eta \nabla v|^{p-1} \, d\mu + \alpha_3' \int B (\eta v)^p \, d\mu.
\]

Here for simplicity we have written $\| \cdot \|_p$ for $\| \cdot \|_{p, B}$.

By Hölder's inequality,
\[
\int_B |v \nabla \eta| |\eta \nabla v|^{p-1} \, d\mu \leq \|v \nabla \eta\|_p \|\eta \nabla v\|^{p-1}_p,
\]
\[
\int_B (\eta v)^p \, d\mu = \|\eta v\|_p \left( \int_B (\eta v)^p \, d\mu \right)^{(p-1)/p}
\]
\[
\leq \|\eta v\|_p \left( \int B (\eta v)^{kp} \, d\mu \right)^{1/k} \left( \int B \, d\mu \right)^{(k-1)/k} \left( \int B \, d\mu \right)^{(p-1)/p}
\]
\[
= \mu(B)^{(k-1)(p-1)/(kp)} \|\eta v\|_p \|\eta v\|^{p-1}_{kp}
\]
\[
\leq c_\mu R^{p-1} \|\eta v\|_p (|v \nabla \eta|^{p-1}_p + |\eta \nabla v|^{p-1}_p),
\]

where we have used Sobolev inequality. By the above inequalities, if we set
\[
z = \frac{\|\eta \nabla v\|_p}{\|v \nabla \eta\|_p}, \quad \zeta = \frac{\|\eta v\|_p}{\|v \nabla \eta\|_p},
\]
then (10) can be written as
\[
\beta z^p \leq c(qz^{p-1} + q^p \zeta (1 + z^{p-1})),
\]

where $c = c(p, \alpha_1, \alpha_2, \alpha_3' R^{p-1}, c_\mu)$. Application of Lemma 4.3 yields
\[
z \leq c(1 + \beta^{-1})(1 + q)^p (1 + \zeta),
\]
that is,
\[
\|\eta \nabla v\|_p \leq c(1 + \beta^{-1})(1 + q)^p (|v \nabla \eta|_p + \|\eta v\|_p).
\]
Finally using Sobolev inequality again, from (11) we obtain the desired estimate. □

Proof of Theorem 4.2: Set $v = \log \bar{u}$. By Lemma 4.4 and Lemma 4.5, there are positive constants $r_0$ and $c_0$ such that
\[
\left( \int_{B_1} e^{rv} d\mu \right) \left( \int_{B_1} e^{-rv} d\mu \right) = \left( \int_{B_1} e^{r_0(v - v_{B_1})} d\mu \right) \left( \int_{B_1} e^{r_0(v_{B_1} - v)} d\mu \right) \leq \left( \int_{B_1} e^{r_0|v - v_{B_1}|} d\mu \right) \leq c_0^2 \{\mu(B_1)\}^2.
\]
Because $B_1$ is any ball contained in $2B$,
\[
\left( \int_{2B} e^{rv} d\mu \right) \left( \int_{2B} e^{-rv} d\mu \right) \leq c_0^2 \{\mu(2B)\}^2.
\]

Hence
\[
\left( \int_{2B} \bar{u}^r d\mu \right)^{1/r} \leq c \{\mu(B)\}^{2/r} \left( \int_{2B} \bar{u}^{-r} d\mu \right)^{-1/r}.
\]

Next, let $0 < h' < h \leq 3R$. Let the function $\eta \in C_0^\infty(B(h))$ be so chosen that $\eta = 1$ in $B(h')$, $0 \leq \eta \leq 1$ in $B(h)$ and $|\nabla \eta| \leq 3(h - h')^{-1}$. Then Lemma 4.6 yields

(i) if $\beta > 0$,
\[
\|\bar{u}^q\|_{kp,B(h')} \leq c \{\mu(B)\}^{(1-k)/kp} R(1 + q)^p (h - h')^{-1}(1 + \beta^{-1}) \|\bar{u}^q\|_{p,B(h)}.
\]

(ii) if $1 - p < \beta < 0$,
\[
\|\bar{u}^q\|_{kp,B(h')} \leq c \{\mu(B)\}^{(1-k)/kp} R(h - h')^{-1}(1 - \beta^{-1}) \|\bar{u}^q\|_{p,B(h)},
\]

(iii) if $\beta < 1 - p$,
\[
\|\bar{u}^q\|_{kp,B(h')} \leq c \{\mu(B)\}^{(1-k)/kp} R(h - h')^{-1}(1 + |q|)^p \|\bar{u}^q\|_{p,B(h)},
\]
where $c$ depends only on $p$, $\alpha_1$, $\alpha_2$, $c_\mu$ and $\alpha'R^{p-1}$.

Putting $r = pq = p + \beta - 1$ in (13) and (14), combining the result in a single inequality, we obtain
\[
\left( \int_{B(h')} \bar{u}^{kr} d\mu \right)^{1/kr} \leq \left\{ c \{\mu(3B)\}^{(1-k)/kp} R(h - h')(1 + |\beta|^{-1})(1 + r)^p \right\}^{p/r} \times \left( \int_{B(h)} \bar{u}^r d\mu \right)^{1/r},
\]
for all $0 < r \neq p - 1$. Let
\[
r_\nu = k^\nu r_0' \quad \nu = 0, 1, 2, \ldots,
\]
and $h_\nu = R(1 + 2^{-\nu})$, $h_{\nu+1}' = h_{\nu+1}$, where $r_0' \leq r_0$ is so chosen that $r_\nu \neq p - 1$ for any $\nu = 0, 1, 2, \ldots$. Thus
\[
|\beta| = |r - (p - 1)| \geq c > 0,
\]
whenever $r = r_\nu$, where $c$ depends only on $p$, $k$, $r_0$. The term $(1 + |\beta|^{-1})$ in (16) can thus be absorbed into the general constant $c$. Hence from (16) we have that
\[
\left( \int_{B(h_\nu')} \bar{u}^{r_{\nu+1}} d\mu \right)^{1/r_{\nu+1}} \leq \left\{ c \{\mu(3B)\}^{(1-k)/kp} 2^{r_{\nu+1}} (1 + r_{\nu})^p \right\}^{p/r_{\nu}} \left( \int_{B(h_\nu)} \bar{u}^{r_{\nu}} d\mu \right)^{1/r_{\nu}}.
\]
By iterating, it follows that

$$\int_{B(h')} \overline{u}^{s} d\mu + 1 \geq c_{1}^{-1} \{\mu(3B)\}^{-1/r_{0}'} \left( \int_{2B} \overline{u}^{r_{0}'} d\mu \right)^{-1/r_{0}'}$$

Let $s_{\nu} = -k^{\nu} r_{0}, h_{\nu} = R(1 + 2^{-\nu})$ and $h_{\nu}' = h_{\nu+1}$. Then

$$\int_{B(h_{\nu})} \overline{u}^{s_{\nu+1}} d\mu \geq c_{1}^{-1/k^{\nu}} c_{2}^{-\nu/k^{\nu}} \{\mu(3B)\}^{-1/k^{\nu}} \left( \int_{B(h_{\nu})} \overline{u}^{s_{\nu}} d\mu \right)^{1/s_{\nu}}.$$

By iterating, we obtain

$$\int_{B(h') \cup B(h_{\nu})} \overline{u}^{s} d\mu \geq c_{1}^{-1} \{\mu(3B)\}^{-1/r_{0}'} \left( \int_{2B} \overline{u}^{r_{0}'} d\mu \right)^{-1/r_{0}'}$$

Finally, by (12), (17), (18), and a simple application of Hölder's inequality, we have

$$\int_{B(h_{\nu})} \overline{u}^{s_{\nu+1}} d\mu \geq c_{1}^{-1/k^{\nu}} c_{2}^{-\nu/k^{\nu}} \{\mu(3B)\}^{-1/k^{\nu}} \left( \int_{B(h_{\nu})} \overline{u}^{s_{\nu}} d\mu \right)^{1/s_{\nu}}.$$

Since $\overline{u} = u + R$, this concludes the proof of Theorem 4.2. 

We apply Theorem 4.4 to show that any solutions of (2) has Hölder continuous representative.

**Theorem 4.7** Let $u$ be a solution of (2) in $\Omega$ and $x_{0}$ be any point of $\Omega$. If $0 < R < \infty$ is such that $B(x_{0}, R) \subset \Omega$ and if $|u| \leq L$ a.e in $B(x_{0}, R)$, then there are constants $c$ and $0 < \lambda < 1$ such that

$$\int_{B(x_{0}, \rho)} u - \int_{B(x_{0}, \rho)} u \leq c \left( \frac{\rho}{R} \right)^{\lambda},$$

whenever $0 < \rho < R$. Here $c$ and $\lambda$ depend only on $n, p, \alpha_{1}, \alpha_{2}, \alpha_{3}, c_{\mu}, R$ and $L$.

**Proof:** We write $B(r) = B(x_{0}, r)$ and

$$M(r) = \int_{B(r)} u, \quad m(r) = \int_{B(r)} u.$$

Then $M(r)$ and $m(r)$ are well defined for $0 < r \leq R$, and

$$\overline{u} = M(r) - u, \quad \tilde{u} = u - m(r)$$
are non-negative in $B(r)$. Obviously $\bar{u}$ is a solution of

$$-\text{div} \bar{A}(x, \nabla \bar{u}) + \bar{B}(x, \bar{u}) = 0$$

where $\bar{A}(x, \bar{h}) = -A(x, -\bar{h})$ and $\bar{B}(x, \bar{t}) = -B(x, M(r) - \bar{t})$. Thus

$$|\bar{B}(x, \bar{t})| \leq \alpha_3 \omega(x)(|\bar{t}|^{p-1} + 1),$$

where $\alpha_3$ is a constant depending only on $\alpha_3, p$ and $L$. By applying Harnack inequality to $\bar{u}$, we have

(19) $M(r) - m(r/3) = \text{ess sup}_{B(r/3)} \bar{u} \leq c(\text{ess inf}_{B(r/3)} \bar{u} + r) = c \{M(r) - M(r/3) + r\}$.

Similarly we have

(20) $M(r/3) - m(r) = \text{ess sup}_{B(r/3)} \bar{u} \leq c(\text{ess inf}_{B(r/3)} \bar{u} + r) = c \{m(r/3) - m(r) + r\}$.

Here $c > 1$ depends on $n, p, \alpha_1, \alpha_2, \alpha_3, c_\mu, R$ and $L$. By (19) and (20),

(21) $M(r/3) - m(r/3) \leq \frac{c - 1}{c + 1} \{M(r) - m(r)\} + \frac{2c}{c + 1} r$.

Thus setting

$$\theta = \frac{c - 1}{c + 1}, \quad \tau = \frac{2cR}{c - 1}$$

and

$$\omega = M(r) - m(r),$$

(21) can be written as

$$\omega(r/3) \leq \theta \{\omega(r) + \tau (r/R)\}.$$ 

Since $\omega(r)$ is an increasing function, for any number $s \geq 3$ we have also

$$\omega(r/s) \leq \theta \{\omega(r) + \tau (r/R)\}, \quad 0 < r \leq R.$$ 

By iterating, we obtain

(22) $\omega(R/s^\nu) \leq \theta^\nu \{\omega(R) + \tau \{1 + (\theta s)^{-1} + \cdots + (\theta s)^{-\nu+1}\}\}$,

for $\nu = 1, 2, 3, \cdots$. Let $s$ be so chosen that $\theta s = 3$. Then (22) implies

(23) $\omega(R/s^\nu) \leq \theta^\nu \{\omega(R) + 2\tau\}$.

For any $\rho$ such that $0 < \rho \leq R/s$ choose $\nu$ such that $R/s^{\nu+1} < \rho \leq R/s^\nu$. Then from (23) we have

(24) $\omega(\rho) \leq \omega(R/s^\nu) \leq \theta^\nu \{\omega(R) + 2\tau\}$.

If we set $\gamma = -\log_3 \theta$, then we have $\theta = s^{-\lambda}$ where $\lambda = \gamma/(\gamma + 1) > 0$. Thus

$$\theta^\nu = \left(\frac{R}{s^{\nu+1} R}\right)^\lambda \leq c \left(\frac{\rho}{R}\right)^\lambda.$$
Hence, since $\omega(R) + 2\tau \leq c(L + R)$, (22) implies
\[
\omega(\rho) \leq c(L + R) \left( \frac{\rho}{R} \right)^{\lambda}, \quad (\rho < R),
\]
as desired. \qed

§5. A regularity at the boundary for solutions

In this section, we are concerned with the continuity of solutions at the boundary. First, we recall the definition of the $(p, \mu)$-capacity which is adopted in [1]. Suppose that $K$ is a compact subset of $\Omega$. Let
\[
\mathcal{W}(K, \Omega) = \{ u \in C^\infty_0(\Omega) : u \geq 1 \text{ on } K \}
\]
and define
\[
\text{cap}_{p, \mu}(K, \Omega) = \inf_{u \in \mathcal{W}(K, \Omega)} \int_\Omega |\nabla u|^p d\mu.
\]
Further, if $U \subset \Omega$ is open, set
\[
\text{cap}_{p, \mu}(U, \Omega) = \sup_{K \subset U \text{ compact}} \text{cap}_{p, \mu}(K, \Omega),
\]
and, finally, for an arbitrary set $E \subset \Omega$
\[
\text{cap}_{p, \mu}(E, \Omega) = \inf_{E \subset U \subset \Omega} \text{cap}_{p, \mu}(U, \Omega).
\]
The number $\text{cap}_{p, \mu}(E, \Omega) \in [0, \infty]$ is called the $(p, \mu)$-capacity of the condenser $(E, \Omega)$. If $u \in H^{1, p}_{\text{loc}}(\Omega; \mu)$, $x_0 \in \partial\Omega$, and $l \in \mathbb{R}$ we say that
\[
(25) \quad u(x_0) \leq l \quad \text{weakly}
\]
if for every $k > l$ there is an $r > 0$ such that $\eta(u - k)^+ \in H^{1, p}_0(\Omega; \mu)$ whenever $\eta \in C^\infty_0(B(x_0, r))$. The condition
\[
(26) \quad u(x_0) \geq l \quad \text{weakly}
\]
is defined analogously and $u(x_0) = l$ weakly if both (25) and (26) hold. Observe that if $f$ is a continuous function on $\mathbb{R}^n \setminus \Omega$, $f \in H^{1, p}_{\text{loc}}(\mathbb{R}^n; \mu)$, and $u - f \in H^{1, p}_0(\Omega; \mu)$, then $u(x) = f(x)$ weakly for every $x \in \partial\Omega$.

**Lemma 5.1** Suppose that $u \in H^{1, p}_{\text{loc}}(\Omega; \mu)$ is a subsolution of (2) in $\Omega$, that $u \leq L$ a.e. in $\Omega$, and that $u(x_0) \leq l$ weakly for $x_0 \in \partial\Omega$. For $k > l$, let
\[
u_k = \begin{cases} 
(u - k)^+ & \text{on } \Omega \\
0 & \text{otherwise}
\end{cases}
\]
and define
\[
M(r) = \text{ess sup}_{B(x_0, r)} \nu_k.
\]
Choose $r_0 > 0$ so small that $\eta u_k \in H^{1, p}_0(\Omega; \mu)$ whenever $\eta \in C^\infty_0(B(x_0, r_0))$. 
Then there is a constant $c$ depending only on $n, p, l, r_0, \alpha_1, \alpha_2, \alpha_3, c_\mu$ and $L$ such that
\[
\int_{B(x_0,r/2)} |\nabla(v^{-1})|^p \mu \leq c(M(r) + r)(M(r) - M(r/2) + r)^{p-1}\mu(B(x_0, r))r^{-p}
\]
where $0 < r \leq r_0/2$, $v^{-1} = M(r) + r - u_k$ and $\eta \in C_0^\infty(B(x_0, r/2))$ with $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq 5/r$.

Before proving Lemma 5.1, we will state its implication.

**Theorem 5.2** Let $u \in H_{loc}^{1,p}(\Omega; \mu)$ be a subsolution of (2) which is bounded above on $\Omega$, $x_0 \in \partial \Omega$, and $u(x_0) \leq l$ weakly. If
\[
(27) \quad \int_0^1 \left( \frac{\text{cap}_{p,\mu}(B(x_0, t) \setminus \{u_k = 0\}, B(x_0, 2t))}{\text{cap}_{p,\mu}(B(x_0, r/4), B(x_0, r/2))} \right)^{1/(p-1)} \frac{dt}{t} = \infty,
\]
then
\[
\text{ess lim sup } u(x) \leq l.
\]

Proof: Since, for any $k > l$, it follows immediately from Theorem 5.1, the definition of $(p, \mu)$-capacity and [1, Lemma 2.14] that
\[
(M(r) + r)\left(\frac{\text{cap}_{p,\mu}(B(x_0, r/4) \cap \{u_k = 0\}, B(x_0, r/2))}{\text{cap}_{p,\mu}(B(x_0, r/4), B(x_0, r/2))}\right)^{1/(p-1)} \leq c(M(r) - M(r/2) + r),
\]
the theorem is proved in the same manner as in the proof of [4, Theorem 2.2].

If $u$ is a supersolution of (2), then $-u$ is a subsolution of
\[-\text{div} \tilde{A}(x, \nabla v) + \tilde{B}(x, v) = 0,
\]
where $\tilde{A}(x, h) = -A(x, -h)$ and $\tilde{B}(x, t) = -B(x, -t)$. Consequently, Theorem 5.2 has the obvious counterpart for supersolutions of (2). These results yield

**Theorem 5.3** Let $u \in H_{loc}^{1,p}(\Omega; \mu)$ be a bounded solution of (2), that $x_0 \in \partial \Omega$, and that $u(x_0) = l$ weakly. If (27) holds, then
\[
\lim_{x \to x_0} u(x) = l.
\]

Proof of Lemma 5.1: Fix $r > 0$ so that $0 < r \leq r_0/2$, let $\eta \in C_0^\infty(B(x_0, r/2))$ with $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq 5/r$. Set
\[
I(r) = (M(r) + r)(M(r) - M(r/2) + r)^{p-1}\mu(B(x_0, r))r^{-p}.
\]
Since
\[
\int |\nabla (\eta v^{-1})|^p d\mu \leq c \left( \int \eta^p |\nabla u_k|^p d\mu + \int v^{-p} |\nabla \eta|^p d\mu \right),
\]
we will show that
\[
\int \eta^p |\nabla u_k|^p d\mu \leq c I(r) \quad \text{and} \quad \int v^{-p} |\nabla \eta|^p d\mu \leq c I(r),
\]
by using following two estimates.

**Estimate 1**  For $(1-p)/p < \alpha \neq 0$
\[
(c m(\alpha))^{-1} \int_{B(x_0, r)} |\nabla (\omega v^\alpha)|^p d\mu \leq \int_{B(x_0, r)} v^\alpha \{ (\omega v)^p + |\nabla \omega|^p \} d\mu,
\]
whenever $\omega \in C_0^\infty(B(x_0, r))$ with $0 \leq \omega \leq 1$, where $c$ is a constant depending on $p$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $l$, $r_0$, and $L$, and
\[
0 < m(\alpha) < 1 + \alpha^p \quad \text{if} \quad \alpha > 0,
\]
\[
m(\alpha) > 0 \quad \text{and} \quad \text{a decreasing function of} \quad \alpha \quad \text{if} \quad (1-p)/p < \alpha < 0.
\]

**Estimate 2**  For $0 < \sigma < p-1$,
\[
\mu(B(x_0, r))^{-1} \left\| v^{-\sigma k} \right\|_{1, B(x_0, r/2)} \leq c(M(r) - M(r/2) + r)^{\sigma k},
\]
where $c$ is a constant depending on $p$, $n$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $l$, $r_0$, $L$ and $\sigma$.

Let us suppose that Estimate 1 and Estimate 2 are true. Fix $\alpha < 0$ such that $1 < (1+\alpha)p < k$, then putting $B = B(x_0, r/2)$, we have
\[
\int_B \eta^{p-1} |\nabla u_k|^{p-1} |\nabla \eta| d\mu = \int_B (\eta v^{1+\alpha} |\nabla u_k|)^{p-1} (v^{-1}(1+\alpha)(p-1) |\nabla \eta|) d\mu
\]
\begin{align*}
= & \ c \int_B (\eta |\nabla v|)^{p-1} (v^{-1}(1+\alpha)(p-1) |\nabla \eta|) d\mu \\
\leq & \ c \left( \int_B (\eta |\nabla v|)^p d\mu \right)^{(p-1)/p} \left( \int_B (v^{-1}(1+\alpha)(p-1) |\nabla \eta|)^p d\mu \right)^{1/p} \\
\leq & \ c \left\{ \left( \int_B |\nabla (\eta v^\alpha)|^p d\mu \right)^{1/p} + \left( \int_B |v^\alpha \nabla \eta|^p d\mu \right)^{1/p} \right\}^{p-1} \\
\times & \left( \int_B (v^{-1}(1+\alpha)(p-1) |\nabla \eta|)^p d\mu \right)^{1/p} \\
\leq & \ c \left( r^{-p} \int_B v^{\alpha p} d\mu \right)^{(p-1)/p} \left( \int_B (v^{-1}(1+\alpha)(p-1) |\nabla \eta|)^p d\mu \right)^{1/p} \\
\leq & \ c \left\{ (M(r) - M(r/2) + r)^{-\alpha p} \mu(B(x_0, r)) r^{-p} \right\}^{(p-1)/p} \\
\times & \left\{ (M(r) - M(r/2) + r)^{(1+\alpha)(p-1) p} \mu(B(x_0, r)) r^{-p} \right\}^{1/p} \\
= & \ c (M(r) - M(r/2) + r)^{(p-1) \mu(B(x_0, r)) r^{-p}},
\end{align*}
in the last inequality we have used Estimate 2 with $\sigma = -\alpha p / k$ and $\sigma = (1+\alpha)(p-1) p / k$ respectively. Also since $\eta \leq 1$,
\begin{align*}
(29) \quad & \int_B \eta^p d\mu \leq \mu(B(x_0, r)) \leq c I(r).
\end{align*}
Hence, by (28) and (29),

\[(30) \int_{B} \eta^{p} \nabla u_{k}^{p} d\mu \leq c (\int_{B} \eta^{p} d\mu + M(r) \int_{B} \eta^{p-1} |\nabla u_{k}|^{p-1} |\nabla \eta| d\mu) \leq c I(r).\]

Here the first inequality has been obtained by using the facts that \(\varphi = \eta^{p} u_{k} \in H^{1,p}_{0}(\Omega; \mu)\), \(\varphi\) is nonnegative, \(u\) is a subsolution and the structure of \(A\) and \(B\). From Estimate 2 with \(\sigma = (p - 1)/k\) again,

\[(31) \int_{B} |v^{-1} \nabla \eta|^{p} d\mu \leq c r^{-p} (M(r) + r) \int_{B} v^{-p+1} d\mu \leq c I(r).\]

Therefore we obtain from (30) and (31)

\[\int_{B} |\nabla (\eta v - 1)|^{p} d\mu \leq c I(r).\]

Finally, we will prove Estimate 1 and Estimate 2. For \(\beta > 0\), let

\[\psi = v^{\beta} - (M(r) + r)^{-\beta}\]

and

\[\varphi = \omega^{p} \psi,\]

where \(\omega \in C^{\infty}_{0}(B(x_0, r))\). Then \(\varphi \in H^{1,p}_{0}(\Omega; \mu)\). Since \(\varphi = 0\) on \(\{u_{k} = 0\}\) and \(\varphi \geq 0\) on \(\Omega\),

\[\int \beta \omega^{p} v^{\beta + 1} A(x, \nabla u) \cdot \nabla u_{k} d\mu + \int p \omega^{p-1} \psi A(x, \nabla u) \cdot \nabla \omega d\mu + \int B(x, u) \varphi d\mu \leq 0,\]

where the integrals are taken over \(B(x_0, r) \cap \{u_{k} > 0\}\). Hereafter we will suppress explicit indication of this domain of integration.

Using (a2), (a3) and (b2) we have

\[c_{1} \beta \int \omega^{p} v^{\beta+1} |\nabla u_{k}|^{p} d\mu \leq \int \omega^{p-1} \psi |\nabla u_{k}|^{p-1} |\nabla \omega| d\mu + c_{2} \int \omega^{p} \psi (|u|^{p-1} + 1) d\mu.\]

Since \(\psi \leq v^{\beta}\), \(v^{-1} \leq M(r_{0}) + r_{0}\) and \(l \leq u \leq L\), we obtain

\[(32) \quad c^{-1} \beta \int \omega^{p} v^{\beta+1} |\nabla u_{k}|^{p} d\mu \leq \int \omega^{p-1} \psi |\nabla u_{k}|^{p-1} |\nabla \omega| d\mu + \int \omega^{p} v^{\beta+1} d\mu,\]

where \(c\) depends on \(p, \alpha_{1}, \alpha_{2}, \alpha_{3}, r_{0}, L\). Application of Young’s inequality yields that

\[\int \omega^{p-1} v^{\beta} |\nabla u_{k}|^{p-1} |\nabla \omega| d\mu \leq \epsilon^{p/(p-1)} (p - 1) \int \omega^{p} v^{\beta+1} |\nabla u_{k}|^{p} d\mu + \epsilon^{-p/(p-1)} \int v^{\beta-1} |\nabla \omega|^{p} d\mu,\]

for any \(\epsilon > 0\). By the above inequality and (32), with an appropriate choice for \(\epsilon\), we have

\[(33) \quad c^{-1} \beta \int \omega^{p} v^{\beta+1} |\nabla u_{k}|^{p} d\mu \leq \int \omega^{p} v^{\beta+1} d\mu + \beta^{-p} \int v^{\beta-1} |\nabla \omega|^{p} d\mu.\]
By letting $\beta = p\alpha + p - 1$ with $0 < \beta \neq p - 1$, we obtain Estimate 1. Next we prove Estimate 2. In (33) letting $\beta = p - 1$,
\[
\int \omega^p|\nabla(\log v)|^p \leq c\{(p - 1)^{-1} \int \omega^p v^p d\mu + (p - 1)^{-p} \int |\nabla \omega|^p d\mu\}.
\]
Since, by using $v \leq 1/r$ and Sobolev inequality,
\[
\int \omega^p v^p d\mu \leq r^{-p} \mu(B(x_0, r))^{(k-1)/k} \left( \int \omega^p d\mu \right)^{1/k} \leq c \int |\nabla \omega|^p d\mu,
\]
we have
\[
\int \omega^p|\nabla(\log v)|^p \leq c \int |\nabla \omega|^p d\mu
\]
whenever $0 \leq \omega \in C_0^\infty(B(x_0, r))$. Using Lemma 4.4 (John-Nirenberg lemma) in the same manner as in the proof of Lemma 4.5 and Theorem 4.2, it follows that there are positive constants $c$ and $\sigma_0$ such that
\[
\int_{B(x_0, s)} v^{-\sigma} d\mu \int_{B(x_0, s)} v^\sigma d\mu \leq c \{\mu(B(x_0, s))\}^2,
\]
whenever $\sigma \leq \sigma_0$ and $0 < s \leq 3r/4$.

Let $r_j = r(2^{-1} + 2^{-j-2})$ for $j = 0, 1, \cdots$. Then since $m(\alpha_0 k^j) \leq c\{k^p\}^j$ for $0 < \alpha_0 \leq \sigma_0^{-1}$, (35) yields that
\[
\left( \int_{B(x_0, r_j+1)} |v^{\alpha_0 k^j} p^p d\mu \right)^{1/k} \leq c \left( \int_{B(x_0, r_j)} v^{\alpha_0 k^j} p^p d\mu \right)^{(1-k)/k} \left( \int_{B(x_0, r_j)} v^p d\mu \right)^{-p} \int_{B(x_0, r_j)} v^{\alpha_0 k^j} p^p d\mu,
\]
and hence
\[
\|v^{\alpha_0 k^j}\|_{p, B(x_0, r_j+1)} \leq c \left\{ \mu(B(x_0, r_j))^{1-k/k} \right\}^{k^{-j}} \left( \int_{B(x_0, r)} v^{\alpha_0 k^j} p^p d\mu \right)^{(1-k)/k} \|v^{\alpha_0 k^j}\|_{1,3r/4}
\]
for $j = 0, 1, \cdots$. Hereafter, for simplicity, we shall write $\|\cdot\|_{p, r}$ for $\|\cdot\|_{p, B(x_0, r)}$. By iterating, we have
\[
(M(r) - M(r/2) + r)^{-p_0} \leq c \left\{ \mu(B(x_0, r)) \right\}^{-1} \|v^{\alpha_0 k^j}\|_{1,3r/4},
\]
whenever $0 < p\alpha_0 \leq \sigma_0$. From (34) and (36), we obtain that
\[
\mu(B(x_0, r))^{-1} \|v^{\alpha_0 k^j}\|_{1,3r/4} \leq c \left( M(r) - M(r/2) + r \right)^{-p_0}
\]
whenever $0 < p\alpha_0 \leq \sigma_0$.

Return to (35) with $1 - p < \alpha_0 < 0$. Let $0 < \sigma < p - 1$ and let $j_0$ is a positive integer such that $p - 1 \leq \sigma_0 k^{j_0}$. Put $\sigma_1 = \sigma k^{j_0}$. Since $0 < \sigma_1 k^j \leq \sigma < p - 1$ for $0 \leq j \leq j_0$, $m(\sigma_1 k^{j-1}) \leq \sigma_0 k^{j_0}$ for $0 \leq j \leq j_0$. 

164
Let $r_j = (r/4)\{3 - j/(j_0 + 1)\}$ for $0 \leq j \leq j_0 + 1$. Then (35) yields that

$$\|v^{-\sigma_1}\|_{k^j+1,r_j+1} \leq \left[ c m(-\sigma p^{-1}) \{\mu(B(x_0, r))\}^{(1-k)/k} \left\{4(j_0 + 1)\right\}^p \right]^{k-j} \|v^{-\sigma_1}\|_{k^j,r_j}. $$

By iterating for $0 \leq j \leq j_0$, we have

$$\mu(B(x_0, r))^{-1}\|v^{-\sigma_1}\|_{k^{j_0+1},r/2} \leq \left[ c m(-\sigma p^{-1}) \{4(j_0 + 1)\}^p \right]^{k(j_0+1-1)} \times \left[ \mu(B(x_0, r))^{-1}\|v^{-\sigma_1}\|_{1,3r/4} \right]^{k^{j_0+1}}.$$

Since $0 < \sigma_1 < \sigma_0$, from (37) we obtain Estimate 2.

Hence Lemma 5.1 follows. \qed

References


