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ON THE THINNESS OF A CLOSED SET IN THE NEIGHBORHOOD OF THE POINT AT INFINITY

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1. Introduction

Let X be a locally compact and non-compact Hausdorff space with countable basis and $G = G(x, y)$ be a continuous function-kernel on X satisfying the complete maximun principle.

For any compact K and for any set A in X, the G-capacity, $cap_G(K)$, of K and the inner G-capacity, $cap_G^i(A)$, of A are defined as usual.

If $cap_G^i(A) < +\infty$, then there exists a compact $K \subset A$ such that

(1.1) $$cap_G^i(A) - \varepsilon \ < \ cap_G(K) \ \leq \ cap_G^i(A)$$

But then, the following inequality

(1.2) $$cap_G^i(A \backslash K) \ < \ \varepsilon$$

does not necessarily hold. Because the inner G-capacity is, indeed, subadditive but not additive in general.

In this paper, we first define the several notions of the thinness of A in the neighbourhood of the point at infinity and investigate the mutual relations holding among them, when A is an unbounded closed set.

Then we consider the conditions on the kernel G and on A under that the inequality (1.2) also holds.

2. Preliminaries

A non-negative function $G = G(x, y)$ on $X\times$X is called a *continuous function-kernel* if $G(x, y)$ is continuous in the extended sence on $X \times X$ and satisfies

$$0 \leq G(x,y) < +\infty \quad for \quad \forall (x,y) \in X \times X \quad s.t. \quad x \neq y,$$
$$0 < G(x,x) \leq +\infty \quad for \quad \forall x \in X.$$

We denote by M the set of all positive measures on X. The *G-potential* $G\mu(x)$ and the *G-energy* $\|\mu\|$ of μ is defined by

$$G\mu(x) = \int G(x,y)d\mu(y),$$
$$\|\mu\|^2 = \int G\mu(x)d\mu(x)$$

respectively.

Put

$$M_o = \{\mu \in M \ ; \ suport \ S\mu \ of \ \mu \ is \ compact \ \},$$
$$E_o = E_o(G) = \{\mu \in M_o \ ; \ \|\mu\| < +\infty\},$$
$$F_o = F_o(G) = \{\mu \in E_o(G) \ ; \ G\mu(x) \ is \ finite \ and \ continuous \ on \ X\}.$$

A Borel measurable set B is said to be *G-negligible* if $\mu(B) = 0$ for every $\mu \in E_o(G)$. We say that a property holds *G-nearly everywhere* on a subset A of X(written symply *G-n.e.* on A), when it holds on A except for a G-negligible set.

A non-negative lower semi-continuous function u on X is said to be *G-superharmonic*, when $u(x) < +\infty$ *G-n.e.* on X and for any $\mu \in E_o(G)$, the inequallity $G\mu(x) \leq u(x)$ *G-n.e.* on $S\mu$ implies the same inequality on the whole space X .

We denote by $S(G)$ the totality of G-superharmonic functions on X and by P_{M_o} (resp. $P_{E_o}(G)$) the totality of G-potentials of measures in M_o (resp. $E_o(G)$).

The potential theoretic principles are stated as follows.

(i) We say that G satisfies the *domination principle* and write simply $G \prec G$ when $P_{M_o}(G) \subset S(G)$.

(ii) We say that G satisfies the *maximun principle* and write simply $G \prec 1$ when $1 \in S(G)$.

(iii) We say that G satisfies the *complete maximun principle* when, for any $c \geq 0$, $P_{M_o}(G) \cup \{c\} \subset S(G)$.

(iv) We say that G satisfies the *balayage principle* if, for any compact K, there exists a measure $\mu'_K \in M_o$, called a *balayaged measure* of μ on K and supported by K satisfying

$$G\mu(x) = G\mu(x) \qquad G\text{-}n.e.\ on\ K,$$

$$G\mu'_K(x) \leq G\mu(x) \qquad on\ X.$$

(v) We say that G satisfies the *equilibrium principle* if, for any compact K, there exists a measure $\gamma_K \in M_o$, called an *equilibrium measure* of K and supported by K satisfying

$$G\gamma_K(x) = 1 \qquad G\text{-}n.e.\ on\ K,$$

$$G\gamma_K(x) \leq 1 \qquad on\ X.$$

(vi) We say that G satisfies the *continuity principle* if, for $\mu \in M_o$, the finite continuity of the restriction of $G\mu(x)$ to $S\mu$ implies the finite continuity of $G\mu(x)$ on the whole space X.

3. Thinness at infinity δ of a closed set with finite capacity

In this section, we define the several notions of thinness of a closed set at δ, the point at infinity, and shall obtain the mutual relations holding among them.

For any compact K and any set A in X, the *G-capacity* $cap_G(K)$ of K and the *G-inner capacity* $cap_G^i(A)$ of A are difined respectively by

$$cap_G(K) = \inf\{\int d\mu \ ; \ \mu \in M_o, \ G\mu(x) \geq 1 \ G-n.e. \text{ on } K \text{ and } S\mu \subset K\},$$

$$cap_G^i(A) = \sup\{cap_G(K) \ ; \ K \text{ is compact set contained in } A\}.$$

For a Borel function u and a closed set F, the *G-reduced function* of u on F and the *G-reduced function* of u on F at infinity δ, are defined respectively by

$$R_G^F(u)(x) = \inf\{v(x) \ ; \ v \in S(G), \ v(x) \geq u(x) \ G\text{-}n.e. \ \ on \ X\},$$

$$R_G^{F,\delta}(u)(x) = \inf_{\omega \in \Omega_o} R_G^{F \cap C\omega} u(x).$$

where Ω_o denotes the totality of all relatively compact open sets in X.

Definition 1. We say that a subset A of X is *G-thin at* infinity δ *in the sence of capacity* (written simply *G-cap. thin at* δ) when we have

$$\inf_{\omega \in \Omega_o} cap_G^i(A \cap C\omega) = 0.$$

For any set $A \subset X$, the subset $S_o(F;G)$ of $S(G)$ is defined by

$$S_o(F;G) = \{u \in S(G) \ ; \ R_G^{F,\delta} u(x) = 0 \ \ G\text{-}n.e. \ \ on \ X\}.$$

In the following, the class $S_o(F;G)$ plays an important role.

Definition 2. We say that a subset A of X is *G-1-thin at infinity* δ when $1 \in S_o(A;G)$.

Definition 3. We say that a subset A of X is *G-thin at infinty* δ, when $P_{M_o}(G) \subset S_o(A;G)$.

Definition 4. We say that, on a subset A, a function u on X *converges to* 0 *in capacity at infinity* δ, if the equality

$$\inf_{\omega \in \Omega_o} cap^i_G(A \cap E \cap C\omega) = 0$$

holds for $\forall c > 0$, where $E = E(u \geq c) = \{x \in \mathrm{X} \; ; \; u(x) \geq c\}$.

Throughout the rest of this paper, G denotes a continuous function-kernel on X for which every non-empty open set in X is not negligible. For simplicity we assume further that G is symmetric.

First we compare the notions of thinness of a closed set with finite G-inner capacity at infinity δ.

Theorem 1. *Suppose that G satisfies the complete maximum principle. Then, for any closed set F in X, the following four statements are equivalent :*

(1) *F is G-cap. thin at infinity δ.*

(2) (i) *$cap^i_G(F) < +\infty$, and*

(ii) *on F , $G\mu(x)$ converges to* 0 *in capacity at infinity δ on F for $\forall \mu \in M_o$.*

(3) (i) *$cap^i_G(F) < +\infty$, and*

(iii) *F is G-1-thin at infinity δ.*

(4) (i) *$cap^i_G(F) < +\infty$, and*

(iv) *F is G-thin at infinity δ.*

Corollary. *Suppose that G satisfies the complete maximum principle. Then for any closed set with finite G-inner capacity, the following five statements are equivalents :*

(1) *F is G-cap. thin at infinity δ.*

(2) *Given $\varepsilon > 0$, there exists a compact K satisfying*

$$cap^i_G(F) - \varepsilon < cap_G(K) \leq cap^i_G(F),$$

$$cap^i_G(F \setminus K) < \varepsilon.$$

(3) *On F, $G\mu(x)$ converges to 0 in capacity at infinity δ for any $\mu \in M_K$.*

(4) *F is G-1-thin at infinity δ.*

(5) *F is G-thin at infinity δ.*

To prove our theorem, first we recall the following lemma obtained in [2].

Lemma 1. *Suppose that G satisfies the domination principle. Then, for a closed set F, every function $u \in S_o(G; F)$ can be balayaged on F, namely, there exists a measure $\mu'_F \in M$ supported by F satisfying*

$$G\mu'_F(x) = u(x) \quad G\text{-n.e. on } F,$$

$$G\mu'_F(x) \leq u(x) \quad \text{in } X.$$

Proof of Theorem 1. The equivalences (1) $\longleftrightarrow$ (3) $\longleftrightarrow$ (4) have been obtained in [3] by using Lemma 1.

The implication (1) $\longrightarrow$ (2) is trivial and therefore it suffices to obtain the implication (2) $\longrightarrow$ (3).

Suppose (2). For any measure $\nu \in M_o$ and $c > 0$, we put

$$E = E(G\nu(x) \geq c) = \{x \in X \ ; \ G\nu(x) \geq c\}.$$

Given a compact K and an open ω, we denote by $\gamma_{F\cap C\omega\cap K}$ (resp. $\gamma_{F\cap C\omega\cap E\cap K}$). By (ii), we can find, for any $\varepsilon > 0$, an open set $\omega_o \in \Omega_o$ veryfying

$$\int d\gamma_{F\cap C\omega\cap E\cap K} < \varepsilon \ \textit{for any open} \ \omega \supset \omega_o. \tag{3.1}$$

Then we have, for $\forall\nu \in F_o(G)$,

$$\int R_G^{F\cap C\omega\cap K}(1)d\nu \ = \ \int G\nu d\gamma_{F\cap C\omega\cap K} \ = \ \int_E \ + \int_{CE} \ . \tag{3.2}$$

We shall estimate the last two integrals. By (3.1), there exists $M > 0$ such that

$$\int_E \ \leq \ \int G\nu d\gamma_{F\cap C\omega\cap E\cap K} \ < \ M\cdot\varepsilon \ \textit{ for any open } \ \omega \supset \omega_o. \tag{3.3}$$

On the other hand, the second integral is estimated as follows.

$$\int_{CE} \ = \ \int_{CE} G\nu d\gamma_{F\cap C\omega\cap K} \ < \ c\cdot cap_G^i(F). \tag{3.4}$$

Let K and ω tend to X and we have

$$\int R_G^{F,\delta}(1)d\nu \ \leq \ M\cdot\varepsilon \ + \ c\cdot cap_G^i(F). \tag{3.5}$$

Further letting c and ε tend to 0, we obtain

$$\int R_G^{F,\delta}(1)d\nu \ = \ 0, \tag{3.6}$$

and hence (iii). This copmletes the proof. ∎

4. Thinness at infinity δ of a closed set with infinte capacity

For a closed set, the following characterizations of G-thinness at infinity δ have been already obtained (cf. [1], [2], [3] and [4]).

Theorem 2. *Suppose that G satisfies the complete maximum principle and that G is non-degenerate,namely, the potentials $G\varepsilon_{x_1}(x)$ and $G\varepsilon_{x_2}(x)$ are not proportional if $x_1 \neq x_2$. Then for any closed set F, the following statements are equivalent :*

(1) *F is G-thin at infinity δ.*

(2) *On F, for $\forall\mu \in M_K$, $G\mu(x)$ converges to 0 at infinity δ.*

(3) *G has the so called dominated convergence property :*

$\{\mu_n\}_{n=1}^{\infty} \subset M$, $S\mu_n \subset F$ *and* $\mu_n \longrightarrow \mu_o$ *vaguely as* $n \longrightarrow +\infty$, *and*

$\exists\nu \in M_o$ *such that* $G\mu_n(x) \leq G\nu(x)$ *on* X *for all* n.

$\Longrightarrow$

$G\mu_o(x) = \liminf_{n\to\infty} G\mu_n(x)$ *G- n.e. on* X.

(4) *G is strongly balayable, namely, for $\forall u \in S(G)$ dominated by a potential in $P_{M_o}(G)$ and for every closed set $F' \subset F$, there exists a positive measure μ' supported by F' and verifying*

$G\mu'(x) = u(x)$ *G-n.e. on* F' ,

$G\mu'(x) \leq u(x)$ *on* X.

By the same methods used in the proof of Theorem 2, we can also characterize the G-1-thinness at infinity δ of a closed set with infinite G-inner capacity.

Theorem 3. *Suppose that G satisfies the complete maximum principle and that G is non-degenerate. Then, for any closed set F in X, the following three statements are equivalent :*

(1) *F is G-1-thin at infinity δ.*

(2) *G has the following dominated convergence property :*

$$\{\mu\}_{n=1}^{\infty} \subset M \ , S\mu_n \subset F, \quad \mu_n \longrightarrow \mu_o \text{ vaguely as } n \longrightarrow +\infty,$$

$$\{G\mu_n(x)\}_{n=1}^{\infty} \text{ is uniformly bounded on } X$$

$$\Longrightarrow$$

$$G\mu_o(x) \quad = \quad \liminf_{n\to+\infty} G\mu_n(x) \quad G\text{-n.e. on } X.$$

(3) *Every bounded G-superharmonic function can be balayaged on every closed set contained in F.*

For the proof of Theorem 3, it suffices to prepare the following two lemmata.

Lemma 2. *Suppose that G satisfies the domination principle and that G is non-degenerate. Then for any closed set F, the following two statements are equivalent:*

(1) $P_{F_o}(G) \subset S_o(F;G)$.

(2) *Every G-superharmonic function dominated by a potential in $P_{F_o}(G)$ can be balayaged on every closed set contained in F.*

Lemma 3. *Suppose that G satisfies the complete maximum principle. Then for any closed set, the following two statements are equivalent :*

(1) *F is G-1-thin at infinity δ .*

(2) (i) *There exists an equilibrium mesrure of F , and*

(ii) $P_{M_o}(G) \subset S_o(F;G)$.

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