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On solutions of quasi-linear partial differential equations
\[-\text{div} A(x, \nabla u) + B(x, u) = 0\]

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§0. Introduction
Recently, a nonlinear potential theory has been developed in [1] for quasi-linear elliptic partial differential equations of second order of the form
\[-\text{div} A(x, \nabla u) = 0,\]
where $A$ is a mapping of $R^n \times R^n$ to $R^n (n \geq 2)$ satisfying a growth condition $A(x, h) \cdot h \approx w(x)|h|^p (1 < p < \infty)$ with a "weight" $w(x)$, which is a nonnegative locally integrable function in $R^n$. A prototype is the so-called weighted $p$-Laplace equations
\[-\text{div}(w(x)|\nabla u|^{p-2}\nabla u) = 0,\]

This purpose of this paper is to extend some of the results in [1] to the equation
\[(*)\]
\[-\text{div} A(x, \nabla u) + B(x, u) = 0,\]
where $B(x, t)$ is a mapping of $R^n \times R$ to $R$, which is non-decreasing in $t$. A prototype equation may be given by
\[-\text{div}(w(x)|\nabla u|^{p-2}\nabla u) + w(x)|u|^{p-2}u = 0.\]

As a matter of fact, we treat the following three topics: (i) Existence and uniqueness of solutions of Dirichlet problems for equation $(*)$ with Sobolev boundary values, or more generally of obstacle problems (section 3); (ii) Harnack inequality and Hölder continuity for solutions of $(*)$ (section 4); (iii) Regularity at the boundary for solutions of $(*)$ (section 5).

We can discuss (i) in the same way as in [1, Appendix I], using a general result of monotone operators. For (ii) and (iii), the methods in [1] are no longer applicable. We follow the discussion in [2] (for (ii)) and those in [4] (for (iii)), in which the unweighted case, namely the case $w = 1$, is treated.

§1. Weighted Sobolev space
We recall the weighted Sobolev spaces $H^{1,p}(\Omega; \mu)$ which are adopted in [1].

Throughout this paper $\Omega$ will denote an open subset of $R^n (n \geq 2)$ and $1 < p < \infty$. We denote $B(x, r) = \{y \in R^n : |x - y| < r\}$, and $\lambda B = B(x, \lambda r)$ if $B = B(x, r)$ and $\lambda > 0$.

Let $w$ be a locally integrable, nonnegative function in $R^n$. Then a Radon measure $\mu$ is canonically associated with the weight $w$:

\[\mu(E) = \int_E w(x)dx.\]
Thus $d\mu(x) = w(x)dx$, where $dx$ is the $n$-dimensional Lebesgue measure. We say that $w$ (or $\mu$) is $p$-admissible if the following four conditions are satisfied:

I. $0 < w < \infty$ almost everywhere in $\mathbb{R}^n$ and the measure $\mu$ is doubling, i.e., there is a constant $C_I > 0$ such that

$$\mu(2B) \leq C_I \mu(B)$$

whenever $B$ is a ball in $\mathbb{R}^n$.

II. If $D$ is an open set and $\varphi_i \in C_0^\infty(D)$ is a sequence of functions such that $\int_D |\varphi_i|^p d\mu \to 0$ and $\int_D |\nabla \varphi_i - v|^p d\mu \to 0 (i \to \infty)$, where $v$ is a vector-valued measurable function in $L^p(D; \mu; \mathbb{R}^n)$, then $v = 0$.

III. (Sobolev inequality) There are constants $k > 1$ and $C_{III} > 0$ such that

$$\left( \frac{1}{\mu(B)} \int_B |\varphi|^{kp} d\mu \right)^{1/kp} \leq C_{III} r \left( \frac{1}{\mu(B)} \int_B |\nabla \varphi|^p d\mu \right)^{1/p}$$

whenever $B = B(x_0, r)$ is a ball in $\mathbb{R}^n$ and $\varphi \in C_0^\infty(B)$.

IV. There is a constant $C_{IV} > 0$ such that

$$\int_B |\varphi - \varphi_B|^p d\mu \leq C_{IV} r^p \int_B |\nabla \varphi|^p d\mu$$

whenever $B = B(x_0, r)$ is a ball in $\mathbb{R}^n$ and $\varphi \in C^\infty(B)$ is bounded. Here

$$\varphi_B = \frac{1}{\mu(B)} \int_B \varphi d\mu.$$

From now on, unless otherwise stated, we assume that $\mu$ is a $p$-admissible measure and $d\mu(x) = w(x)dx$.

In this paper, both condition IV and the following inequality are called the Poincaré inequality.

**Poincaré inequality** ([1, p.9])

If $\Omega$ is bounded, then

$$\int_{\Omega} |\varphi|^p d\mu \leq C_{III}^p (\text{diam } \Omega)^p \int_{\Omega} |\nabla \varphi|^p d\mu$$

for $\varphi \in C_0^\infty(\Omega)$.

Throughout this paper, let $c_\mu$ denote constants depending on $C_I$, $C_{II}$, $C_{III}$, $k$ and $C_{IV}$.

For a $\mu$-measurable function $f$ defined on an open set $\Omega$, $L^p$-norm of $f$ is defined by

$$\|f\|_{p, \Omega} = \left( \int_{\Omega} |f|^p d\mu \right)^{1/p}.$$

For a function $\varphi \in C^\infty(\Omega)$ we let

$$\|\varphi\|_{1,p, \Omega} = \left( \int_{\Omega} |\varphi|^p d\mu \right)^{1/p} + \left( \int_{\Omega} |\nabla \varphi|^p d\mu \right)^{1/p},$$
where, we recall, $\nabla \varphi = (\partial_1 \varphi, \cdots, \partial_n \varphi)$ is the gradient of $\varphi$. The Sobolev space $H^{1,p}(\Omega; \mu)$ is defined to be the completion of

$$\{ \varphi \in C^\infty(\Omega) : ||\varphi||_{1,p;\Omega} < \infty \}$$

with respect to norm $|| \cdot ||_{1,p;\Omega}$. In other words, a function $u$ is in $H^{1,p}(\Omega; \mu)$ if and only if $u$ is in $L^p(\Omega; \mu)$ and there is a vector-valued function $v$ in $L^p(\Omega; \mu; \mathbb{R}^n)$ such that for some sequence $\varphi_i \in C^\infty(\Omega)$

$$\int_{\Omega} |\varphi_i - u|^p d\mu \to 0$$

and

$$\int_{\Omega} |\nabla \varphi_i - v|^p d\mu \to 0$$
as $i \to \infty$. The function $v$ is called the gradient of $u$ in $H^{1,p}(\Omega; \mu)$ and denoted by $\nabla u$.

The space $H^1_0(\Omega; \mu)$ is the closure of $C_0^\infty(\Omega)$ in $H^{1,p}(\Omega; \mu)$. The corresponding local space $H^{1,p}_{loc}(\Omega; \mu)$ is defined in the obvious manner.

§2. Quasilinear PDE's

$A$ is a mapping of $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$ satisfying the following assumptions for some constants $0 < \alpha_1 \leq \alpha_2 < \infty$:

(a1) the mapping $x \mapsto A(x, h)$ is measurable for all $h \in \mathbb{R}^n$ and

the mapping $h \mapsto A(x, h)$ is continuous for a.e. $x \in \mathbb{R}^n$;

for all $h \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$

(a2) $A(x, h) \cdot h \geq \alpha_1 w(x)|h|^p$,

(a3) $|A(x, h)| \leq \alpha_2 w(x)|h|^{p-1}$,

(a4) $(A(x, h_1) - A(x, h_2)) \cdot (h_1 - h_2) > 0$

whenever $h_1, h_2 \in \mathbb{R}^n$, $h_1 \neq h_2$.

$B$ is a mapping of $\mathbb{R}^n \times R$ to $\mathbb{R}$ satisfying the following assumptions for a constant $0 < \alpha_3 < \infty$:

(b1) the mapping $x \mapsto B(x, t)$ is measurable for all $t \in \mathbb{R}$ and

the mapping $t \mapsto B(x, t)$ is continuous for a.e. $x \in \mathbb{R}^n$;

for all $t \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^n$

(b2) $|B(x, t)| \leq \alpha_3 w(x)(|t|^{p-1} + 1),$

(b3) $(B(x, t_1) - B(x, t_2))(t_1 - t_2) \geq 0.$

whenever $t_1, t_2 \in \mathbb{R}^n$. Using $A$ and $B$ we consider the quasilinear elliptic equation

(2) $-\text{div}A(x, \nabla u) + B(x, u) = 0.$
A function \( u \in H^{1,p}_{\text{loc}}(\Omega; \mu) \) is a (weak) solution of (2) if
\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx + \int_{\Omega} B(x, u) \varphi \, dx = 0
\]
whenever \( \varphi \in C^\infty_0(\Omega) \). A function \( u \in H^{1,p}_{\text{loc}}(\Omega; \mu) \) is a supersolution of (2) in \( \Omega \) if
\[
-\text{div} A(x, \nabla u) + B(x, u) \geq 0
\]
weakly in \( \Omega \), i.e.
\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx + \int_{\Omega} B(x, u) \varphi \, dx \geq 0
\]
whenever \( \varphi \in C^\infty_0(\Omega) \) is nonnegative. A function \( u \in H^{1,p}_{\text{loc}}(\Omega; \mu) \) is a subsolution in \( \Omega \) if (4) holds for all nonpositive \( \varphi \in C^\infty_0(\Omega) \).

**Lemma 2.1** If \( u \in H^{1,p}(\Omega; \mu) \) is a solution (respectively, a supersolution) of (2) in \( \Omega \), then
\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx + \int_{\Omega} B(x, u) \varphi \, dx = 0 \quad (\text{respectively,} \geq 0)
\]
for all \( \varphi \in H^{1,p}_{0}(\Omega; \mu) \) (respectively, for all nonnegative \( \varphi \in H^{1,p}_{0}(\Omega; \mu) \) ) with compact support.

**Proof:** Let \( \Omega' \) be an open set such that \( \text{spt} \varphi \subset \Omega' \subset \subset \Omega \). Since \( \varphi \in H^{1,p}_{0}(\Omega'; \mu) \), we can choose a sequence of functions \( \varphi_i \in C^\infty_0(\Omega') \) such that \( \varphi_i \to \varphi \) in \( H^{1,p}_{0}(\Omega'; \mu) \). If \( \varphi \) is nonnegative, pick nonnegative functions \( \varphi_i \) ([1, Lemma 1.23, p.21]). Then by (a3)
\[
\left| \int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx + \int_{\Omega} B(x, u) \varphi \, dx - \left( \int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi_i \, dx + \int_{\Omega} B(x, u) \varphi_i \, dx \right) \right|
\leq \alpha_2 \int_{\Omega'} |u|^{p-1} |\nabla \varphi - \nabla \varphi_i| \, d\mu + \alpha_3 \int_{\Omega'} (|u|^{p-1} + 1)|\varphi - \varphi_i| \, d\mu
\leq \alpha_2 \left( \int_{\Omega'} |u|^{p} \, d\mu \right)^{(p-1)/p} \left( \int_{\Omega'} |\nabla \varphi - \nabla \varphi_i| \, d\mu \right)^{1/p}
+ \alpha_3 \left( \int_{\Omega'} (|u| + 1)^p \, d\mu \right)^{(p-1)/p} \left( \int_{\Omega'} |\varphi - \varphi_i| \, d\mu \right)^{1/p}.
\]
Because the last integral tends to zero as \( i \to 0 \), we have
\[
\int_{\Omega} A(x, \nabla u) \cdot \nabla \varphi \, dx + \int_{\Omega} B(x, u) \varphi \, dx = \lim_{i \to \infty} \left( \int_{\Omega'} A(x, \nabla u) \cdot \nabla \varphi_i \, dx + \int_{\Omega'} B(x, u) \varphi_i \, dx \right) = (\geq) 0,
\]
and the lemma follows. \( \square \)

The proof of Lemma 2.1 implies that (5) holds for all (nonnegative) \( \varphi \in H^{1,p}_{0}(\Omega; \mu) \) if \( \Omega \) is bounded.

A function \( u \) is a solution of (2) if and only if \( u \) is a supersolution and a subsolution. Indeed, if \( u \) is a supersolution and a subsolution of (2), since the positive part \( \varphi^+ \) of a test function \( \varphi \in C^\infty_0(\Omega) \), belongs \( H^{1,p}_{0}(\Omega; \mu) \) and has compact support, \( u \) satisfies (3) for \( \varphi^+ \). Similarly, \( u \) satisfies (3) for the negative part of \( \varphi \). Hence \( u \) is a solution of (2).
§3. The existence of solutions

In this section, the existence of solutions of Dirichlet problems for equation (2) with Sobolev boundary values will be proved, using a general result in the theory of monotone operators.

Let $X$ be a reflexive Banach space with dual $X'$ and let $\langle \cdot , \cdot \rangle$ denote a pairing between $X'$ and $X$. If $K \subset X$ is a closed convex set, then a mapping $\mathfrak{S} : K \to X'$ is called monotone if

$$\langle \mathfrak{S}u - \mathfrak{S}v , u - v \rangle \geq 0$$

for all $u, v \in K$. Further, $\mathfrak{S}$ is called coercive on $K$ if there exists $\varphi \in K$ such that

$$\frac{\langle \mathfrak{S}u_j - \mathfrak{S}\varphi , u_j - \varphi \rangle}{\| u_j - \varphi \|} \to \infty$$

whenever $u_j$ is a sequence in $K$ with $\| u_j \| \to \infty$.

We recall the following proposition. ([3, Corollary III.1.8, p.87]).

**Proposition 3.1** Let $K$ be a nonempty closed convex subset of $X$ and let $\mathfrak{S} : K \to X'$ be monotone, coercive, and weakly continuous on $K$. Then there exists an element $u$ in $K$ such that

$$\langle \mathfrak{S}u , v - u \rangle \geq 0$$

whenever $v \in K$.

Throughout this section, we assume that $\Omega$ is bounded.

Suppose that $\psi$ is any function in $\Omega$ with values in the extended reals $[-\infty, \infty]$, and that $\theta \in H^{1,p}(\Omega; \mu)$. Let

$$\mathcal{K}_{\psi, \theta} = \mathcal{K}_{\psi, \theta}(\Omega) = \{ v \in H^{1,p}(\Omega; \mu) : v \geq \psi \text{ a.e in } \Omega , \ v - \theta \in H^{1,p}_0(\Omega; \mu) \}.$$  

Set $X = L^p(\Omega; \mu; \mathbb{R}^n) \times L^p(\Omega; \mu; \mathbb{R})$ and $K = \{(\nabla v, v) : v \in \mathcal{K}_{\psi, \theta}(\Omega) \}$. Let $\langle \cdot , \cdot \rangle$ be the pairing between $X$ and $X'$,

$$\langle (f, u), (g, v) \rangle = \int_{\Omega} f \cdot g d\mu + \int_{\Omega} uv d\mu ,$$

where $(f, u)$ is in $X$ and $(g, v)$ in $X' = L^{p/(p-1)}(\Omega; \mathbb{R}^n) \times L^{p/(p-1)}(\Omega; \mathbb{R})$.

A mapping $\mathfrak{S} : K \to X'$ is well defined by the formula

$$\langle \mathfrak{S}(\nabla v, v) , (f, u) \rangle = \int_{\Omega} A(x, \nabla v(x)) \cdot f(x) dx + \int_{\Omega} B(x, v(x)) u(x) dx$$

Lemma 3.2 $K$ is a closed convex set in $X$.

Proof : $K$ is clearly convex. To show the closedness, let $(\nabla v_i, v_i) \in K$ be a sequence converging to $(f, u)$ in $X$. By $\nabla v_i \to f$ in $L^p(\Omega; \mu; \mathbb{R}^n)$ and $v_i \to u$ in $L^p(\Omega; \mu; \mathbb{R})$, $v_i$ is a bounded sequence in $H^{1,p}(\Omega; \mu)$. Since $\mathcal{K}_{\psi, \theta}$ is a convex and closed subset of $H^{1,p}(\Omega; \mu)$, there is a function $v \in \mathcal{K}_{\psi, \theta}$ such that $v = u$ and $\nabla v = f$ ([1, Theorem 1.31, p.25]). Thus $(f, u) \in K$. The lemma is proved. \Box

Let $\langle \cdot , \cdot \rangle$ be the pairing between $X$ and $X'$,
for \((f, u) \in X\); indeed, by (a3) and (b2),
\[
|\int_{\Omega}A(x, \nabla v) \cdot f dx| \leq \alpha_{2} \left( \int_{\Omega} |\nabla v|^{p} d\mu \right)^{(p-1)/p} \left( \int_{\Omega} |f|^{p} d\mu \right)^{1/p} \\
|\int_{\Omega} B(x, v) u dx| \leq 2\alpha_{3} \left( \int_{\Omega} (|v| + 1)^{p} d\mu \right)^{(p-1)/p} \left( \int_{\Omega} |u|^{p} d\mu \right)^{1/p}
\]

**Lemma 3.3** \(\mathcal{S}\) is monotone, coercive, and weakly continuous on \(K\).

**Proof**: By (a4) and (b3), \(\mathcal{S}\) is monotone.

Next we show that \(\mathcal{S}\) is coercive on \(K\). Fix \((\nabla \varphi, \varphi) \in K\). Hereafter, for simplicity, we shall write \(\| \cdot \|\) for \(\| \cdot \|_{p, \Omega}\). By (a2), (a3) and (b3)
\[
\langle \mathcal{S}(\nabla u - \nabla \varphi, u - \varphi) \rangle
= \int_{\Omega} (A(x, \nabla u) - A(x, \nabla \varphi)) \cdot (\nabla u - \nabla \varphi) dx + \int_{\Omega} (B(x, u) - B(x, \varphi)) (u - \varphi) dx
\]
\[
\geq \alpha_{1} (\|\nabla u\|^{p} + \|\nabla \varphi\|^{p}) - \alpha_{2} (\|\nabla u\|^{p-1} \|\nabla \varphi\| - \|\nabla u\| \|\nabla \varphi\|^{p-1})
\]
\[
\geq \|\nabla u - \nabla \varphi\|^{p-1} (\|\nabla \varphi\| + \|\nabla u - \nabla \varphi\|).
\]

Since \(u - \varphi \in H_{0}^{1, p}(\Omega; \mu)\),
\[
\|u - \varphi\| \leq c \|\nabla u - \nabla \varphi\|.
\]

By (6) and (7), \(\mathcal{S}\) is coercive on \(K\).

Finally, to show that \(\mathcal{S}\) is weakly continuous on \(K\), let \((\nabla u_{i}, u_{i}) \in K\) be a sequence that converges to an element \((\nabla u, u) \in K\) in \(X\). For any subsequence \((\nabla u_{i_{j}}, u_{i_{j}})\) of \((\nabla u_{i}, u_{i})\), there is a subsequence \((\nabla u_{i_{j}'}, u_{i_{j}'})\) of \((\nabla u_{i_{j}}, u_{i_{j}})\) such that \((\nabla u_{i_{j}'}, u_{i_{j}'}) \rightarrow (\nabla u, u)\) a.e. in \(\Omega\). By (a1) and (b1), we have
\[
A(x, \nabla u_{i_{j}'})w^{-1/p}(x) \rightarrow A(x, \nabla u(x))w^{-1/p}(x)
\]
\[
B(x, u_{i_{j}'})w^{-1/p}(x) \rightarrow B(x, u(x))w^{-1/p}(x)
\]
a.e. in \(\Omega\). Since
\[
\int_{\Omega} |A(x, \nabla u_{i})w^{-1/p}|^{p/(p-1)} dx \leq \alpha_{2}^{p/(p-1)} \int_{\Omega} |\nabla u_{i}|^{p} d\mu
\]
\[
\int_{\Omega} |B(x, u_{i})w^{-1/p}|^{p/(p-1)} dx \leq 2\alpha_{3}^{p/(p-1)} \int_{\Omega} (|u_{i}| + 1)^{p} d\mu,
\]
\(L^{p/(p-1)}(\Omega; dx)\)-norms of \(A(x, \nabla u_{i})w^{-1/p}\) and \(B(x, u_{i})w^{-1/p}\) are uniformly bounded. Therefore
\[
A(x, \nabla u_{i_{j}'})w^{-1/p} \rightarrow A(x, \nabla u)w^{-1/p}
\]
\[
B(x, u_{i_{j}'})w^{-1/p} \rightarrow B(x, u)w^{-1/p}
\]
weakly in \(L^{p/(p-1)}(\Omega; dx)\). Since the weak limit is independent of \((\nabla u_{i_{j}}, u_{i_{j}})\),
\[
A(x, \nabla u_{i})w^{-1/p} \rightarrow A(x, \nabla u)w^{-1/p}
\]
\[
B(x, u_{i})w^{-1/p} \rightarrow B(x, u)w^{-1/p}.
\]
weakly in $L^{p/(p-1)}(\Omega; dx)$. Hence we have for all $(f, g) \in X$ that

$$\langle \Im(\nabla u_i, u_i), (f, g) \rangle = \int_{\Omega} A(x, \nabla u_i) \cdot f dx + \int_{\Omega} B(x, u_i) g dx$$

$$= \int_{\Omega} A(x, \nabla u_i) w^{-1/p} \cdot f w^{1/p} dx + \int_{\Omega} B(x, u_i) w^{-1/p} g w^{1/p} dx$$

$$\rightarrow \int_{\Omega} A(x, \nabla u) w^{-1/p} \cdot f w^{1/p} dx + \int_{\Omega} B(x, u) w^{-1/p} g w^{1/p} dx$$

$$= \langle \Im(\nabla u, u), (f, g) \rangle.$$
The uniqueness of solutions of Dirichlet problems for equation (2) and obstacle problems in $\mathcal{K}_{\psi,\theta}$ follows from the following comparison principle Lemma 3.6 and Lemma 3.7 respectively.

**Lemma 3.6** Let $u \in H^{1,p}(\Omega; \mu)$ be a supersolution and $v \in H^{1,p}(\Omega; \mu)$ a subsolution of (2) in $\Omega$. If $\eta = \min(u - v, 0) \in H^{1,p}_0(\Omega; \mu)$, then $u \geq v$ a.e. in $\Omega$.

**Proof:** By (a4) and (b3),
\[
\int_{\Omega} (A(x, \nabla v) - A(x, \nabla u)) \cdot \nabla \eta \, dx \leq - \int_{\{u < v\}} (A(x, \nabla v) - A(x, \nabla u)) \cdot (\nabla v - \nabla u) \, dx \leq 0,
\]
\[
\int_{\Omega} (B(x, v) - B(x, u)) \eta \, dx \leq - \int_{\{v\}} (B(x, v) - B(x, u))(v - u) \, dx \leq 0.
\]
From this we have
\[
0 \leq \int_{\Omega} A(x, \nabla v) \cdot \nabla \eta \, dx + \int_{\Omega} B(x, v) \eta \, dx - \left( \int_{\Omega} A(x, \nabla u) \cdot \nabla \eta \, dx + \int_{\Omega} B(x, u) \eta \, dx \right) \leq 0.
\]
and, hence
\[
\int_{\Omega} (A(x, \nabla v) - A(x, \nabla u)) \cdot \nabla \eta \, dx = 0
\]
and
\[
\int_{\Omega} (B(x, v) - B(x, u)) \eta \, dx = 0.
\]
Therefore $\nabla \eta = 0$ a.e. in $\Omega$. Because $\eta \in H^{1,p}_0(\Omega; \mu)$, $\eta = 0$ a.e. in $\Omega$ ([1, Lemma 1.17, p.18]). The lemma follows.

**Lemma 3.7** Suppose that $u$ is a solution to the obstacle problem in $\mathcal{K}_{\psi,\theta}(\Omega)$. If $v \in H^{1,p}(\Omega; \mu)$ is a supersolution of (2) in $\Omega$ such that $\min(u, v) \in \mathcal{K}_{\psi,\theta}(\Omega)$, then $v \geq u$ a.e. in $\Omega$.

**Proof:** Since $u - \min(u, v) \in H^{1,p}_0(\Omega; \mu)$ and is nonnegative, the lemma is proved in the same manner as in the proof of Lemma 3.6.

§4. **The local behavior of solutions**

In this section, we study the local behavior of solutions of (2).

The next theorem can be shown in the same manner as [2, Theorem 1].

**Theorem 4.1** Each solution of (2) in $\Omega$ is locally bounded.

We obtain, using the Moser iteration technique, the following Harnack inequality.

Let $B(R)$ denote an open ball of radius $R$.

**Theorem 4.2** Let $u$ be a nonnegative solution of equation (2) in $\Omega$. Given $R_0 > 0$ there is a constant $c > 0$ such that
\[
\text{ess sup}_{B(R)} u \leq c \text{ ess inf}_{B(R)} (u + R)
\]
whenever $B(R)$ is a ball in $\Omega$ such that $3B(R) \subset \Omega$ and $R \leq R_0$. Here $c$ depends only on $n$, $p$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $c_\mu$ and $R_0$. 
We require some lemmas to prove Theorem 4.2.

**Lemma 4.3** ([2, Lemma 2, p.252]) Let \( a \) be a positive exponent, and let \( a_i, b_i \) \((i = 1, \cdots, N)\), be two sets of \( N \) real numbers such that \( 0 < a_i < \infty \) and \( 0 \leq b_i < a \). Suppose that \( z \) is a positive number satisfying

\[
    z^a \leq \sum a_i z^{b_i}.
\]

Then

\[
    z \leq c \sum (a_i)^{\gamma_i}
\]

where \( c \) depends only on \( N, a, \) and \( b_i \), and where \( \gamma_i = (a - b_i)^{-1} \).

**Lemma 4.4** (John-Nirenberg lemma) ([1, Appendix II]) Suppose that \( v \) is a locally \( \mu \)-integrable function in \( \Omega \) with

\[
    \sup \frac{1}{\mu(B)} \int_B |v - v_B|d\mu \leq c_0,
\]

where

\[
    v_B = \frac{1}{\mu(B)} \int_B v d\mu
\]

and the supremum is taken over all balls \( B \subset \subset \Omega \). Then there are positive constants \( c_1 \) and \( c_2 \) depending on \( c_0, n, \) and \( c_\mu \) such that

\[
    \sup \frac{1}{\mu(B)} \int_B e^{c_1|v - v_B|}d\mu \leq c_2,
\]

where the supremum is taken over all balls \( B \subset \subset \Omega \).

Let \( u \) be a nonnegative solution of equation (2) in \( \Omega \) and \( B = B(R) \) is a ball in \( \Omega \). We set \( \bar{u} = u + R \). Thus, by Theorem 4.1, if \( \eta \in C_0^\infty(B) \) is nonnegative, then \( \varphi(x) = \eta^p \bar{u}^\beta \in H^{1,p}_0(B; \mu) \) for any real value of \( \beta \). Moreover,

\[
    |B(x, u)| \leq 2\alpha_3 w \max(1, 1/R^{p-1}) \bar{u}^{p-1}.
\]

We set \( \alpha_3' = 2\alpha_3 \max(1, 1/R^{p-1}) \).

Next lemma guarantees that \( v = \log \bar{u} \) satisfies the hypothesis of John-Nirenberg lemma.

**Lemma 4.5** Suppose that \( u \) is a nonnegative solution of equation (2) in \( \Omega \) and \( B = B(R) \) is a ball in \( \Omega \) such \( 3B \subset \Omega \). Then there is a constant \( c > 0 \) such that

\[
    \int_{B_1} |v - v_{B_1}|d\mu \leq c \mu(B_1) \quad (v = \log \bar{u}),
\]

whenever \( B_1 \) is a ball with \( B_1 \subset 2B \). Here \( c \) depends on \( p, \alpha_1, \alpha_2, \alpha_3' R^p \) and \( c_\mu \).

Proof: Setting \( \varphi = \eta^p \bar{u}^{1-p} \), we have

\[
    0 = \int_{3B} A(x, \nabla u) \cdot \nabla \varphi dx + \int_{3B} B(x, u) \varphi dx
\]
$$= \int_{3B} A(x, \nabla u) \cdot \{p(\eta/\overline{u})^{p-1} \nabla \eta + (1 - p)(\eta/\overline{u})^{p} \nabla u\} dx + \int_{3B} B(x, u) \eta^{p} \overline{u}^{1-p} dx$$

$$\leq -\alpha_{1}(p-1) \int_{3B} (\eta/\overline{u})^{p}|\nabla u|^{p} d\mu + \alpha_{2} p \int_{3B} |\nabla \eta||\eta/\overline{u})^{p-1}|\nabla u|^{p-1} d\mu

+ \alpha_{3}' \int_{3B} \eta^{p-1} \overline{u}^{1} |\nabla \eta|^{p-1} d\mu$$

$$= -\alpha_{1}(p-1) \int_{3B} |\eta v|^p d\mu + \alpha_{2} p \int_{3B} |\nabla \eta||\eta v|^p d\mu + \alpha_{3}' \int_{3B} \eta^{p} d\mu,$$ 

where $v = \log \overline{u}$. Hence

(9) \[\alpha_{1}(p-1)||\eta v||_{p,3B} \leq \alpha_{2} p \int_{3B} |\nabla \eta||\eta v|^p d\mu + \alpha_{3}' \int_{3B} \eta^{p} d\mu.\]

Let $B_1 \subset 2B$ be any open ball of radius $h$. Let $\eta$ be so chosen that $\eta = 1$ in $B_1$, $0 \leq \eta \leq 1$ in $3B \setminus B_1$, the support of $\eta$ is contained in $(3/2)B_1$, and $|\nabla \eta| \leq 3/h$. Then by Hölder’s inequality we obtain

$$\int_{3B} |\nabla \eta||\eta v|^p d\mu \leq \left(\int_{(3/2)B_1} |\nabla \eta|^p d\mu\right)^{1/p} \left(\int_{(3/2)B_1} |\eta v|^p d\mu\right)^{(p-1)/p}$$

$$\leq \frac{3}{h} \{\mu((3/2)B_1)\}^{1/p} ||\eta v||_{p,3B}^{p-1},$$

$$\int_{3B} \eta^{p} d\mu \leq \mu((3/2)B_1).$$

By the above inequalities and (9) we have

$$\alpha_{1}(p-1)||\eta v||_{p,3B} \leq \frac{3\alpha_{2} p}{h} \{\mu((3/2)B_1)\}^{1/p} ||\eta v||_{p,3B}^{p-1} + \alpha_{3}'(3R)^{p} h^{p} \mu((3/2)B_1).$$

Application of Lemma 4.3 yields,

$$||v||_{p,B} \leq ch^{-1}\mu((3/2)B_1)^{1/p},$$

where $\eta = 1$ in $B_1$ have been used. Finally by the the doubling property, Hölder’s inequality and Poincaré inequality we have

$$\int_{B_1} |v - v_{B_1}| d\mu \leq c\{\mu((3/2)B_1)\}^{(p-1)/p} h^{p} \left(\int_{B_1} |\nabla v|^p d\mu\right)^{1/p} \leq c \mu(B_1) \quad (v = \log \overline{u}),$$

where $c = c(p, \alpha_{1}, \alpha_{2}, \alpha_{3}' R^p, c).$ \[\square\]

The following estimates will be used when we apply to the Moser iteration technique. 

**Lemma 4.6** Suppose that $u$ is a nonnegative solution of equation (2) in $\Omega$ and $B = B(R)$ is a ball in $\Omega$. For $\beta \neq 0$, $p - 1$, let $q$ satisfying $pq = p + \beta - 1$ and $v = \overline{u}^{q}$. Then there is a constant $c > 0$ such that

(i) if $\beta > 0,$

$$||\eta v||_{p,B} \leq c\{\mu(B)\}^{(1-k)/k} R(1 + \beta^{-1})(1 + q)^p (||v\nabla \eta||_{p,B} + ||\eta v||_{p,B}),$$
(ii) if $1 - p < \beta < 0$,
\[ \|\eta v\|_{kp,B} \leq c\{\mu(B)\}(1 - \beta^{-1})(\|v \nabla \eta\|_{p,B} + \|\eta v\|_{p,B}), \]
(iii) if $\beta < 1 - p$,
\[ \|\eta v\|_{kp,B} \leq c\{\mu(B)\}(1 - \beta^{-1})(\|v \nabla \eta\|_{p,B} + \|\eta v\|_{p,B}), \]
where $c$ depends only on $p$, $\alpha_1$, $\alpha_2$, $c_\mu$ and $\alpha'_3R^{p-1}$.

Proof: We prove only (i), the proofs of (ii) and (iii) being similar. For $\varphi = \eta^p u^\beta$, we have
\[ 0 = \int_B A(x, \nabla u) \cdot \nabla \varphi \, dx + \int_B B(x, u) \varphi \, dx \]
\[ = \int_B A(x, \nabla u) \cdot (p^p \eta^{p-1} u^\beta \nabla \eta + \beta \eta^p u^{p\beta-1} \nabla u) \, dx + \int_B B(x, u) \eta^p u^\beta \, dx \]
\[ \geq \alpha_1 \beta \int_B \eta^p u^{p-1} \nabla u \cdot \nabla \eta \, dx - p\alpha_2 \int_B |\nabla u|^{p-1} |\nabla \eta| \, dx - \mu \alpha'_3 \int_B \eta^p u^\beta \, dx. \]

Since $pq = p + \beta - 1$ and $v = \bar{u}^q$,
\[ (10) \quad \frac{\alpha_1 \beta}{q^p} \|\eta \nabla v\|^p \leq \frac{p\alpha_2}{q^{p-1}} \int_B |v \nabla \eta| \|\eta \nabla v\|^p \, dx + \alpha'_3 \int_B (\eta v)^p \, dx. \]

Here for simplicity we have written $\| \cdot \|_p$ for $\| \cdot \|_{p,B}$.

By Hölder's inequality,
\[ \int_B |v \nabla \eta| \|\eta \nabla v\|^p \, dx \leq \|v \nabla \eta\|_p \|\eta \nabla v\|_{p-1}^p, \]
\[ \int_B (\eta v)^p \, dx \leq \|\eta v\|_p \left( \int_B (\eta v)^{p-1} \, dx \right)^{1/p} \]
\[ \leq \|\eta v\|_p \left\{ \left( \int_B (\eta v)^{kp} \, dx \right)^{1/k} \left( \int_B p^{p-1} \, dx \right)^{(k-1)/kp} \right\} \]
\[ \leq \mu(B)^{1/(p-1)(kp)} \|\eta v\|_p \|\eta v\|_{kp} \]
\[ \leq c \mu R^{p-1} \|\eta v\|_p \left( \|v \nabla \eta\|_{p-1} + \|\eta \nabla v\|_{p-1} \right), \]

where we have used Sobolev inequality. By the above inequalities, if we set
\[ z = \frac{\|\eta \nabla v\|_p}{\|v \nabla \eta\|_p}, \quad \zeta = \frac{\|\eta v\|_p}{\|v \nabla \eta\|_p}, \]
then (10) can be written as
\[ \beta z^p \leq c \left\{ qz^{p-1} + q^p \zeta (1 + z^{p-1}) \right\}, \]
where $c = c(p, \alpha_1, \alpha_2, \alpha'_3 R^{p-1}, c_\mu)$. Application of Lemma 4.3 yields
\[ z \leq c(1 + \beta^{-1})(1 + q)^p(1 + \zeta), \]
that is,
\[ (11) \quad \|\eta \nabla v\|_p \leq c(1 + \beta^{-1})(1 + q)^p(\|v \nabla \eta\|_p + \|\eta v\|_p). \]
Finally using Sobolev inequality again, from (11) we obtain the desired estimate. □

Proof of Theorem 4.2: Set \( v = \log \overline{u} \). By Lemma 4.4 and Lemma 4.5, there are positive constants \( r_0 \) and \( c_0 \) such that
\[
\left( \int_{B_1} e^{r_0 u} d\mu \right) \left( \int_{B_1} e^{-r_0 v} d\mu \right) = \left( \int_{B_1} e^{r_0 (v - \mu v_1)} d\mu \right) \left( \int_{B_1} e^{r_0 (\mu v_1 - v)} d\mu \right) \leq \left( \int_{B_1} e^{r_0 (v - \mu v_1)} d\mu \right) \leq c_0^2 \{ \mu(B_1) \}^2.
\]
Because \( B_1 \) is any ball contained in \( 2B \),
\[
\left( \int_{2B} e^{r_0 u} d\mu \right) \left( \int_{2B} e^{-r_0 v} d\mu \right) \leq c_0^2 \{ \mu(2B) \}^2.
\]
Hence
\[
\left( \int_{2B} \overline{u}^{r_0} d\mu \right)^{1/r_0} \leq c \{ \mu(2B) \} \left( \int_{2B} \overline{u}^{-r_0} d\mu \right)^{-1/r_0}.
\]

Next, let \( 0 < h' < h \leq 3R \). Let the function \( \eta \in C_0^\infty(B(h)) \) be so chosen that \( \eta = 1 \) in \( B(h') \), \( 0 \leq \eta \leq 1 \) in \( B(h) \) and \( |
\nabla \eta| \leq 3(h - h')^{-1} \). Then Lemma 4.6 yields
(i) if \( \beta > 0 \),
\[
\| \overline{u}^q \|_{k p, B(h')} \leq c \{ \mu(B) \}^{(1-k)/kp} R(1 + q)^p (h - h')^{-1} (1 + \beta^{-1}) \| \overline{u}^q \|_{p, B(h)},
\]
(ii) if \( 1 - p < \beta < 0 \),
\[
\| \overline{u}^q \|_{k p, B(h')} \leq c \{ \mu(B) \}^{(1-k)/kp} R(h - h')^{-1} (1 - \beta^{-1}) \| \overline{u}^q \|_{p, B(h)},
\]
(iii) if \( \beta < 1 - p \),
\[
\| \overline{u}^q \|_{k p, B(h')} \leq c \{ \mu(B) \}^{(1-k)/kp} R(h - h')^{-1} (1 + |q|)^p \| \overline{u}^q \|_{p, B(h)},
\]
where \( c \) depends only on \( p, \alpha_1, \alpha_2, c_\mu \) and \( \alpha_3'R^{p-1} \).

Putting \( r = pq = p + \beta - 1 \) in (13) and (14), combining the result in a single inequality, we obtain
\[
\left( \int_{B(h')} \overline{u}^{r \nu} d\mu \right)^{1/r \nu} \leq \left\{ c \{ \mu(3B) \}^{(1-k)/kp} R(h - h') (1 + |\beta|^{-1}) (1 + r)^p \right\}^{p/r} \times \left( \int_{B(h)} \overline{u}^r d\mu \right)^{1/r},
\]
for all \( 0 < r \neq p - 1 \). Let
\[
r_\nu = k^\nu r_0' \quad \nu = 0, 1, 2, \ldots,
\]
and \( h_\nu = R(1 + 2^{-\nu}) \), \( h'_\nu = h_{\nu+1} \), where \( r_0' \leq r_0 \) is so chosen that \( r_\nu \neq p - 1 \) for any \( \nu = 0, 1, 2, \ldots \). Thus
\[
|\beta| = |r - (p - 1)| \geq c > 0,
\]
whenever \( r = r_\nu \), where \( c \) depends only on \( p, k, r_0 \). The term \( (1 + |\beta|^{-1}) \) in (16) can thus be absorbed into the general constant \( c \). Hence from (16) we have that
\[
\left( \int_{B(h_\nu)} \overline{u}^{r_{\nu+1}} d\mu \right)^{1/r_{\nu+1}} \leq \left\{ c \{ \mu(3B) \}^{(1-k)/kp 2^{\nu+1}} (1 + r_\nu)^p \right\}^{p/r_\nu} \left( \int_{B(h_\nu)} \overline{u}^{r_\nu} d\mu \right)^{1/r_\nu}.
\]
\[ c^{1/k'} \{ \mu(3B) \}^{(1-k)/kr_0 k'} (1 + r_0' k')^{p/r_0'} \left( \int_{B(h_\nu)} \bar{u}^{r_0'} \, d\mu \right)^{1/r_0'} \leq c_1^{1/k'} c_2^{\nu/kr_0'} \{ \mu(3B) \}^{(1-k)/kr_0 k'} \left( \int_{B(h_\nu)} \bar{u}^{r_0'} \, d\mu \right)^{1/r_0'}. \]

By iterating, it follows that

\[ \text{ess sup}_B \bar{u} \leq c \{ \mu(3B) \}^{-1/r_0'} \left( \int_{2B} \bar{u}^{r_0'} \, d\mu \right)^{1/r_0'}. \]

Setting \( s = pq \) in (15), since \( s \) and \( q \) are negative, we obtain

\[ \left( \int_{B(h')} \bar{u}^{k\nu} \, d\mu \right)^{1/k\nu} \geq \left\{ c \{ \mu(3B) \}^{(1-k)/kp} R(h-h')^{-1} (1+|s|)^p \right\}^{\nu/k^p} \left( \int_{B(h')} \bar{u}^{k\nu} \, d\mu \right)^{1/k\nu}. \]

Let \( s_\nu = -k^\nu r_0, \ h_\nu = R(1+2^{-\nu}) \) and \( h_\nu' = h_{\nu+1} \). Then

\[ \left( \int_{B(h'_\nu)} \bar{u}^{s_{\nu+1}} \, d\mu \right)^{1/s_{\nu+1}} \geq c_1^{-1/k'} c_2^{\nu/kr_0'} \{ \mu(3B) \}^{-(1-k)/kr_0 k'} \left( \int_{B(h_\nu)} \bar{u}^{s_\nu} \, d\mu \right)^{1/s_\nu}. \]

By iterating, we obtain

\[ \text{ess inf}_B \bar{u} \geq \left( \frac{\rho}{R} \right)^{\lambda} \text{ess sup}_B \bar{u} \geq \left( \frac{\rho}{R} \right)^{\lambda} \text{ess inf}_B \bar{u}. \]

Finally, by (12), (17), (18), and a simple application of Hölder's inequality, we have

\[ \text{ess sup}_B \bar{u} \leq c \{ \mu(3B) \}^{-1/r_0'} \left( \int_{2B} \bar{u}^{-r_0} \, d\mu \right)^{-1/r_0} \leq c \{ \mu(3B) \}^{-1/r_0} \left( \int_{2B} \bar{u}^{-r_0} \, d\mu \right)^{-1/r_0} \]

\[ \leq c \{ \mu(3B) \}^{1/r_0} \left( \int_{2B} \bar{u}^{-r_0} \, d\mu \right)^{-1/r_0} \leq c \text{ ess inf}_B \bar{u}. \]

Since \( \bar{u} = u + R \), this concludes the proof of Theorem 4.2. \( \square \)

We apply Theorem 4.4 to show that any solutions of (2) has Hölder continuous representative.

**Theorem 4.7** Let \( u \) be a solution of (2) in \( \Omega \) and \( x_0 \) be any point of \( \Omega \). If \( 0 < R < \infty \) is such that \( B(x_0, R) \subset \Omega \) and if \( |u| \leq L \) a.e in \( B(x_0, R) \), then there are constants \( c \) and \( 0 < \lambda < 1 \) such that

\[ \text{ess sup}_{B(x_0, \rho)} u - \text{ess inf}_{B(x_0, \rho)} u \leq c \left( \frac{\rho}{R} \right)^{\lambda}, \]

whenever \( 0 < \rho < R \). Here \( c \) and \( \lambda \) depend only on \( n, p, \alpha_1, \alpha_2, \alpha_3, c_\mu, R \) and \( L \).

**Proof:** We write \( B(r) = B(x_0, r) \) and

\[ M(r) = \text{ess sup}_{B(r)} u, \quad m(r) = \text{ess inf}_{B(r)} u. \]

Then \( M(r) \) and \( m(r) \) are well defined for \( 0 < r \leq R \), and

\[ \bar{u} = M(r) - u, \quad \bar{u} = u - m(r) \]
are non-negative in $B(r)$. Obviously $\bar{u}$ is a solution of

$$-\text{div}\bar{A}(x, \nabla \bar{u}) + \bar{B}(x, \bar{u}) = 0$$

where $\bar{A}(x, \bar{h}) = -A(x, -\bar{h})$ and $\bar{B}(x, \bar{t}) = -B(x, M(r) - \bar{t})$. Thus

$$|\bar{B}(x, \bar{t})| \leq \alpha'_3 w(x)(|\bar{t}|^{p_1} + 1),$$

where $\alpha'_3$ is a constant depending only on $\alpha_3, p$ and $L$. By applying Harnack inequality to $\bar{u}$, we have

(19) $M(r) - m(r/3) = \text{ess sup}_{B(r/3)} \bar{u} \leq c(\text{ess inf}_{B(r/3)} \bar{u} + r) = c\{M(r) - M(r/3) + r\}.$

Similarly we have

(20) $M(r/3) - m(r) = \text{ess sup}_{B(r/3)} \bar{u} \leq c(\text{ess inf}_{B(r/3)} \bar{u} + r) = c\{m(r/3) - m(r) + r\}.$

Here $c > 1$ depends on $n, p, \alpha_1, \alpha_2, \alpha_3, c_\mu, R$ and $L$. By (19) and (20),

(21) $M(r/3) - m(r/3) \leq \frac{c-1}{c+1}\{M(r) - m(r)\} + \frac{2c}{c+1} r.$

Thus setting

$$\theta = \frac{c-1}{c+1}, \quad \tau = \frac{2cR}{c-1}$$

and

$$\omega = M(r) - m(r),$$

(21) can be written as

$$\omega(r/3) \leq \theta\{\omega(r) + \tau(r/R)\}.$$

Since $\omega(r)$ is an increasing function, for any number $s \geq 3$ we have also

$$\omega(r/s) \leq \theta\{\omega(r) + \tau(r/R)\}, \quad 0 < r \leq R.$$

By iterating, we obtain

(22) $\omega(R/s^\nu) \leq \theta^\nu\{\omega(R) + \tau\{1 + (\theta s)^{-1} + \cdots + (\theta s)^{-\nu+1}\}\},$

for $\nu = 1, 2, 3, \cdots$. Let $s$ be so chosen that $\theta s = 3$. Then (22) implies

(23) $\omega(R/s^\nu) \leq \theta^\nu\{\omega(R) + 2\tau\}.$

For any $\rho$ such that $0 < \rho \leq R/s$ choose $\nu$ such that $R/s^{\nu+1} < \rho \leq R/s^\nu$. Then from (23) we have

(24) $\omega(\rho) \leq \omega(R/s^\nu) \leq \theta^\nu(\omega(R) + 2\tau).$

If we set $\gamma = -\log_3 \theta$, then we have $\theta = s^{-\lambda}$ where $\lambda = \gamma/(\gamma+1) > 0$. Thus

$$\theta^\nu = \left(\frac{R}{s^{\nu+1}}\frac{s}{R}\right)^\lambda \leq c\left(\frac{\rho}{R}\right)^\lambda.$$
Hence, since $\omega(R) + 2\tau \leq c(L + R)$, (22) implies

$$\omega(\rho) \leq c(L + R) \left( \frac{\rho}{R} \right)^\lambda, \quad (\rho < R),$$

as desired. $\square$

§5. A regularity at the boundary for solutions

In this section, we are concerned with the continuity of solutions at the boundary. First, we recall the definition of the $(p, \mu)$-capacity which is adopted in [1]. Suppose that $K$ is a compact subset of $\Omega$. Let

$$W(K, \Omega) = \{ u \in C_0^\infty(\Omega) : u \geq 1 \text{ on } K \}$$

and define

$$\text{cap}_{p, \mu}(K, \Omega) = \inf_{u \in W(K, \Omega)} \int_\Omega |\nabla u|^p d\mu.$$ 

Further, if $U \subset \Omega$ is open, set

$$\text{cap}_{p, \mu}(U, \Omega) = \sup_{K \subset U \text{ compact}} \text{cap}_{p, \mu}(K, \Omega),$$

and, finally, for an arbitrary set $E \subset \Omega$

$$\text{cap}_{p, \mu}(E, \Omega) = \inf_{E \subset U \subset \Omega} \text{cap}_{p, \mu}(U, \Omega).$$

The number $\text{cap}_{p, \mu}(E, \Omega) \in [0, \infty]$ is called the $(p, \mu)$-capacity of the condenser $(E, \Omega)$. If $u \in H^1_{loc}(\Omega; \mu)$, $x_0 \in \partial \Omega$, and $l \in R$ we say that

(25) \hspace{1cm} u(x_0) \leq l \text{ weakly}

if for every $k > l$ there is an $r > 0$ such that $\eta(u - k)^+ \in H^1_{0}(\Omega; \mu)$ whenever $\eta \in C_0^\infty(B(x_0, r))$. The condition

(26) \hspace{1cm} u(x_0) \geq l \text{ weakly}

is defined analogously and $u(x_0) = l$ weakly if both (25) and (26) hold. Observe that if $f$ is a continuous function on $R^n \setminus \Omega$, $f \in H^1_{loc}(R^n; \mu)$, and $u - f \in H^1_{0}(\Omega; \mu)$, then $u(x) = f(x)$ weakly for every $x \in \partial \Omega$.

Lemma 5.1 Suppose that $u \in H^1_{loc}(\Omega; \mu)$ is a subsolution of (2) in $\Omega$, that $u \leq L$ a.e. in $\Omega$, and that $u(x_0) \leq l$ weakly for $x_0 \in \partial \Omega$. For $k > l$, let

$$u_k = \begin{cases} (u - k)^+ & \text{on } \Omega \\ 0 & \text{otherwise} \end{cases}$$

and define

$$M(r) = \text{ess sup}_{B(x_0, r)} u_k.$$ 

Choose $r_0 > 0$ so small that $\eta u_k \in H^1_{0}(\Omega; \mu)$ whenever $\eta \in C_0^\infty(B(x_0, r_0))$. 

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Then there is a constant $c$ depending only on $n$, $p$, $l$, $r_0$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $c_\mu$ and $L$ such that
\[
\int_{B(x_0,r/2)} |\nabla (\eta v^{-1})|^p d\mu \leq c(M(r) + r)(M(r) - M(r/2) + r)^{p-1}\mu(B(x_0, r))r^{-p}
\]
where $0 < r \leq r_0/2$, $v^{-1} = M(r) + r - u_k$ and $\eta \in C_0^\infty(B(x_0, r/2))$ with $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq 5/r$.

Before proving Lemma 5.1, we will state its implication.

**Theorem 5.2** Let $u \in H_{loc}^{1,p}(\Omega; \mu)$ be a subsolution of (2) which is bounded above on $\Omega$, $x_0 \in \partial \Omega$, and $u(x_0) \leq l$ weakly. If
\[
\int_0^1 \left( \frac{\text{cap}_{p,\mu}(B(x_0, t) \setminus \{u_k = 0\}, B(x_0, 2t))}{\text{cap}_{p,\mu}(B(x_0, r/4), B(x_0, r/2))} \right)^{1/(p-1)} dt = \infty,
\]
then
\[
\text{ess lim sup}_{x \to x_0} u(x) \leq l.
\]

Proof: Since, for any $k > l$, it follows immediately from Theorem 5.1, the definition of $(p, \mu)$-capacity and [1, Lemma 2.14] that
\[
(M(r) + r)\left( \frac{\text{cap}_{p,\mu}(B(x_0, r/4) \setminus \{u_k = 0\}, B(x_0, r/2))}{\text{cap}_{p,\mu}(B(x_0, r/4), B(x_0, r/2))} \right)^{1/(p-1)} \\
\leq c(M(r) - M(r/2) + r),
\]
the theorem is proved in the same manner as in the proof of [4, Theorem 2.2]. \hfill \square

If $u$ is a supersolution of (2), then $-u$ is a subsolution of
\[
-\text{div}\bar{A}(x, \nabla v) + \bar{B}(x, v) = 0,
\]
where $\bar{A}(x, h) = -A(x, -h)$ and $\bar{B}(x, t) = -B(x, -t)$. Consequently, Theorem 5.2 has the obvious counterpart for supersolutions of (2). These results yield

**Theorem 5.3** Let $u \in H_{loc}^{1,p}(\Omega; \mu)$ be a bounded solution of (2), that $x_0 \in \partial \Omega$, and that $u(x_0) = l$ weakly. If (27) holds, then
\[
\lim_{x \to x_0} u(x) = l.
\]

Proof of Lemma 5.1: Fix $r > 0$ so that $0 < r \leq r_0/2$, let $\eta \in C_0^\infty(B(x_0, r/2))$ with $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq 5/r$. Set
\[
I(r) = (M(r) + r)(M(r) - M(r/2) + r)^{p-1}\mu(B(x_0, r))r^{-p}.
\]
Since
\[ \int |\nabla(\eta v^{-1})|^{p}d\mu \leq c \left( \int \eta^{p}|\nabla u_{k}|^{p}d\mu + \int v^{-p}|\nabla \eta|^{p}d\mu \right), \]
we will show that
\[ \int \eta^{p}|\nabla u_{k}|^{p}d\mu \leq c I(r) \quad \text{and} \quad \int v^{-p}|\nabla \eta|^{p}d\mu \leq c I(r), \]
by using following two estimates.

Estimate 1 For $0 < \sigma < p - 1$,
\[ (c m(\alpha))^{-1} \int_{B(x_{0}, r)} |\nabla(\omega v^{\alpha})|^{p}d\mu \leq \int_{B(x_{0}, r)} v^{p\alpha}\{\omega v^{\alpha} + |\nabla \omega|^{p}\}d\mu, \]
whenever $\omega \in C_{0}^{\infty}(B(x_{0}, r))$ with $0 \leq \omega \leq 1$, where $c$ is a constant depending on $p$, $\alpha_{1}$, $\alpha_{2}$, $\alpha_{3}$, $l$, $r_{0}$, and $L$, and
\[ 0 < m(\alpha) < 1 + \alpha^{p} \text{ if } \alpha > 0, \]
\[ m(\alpha) > 0 \text{ and a decreasing function of } \alpha \text{ if } (1 - p)/p < \alpha < 0. \]

Estimate 2 For $0 < \sigma < p - 1$,
\[ \mu(B(x_{0}, r))^{-1}\|v^{-\sigma k}\|_{1,B(x_{0}, r/2)} \leq c(M(r) - M(r/2) + r)^{\sigma k}, \]
where $c$ is a constant depending on $p$, $n$, $\alpha_{1}$, $\alpha_{2}$, $\alpha_{3}$, $l$, $r_{0}$, $L$ and $\sigma$.

Let us suppose that Estimate 1 and Estimate 2 are true. Fix $\alpha < 0$ such that $1 < (1 + \alpha)p < k$, then putting $B = B(x_{0}, r/2)$, we have
\[ \int_{B} \eta^{p} - |\nabla u_{k}|^{p - 1}|\nabla \eta|d\mu = \int_{B} \eta^{p + \alpha}|\nabla u_{k}|^{p - 1}(v^{-(1 + \alpha)(p - 1)}|\nabla \eta|)d\mu \]
(28)
\[ = c \int_{B} (\eta|\nabla v^{\alpha}|^{p - 1}(v^{-(1 + \alpha)(p - 1)}|\nabla \eta|)d\mu \]
\[ \leq c \left( \int_{B} (\eta|\nabla v^{\alpha}|^{p}d\mu)^{(p - 1)/p} \left( \int_{B} (v^{-(1 + \alpha)(p - 1)}|\nabla \eta|^{p}d\mu)^{1/p} \right)^{p - 1} \right) \]
\[ \times \left( \int_{B} (v^{-(1 + \alpha)(p - 1)}|\nabla \eta|^{p}d\mu)^{1/p} \right) \]
\[ \leq c \left( r^{-p} \int_{B} \omega^{p\alpha}d\mu \right)^{(p - 1)/p} \left( \int_{B} (v^{-(1 + \alpha)(p - 1)}|\nabla \eta|^{p}d\mu)^{1/p} \right)^{p - 1} \]
\[ \leq c \left\{ (M(r) - M(r/2) + r)^{-\alpha p} \mu(B(x_{0}, r)) \right\}^{(p - 1)/p} \]
\[ \times \left\{ (M(r) - M(r/2) + r)^{(1 + \alpha)(p - 1)p} \mu(B(x_{0}, r)) r^{-p} \right\}^{1/p} \]
\[ = c (M(r) - M(r/2) + r)^{(p - 1)} \mu(B(x_{0}, r)) r^{-p}, \]
in the last inequality we have used Estimate 2 with $\sigma = -\alpha p/k$ and $\sigma = (1 + \alpha)(p - 1)p/k$ respectively. Also since $\eta \leq 1$,
\[ \int_{B} \eta^{p}d\mu \leq \mu(B(x_{0}, r)) \leq c I(r). \]

(29)
Hence, by (28) and (29),
\begin{equation}
\int_{B} \eta^{p} |\nabla u_{k}|^{p} d\mu \leq c \left( \int_{B} \eta^{p} d\mu + M(r) \int_{B} \eta^{p-1} |\nabla u_{k}|^{p-1} |\nabla \eta| d\mu \right) \leq c I(r).
\end{equation}

Here the first inequality has been obtained by using the facts that \( \varphi = \eta^{p} u_{k} \in H_{0}^{1,p}(\Omega; \mu) \), \( \varphi \) is nonnegative, \( u \) is a subsolution and the structure of \( A \) and \( B \). From Estimate 2 with \( \sigma = (p-1)/k \) again
\begin{equation}
\int_{B} |v^{-1} \nabla \eta|^{p} d\mu \leq c r^{-p} (M(r) + r) \int_{B} v^{-p+1} d\mu \leq c I(r).
\end{equation}

Therefore we obtain from (30) and (31)
\begin{equation}
\int_{B} |\nabla (\eta v^{-1})|^{p} d\mu \leq c I(r).
\end{equation}

Finally, we will prove Estimate 1 and Estimate 2. For \( \beta > 0 \), let
\begin{equation*}
\psi = v^{\beta} - (M(r) + r)^{-\beta}
\end{equation*}
and
\begin{equation*}
\varphi = \omega^{\beta} \psi,
\end{equation*}
where \( \omega \in C_{0}^{\infty}(B(x_{0}, r)) \). Then \( \varphi \in H_{0}^{1,p}(\Omega; \mu) \). Since \( \varphi = 0 \) on \( \{u_{k} = 0\} \) and \( \varphi \geq 0 \) on \( \Omega \),
\begin{equation*}
\int_{B} \omega^{\beta} v^{\beta+1} A(x, \nabla u) \cdot \nabla u_{k} dx + \int p \omega^{p-1} \omega A(x, \nabla u) \cdot \nabla \omega dx + \int B(x, u) \varphi dx \leq 0,
\end{equation*}
where the integrals are taken over \( B(x_{0}, r) \cap \{u_{k} > 0\} \). Hereafter we will suppress explicit indication of this domain of integration.

Using (a2), (a3) and (b2) we have
\begin{equation*}
\alpha_{1} \beta \int \omega^{p} v^{\beta+1} |\nabla u_{k}|^{p} d\mu \leq \alpha_{2} \int \omega^{p-1} \psi |\nabla u_{k}|^{p-1} |\nabla \omega| d\mu + \alpha_{3} \int \omega^{p} \psi (|u|^{p-1} + 1) d\mu.
\end{equation*}

Since \( \psi \leq v^{\beta} \), \( v^{-1} \leq M(r_{0}) + r_{0} \) and \( l \leq u \leq L \), we obtain
\begin{equation}
c^{-1} \beta \int \omega^{p} v^{\beta+1} |\nabla u_{k}|^{p} d\mu \leq c \int \omega^{p} v^{\beta+1} |\nabla u_{k}|^{p-1} |\nabla \omega| d\mu + \int \omega^{p} v^{\beta+1} d\mu,
\end{equation}
where \( c \) depends on \( p, \alpha_{1}, \alpha_{2}, \alpha_{3}, r_{0}, L \). Application of Young's inequality yields that
\begin{equation*}
\int \omega^{p} v^{\beta+1} |\nabla u_{k}|^{p-1} |\nabla \omega| d\mu \leq \varepsilon^{p/(p-1)} (p-1)^{-1} \int \omega^{p} v^{\beta+1} |\nabla u_{k}|^{p} d\mu + \varepsilon^{-p+1} \int v^{\beta-p+1} |\nabla \omega|^{p} d\mu,
\end{equation*}
for any \( \varepsilon > 0 \). By the above inequality and (32), with an appropriate choice for \( \varepsilon \), we have
\begin{equation}
c^{-1} \beta \int \omega^{p} v^{\beta+1} |\nabla u_{k}|^{p} d\mu \leq c \int \omega^{p} v^{\beta+1} d\mu + \beta^{1-p} \int v^{\beta-p+1} |\nabla \omega|^{p} d\mu.
\end{equation}
By letting $\beta = p\alpha + p - 1$ with $0 < \beta \neq p - 1$, we obtain Estimate 1.

Next we prove Estimate 2. In (33) letting $\beta = p - 1$,

$$\int \omega^p|\nabla (\log v)|^p \leq c\{(p - 1)^{-1} \int \omega^p v^p d\mu + (p - 1)^{-p} \int |\nabla \omega|^p d\mu\}.$$

Since, by using $v \leq 1/r$ and Sobolev inequality,

$$\int \omega^p v^p d\mu \leq r^{-p} \mu(B(x_0, r))^{(k-1)/k} \left( \int \omega^p d\mu \right)^{1/k} \leq c \int |\nabla \omega|^p d\mu,$$

we have

$$\int \omega^p|\nabla (\log v)|^p \leq c \int |\nabla \omega|^p d\mu$$

whenever $0 \leq \omega \in C_0^\infty(B(x_0, r))$. Using Lemma 4.4 (John-Nirenberg lemma) in the same manner as in the proof of Lemma 4.5 and Theorem 4.2, it follows that there are positive constants $c$ and $\sigma_0$ such that

$$(34) \int_{B(x_0, s)} v^{-\sigma} d\mu \int_{B(x_0, s)} v^\sigma d\mu \leq c \left\{\mu(B(x_0, s))\right\}^2,$$

whenever $\sigma \leq \sigma_0$ and $0 < s \leq 3r/4$.

Let $0 < s < t \leq r$ and let a function $\omega \in C_0^\infty(B(x_0, t))$ be chosen such that $0 \leq \omega \leq 1$, $\omega = 1$ on $B(x_0, s)$ and $|\nabla \omega| \leq 2(t - s)^{-1}$. Then $(\omega v)^p \leq v^p \leq r^{-p} \leq 2(t - s)^{-p}$. Hence, from Sobolev inequality and Estimate 1,

$$(35) \left( \int_{B(x_0, s)} |v^{\alpha k^p} d\mu \right)^{1/k} \leq c \mu(B(x_0, r))^{(1-k)/k} r^p (t - s)^{-p} \int_{B(x_0, t)} v^{p0} d\mu,$$

whenever $0 < s < t \leq r$ and $(1 - p)p^{-1} < \alpha \neq 0$.

Let $r_j = r(2^{-1} + 2^{-j-2})$ for $j = 0, 1, \cdots$. Then since $m(\alpha_0 k^j) \leq c (k^p)^j$ for $0 < \alpha_0 \leq \sigma_0 p^{-1}$, (35) yields that

$$\left( \int_{B(x_0, r_j+1)} |v^{\alpha k^j}|^p d\mu \right)^{1/k} \leq c (k^p)^j \mu(B(x_0, r))^{(1-k)/k} (2p)^j \int_{B(x_0, r_j)} v^{p0k^j} d\mu,$$

and hence

$$\|v^{p0}\|_{k^j+1, B(x_0, r_j+1)} \leq c \left\{ \mu(B(x_0, r))^{(1-k)/k} \right\}^{k^{-j}} (2p)^j \|v^{p0}\|_{k^j, B(x_0, r_j)}$$

for $j = 0, 1, \cdots$. Hereafter, for simplicity, we shall write $\| \cdot \|_{p,r}$ for $\| \cdot \|_{p, B(x_0, r)}$. By iterating, we have

$$(36) (M(r) - M(r/2) + r)^{-p\alpha_0} \leq c \{\mu(B(x_0, r))\}^{-1}\|v^{p0}\|_{1,3r/4}^{-1},$$

whenever $0 < p\alpha_0 \leq \sigma_0$. From (34) and (36), we obtain that

$$(37) \mu(B(x_0, r))^{-1}\|v^{-p\alpha_0}\|_{1,3r/4} \leq c (M(r) - M(r/2) + r)^{p\alpha_0}$$

whenever $0 < p\alpha_0 \leq \sigma_0$.

Return to (35) with $1 - p < p\alpha < 0$. Let $0 < \sigma < p - 1$ and let $j_0$ is a positive integer such that $p - 1 \leq \sigma_0 k^{j_0}$. Put $\sigma_1 = \sigma k^{-j_0}$. Since $0 < \sigma_1 k^{j} \leq \sigma < p - 1$ for $0 \leq j \leq j_0$, $m(-\sigma_1 k^{j} p^{-1}) \leq m(-\sigma p^{-1})$ for $0 \leq j \leq j_0$. 


Let $r_j = (r/4)\{3 - j/(j_0 + 1)\}$ for $0 \leq j \leq j_0 + 1$. Then (35) yields that

$$\|v^{-\sigma_1}\|_{k_{j+1}, r_{j+1}} \leq \left[ c m(-\sigma p^{-1})\{\mu(B(x_0, r))\}^{(1-k)/k}\{4(j_0 + 1)\}^p\right]^{k-j} \|v^{-\sigma_1}\|_{k_j, r_j}.$$

By iterating for $0 \leq j \leq j_0$, we have

$$\mu(B(x_0, r))^{-1}\|v^{-\sigma_1}\|_{k_{j_0+1}, r/2} \leq \left[ c m(-\sigma p^{-1})\{4(j_0 + 1)\}^p\right]^{k_{j_0+1} - 1} \times \left[ \{\mu(B(x_0, r))\}^{-1}\|v^{-\sigma_1}\|_{1,3r/4}\right]^{k_{j_0+1}}.$$

Since $0 < \sigma_1 < \sigma_0$, from (37) we obtain Estimate 2.

Hence Lemma 5.1 follows. □

References


