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京都大学
ON THE THINNESS OF A CLOSED SET
IN THE NEIGHBORHOOD OF THE POINT AT INFINITY

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1. Introduction

Let $X$ be a locally compact and non-compact Hausdorff space with countable basis and $G = G(x, y)$ be a continuous function-kernel on $X$ satisfying the complete maximum principle.

For any compact $K$ and for any set $A$ in $X$, the $G$-capacity, $cap_G(K)$, of $K$ and the inner $G$-capacity, $cap_G^i(A)$, of $A$ are defined as usual.

If $cap_G^i(A) < +\infty$, then there exists a compact $K \subset A$ such that

(1.1) $cap_G^i(A) - \varepsilon < cap_G(K) \leq cap_G^i(A)$

But then, the following inequality

(1.2) $cap_G^i(A \setminus K) < \varepsilon$

does not necessarily hold. Because the inner $G$-capacity is, indeed, subadditive but not additive in general.

In this paper, we first define the several notions of the thinness of $A$ in the neighbourhood of the point at infinity and investigate the mutual relations holding among them, when $A$ is an unbounded closed set.

Then we consider the conditions on the kernel $G$ and on $A$ under that the inequality (1.2) also holds.
2. Preliminaries

A non-negative function $G = G(x, y)$ on $X \times X$ is called a continuous function-kernel if $G(x, y)$ is continuous in the extended sense on $X \times X$ and satisfies

\[
0 \leq G(x, y) < +\infty \quad \text{for} \quad \forall (x, y) \in X \times X \quad \text{s.t.} \quad x \neq y,
\]

\[
0 < G(x, x) \leq +\infty \quad \text{for} \quad \forall x \in X.
\]

We denote by $M$ the set of all positive measures on $X$. The $G$-potential $G\mu(x)$ and the $G$-energy $||\mu||$ of $\mu$ is defined by

\[
G\mu(x) = \int G(x, y)d\mu(y),
\]

\[
||\mu||^2 = \int G\mu(x)d\mu(x)
\]

respectively.

Put

\[
M_o = \{\mu \in M ; \text{suport } S\mu \text{ of } \mu \text{ is compact } \},
\]

\[
E_o = E_o(G) = \{\mu \in M_o ; ||\mu|| < +\infty \},
\]

\[
F_o = F_o(G) = \{\mu \in E_o(G) ; G\mu(x) \text{ is finite and continuous on } X \}.
\]

A Borel measurable set $B$ is said to be $G$-negligible if $\mu(B) = 0$ for every $\mu \in E_o(G)$. We say that a property holds $G$-nearly everywhere on a subset $A$ of $X$ (written symply $G$-n.e. on $A$), when it holds on $A$ except for a $G$-negligible set.

A non-negative lower semi-continuous function $u$ on $X$ is said to be $G$-superharmonic, when $u(x) < +\infty$ $G$-n.e. on $X$ and for any $\mu \in E_o(G)$, the inequallity $G\mu(x) \leq u(x)$ $G$-n.e. on $S\mu$ implies the same inequality on the whole space $X$. 
We denote by $S(G)$ the totality of $G$-superharmonic functions on $X$ and by $P_{M_{o}}$ (resp. $P_{E_{o}}(G)$) the totality of $G$-potentials of measures in $M_{o}$ (resp. $E_{o}(G)$).

The potential theoretic principles are stated as follows.

(i) We say that $G$ satisfies the domination principle and write simply $G \prec G$ when $P_{M_{o}}(c) \subset S(G)$.

(ii) We say that $G$ satisfies the maximum principle and write simply $G \prec 1$ when $1 \in S(G)$.

(iii) We say that $G$ satisfies the complete maximum principle when, for any $c \geq 0$, $P_{M_{o}}(G) \cup \{c\} \subset S(G)$.

(iv) We say that $G$ satisfies the balayage principle if, for any compact $K$, there exists a measure $\mu'_{K} \in M_{o}$, called a balayaged measure of $\mu$ on $K$ and supported by $K$ satisfying

\[
G\mu(x) = G\mu(x) \quad \text{G-n.e. on } K,
\]

\[
G\mu'_{K}(x) \leq G\mu(x) \quad \text{on } X.
\]

(v) We say that $G$ satisfies the equilibrium principle if, for any compact $K$, there exists a measure $\gamma_{K} \in M_{o}$, called an equilibrium measure of $K$ and supported by $K$ satisfying

\[
G\gamma_{K}(x) = 1 \quad \text{G-n.e. on } K,
\]

\[
G\gamma_{K}(x) \leq 1 \quad \text{on } X.
\]

(vi) We say that $G$ satisfies the continuity principle if, for $\mu \in M_{o}$, the finite continuity of the restriction of $G\mu(x)$ to $S\mu$ implies the finite continuity of $G\mu(x)$ on the whole space $X$. 
3. Thinness at infinity $\delta$ of a closed set with finite capacity

In this section, we define the several notions of thinness of a closed set at $\delta$, the point at infinity, and shall obtain the mutual relations holding among them.

For any compact $K$ and any set $A$ in $X$, the $G$-capacity $\text{cap}_G(K)$ of $K$ and the $G$-inner capacity $\text{cap}^i_G(A)$ of $A$ are defined respectively by

$$\text{cap}_G(K) = \inf \{ \int d\mu ; \mu \in M_\circ, \ G\mu(x) \geq 1 \text{ G-n.e. on } K \text{ and } S\mu \subset K \} ,$$
$$\text{cap}^i_G(A) = \sup \{ \text{cap}_G(K) ; K \text{ is compact set contained in } A \} .$$

For a Borel function $u$ and a closed set $F$, the $G$-reduced function of $u$ on $F$ and the $G$-reduced function of $u$ on $F$ at infinity $\delta$, are defined respectively by

$$R_G^{F}(u)(x) = \inf \{ v(x) ; v \in S(G), \ v(x) \geq u(x) \text{ G-n.e. on } X \} ,$$
$$R_G^{F,\delta}(u)(x) = \inf_{\omega \in \Omega_\circ} R_G^{F \cap C\omega} u(x).$$

where $\Omega_\circ$ denotes the totality of all relatively compact open sets in $X$.

**Definition 1.** We say that a subset $A$ of $X$ is $G$-thin at infinity $\delta$ in the sense of capacity (written simply $G$-cap. thin at $\delta$) when we have

$$\inf_{\omega \in \Omega_\circ} \text{cap}^i_G(A \cap C\omega) = 0.$$ 

For any set $A \subset X$, the subset $S_\circ(F;G)$ of $S(G)$ is defined by

$$S_\circ(F;G) = \{ u \in S(G) ; R_G^{F,\delta} u(x) = 0 \text{ G-n.e. on } X \}.$$ 

In the following, the class $S_\circ(F;G)$ plays an important role.
Definition 2. We say that a subset $A$ of $X$ is $G$-1-thin at infinity $\delta$ when $1 \in S_0(A;G)$.

Definition 3. We say that a subset $A$ of $X$ is $G$-thin at infinity $\delta$, when $P_{M_o}(G) \subset S_0(A;G)$.

Definition 4. We say that, on a subset $A$, a function $u$ on $X$ converges to 0 in capacity at infinity $\delta$, if the equality

$$\inf_{\omega \in \Omega_o} \text{cap}^i_G(A \cap E \cap C\omega) = 0$$

holds for $\forall c > 0$, where $E = E(u \geq c) = \{x \in X : u(x) \geq c\}$.

Throughout the rest of this paper, $G$ denotes a continuous function-kernel on $X$ for which every non-empty open set in $X$ is not negligible. For simplicity we assume further that $G$ is symmetric.

First we compare the notions of thinness of a closed set with finite $G$-inner capacity at infinity $\delta$.

Theorem 1. Suppose that $G$ satisfies the complete maximum principle. Then, for any closed set $F$ in $X$, the following four statements are equivalent:

(1) $F$ is $G$-cap. thin at infinity $\delta$.

(2) (i) $\text{cap}^i_G(F) < +\infty$, and

(ii) on $F$, $G\mu(x)$ converges to 0 in capacity at infinity $\delta$ on $F$ for $\forall \mu \in M_o$.

(3) (i) $\text{cap}^i_G(F) < +\infty$, and

(iii) $F$ is $G$-1-thin at infinity $\delta$.

(4) (i) $\text{cap}^i_G(F) < +\infty$, and

(iv) $F$ is $G$-thin at infinity $\delta$. 
Corollary. Suppose that $G$ satisfies the complete maximum principle. Then for any closed set with finite $G$-inner capacity, the following five statements are equivalents:

(1) $F$ is $G$-cap. thin at infinity $\delta$.

(2) Given $\epsilon > 0$, there exists a compact $K$ satisfying

$$\text{cap}_G^i(F) - \epsilon < \text{cap}_G(K) \leq \text{cap}_G^i(F),$$

$$\text{cap}_G^i(F \setminus K) < \epsilon.$$

(3) On $F$, $G\mu(x)$ converges to 0 in capacity at infinity $\delta$ for any $\mu \in M_K$.

(4) $F$ is $G$-1-thin at infinity $\delta$.

(5) $F$ is $G$-thin at infinity $\delta$.

To prove our theorem, first we recall the following lemma obtained in [2].

Lemma 1. Suppose that $G$ satisfies the domination principle. Then, for a closed set $F$, every function $u \in S_o(G;F)$ can be balayaged on $F$, namely, there exists a measure $\mu'_F \in M$ supported by $F$ satisfying

$$G\mu'_F(x) = u(x) \quad G\text{-n.e. on } F,$$

$$G\mu'_F(x) \leq u(x) \quad \text{in } X.$$ 

Proof of Theorem 1. The equivalences $(1) \iff (3) \iff (4)$ have been obtained in [3] by using Lemma 1.

The implication $(1) \rightarrow (2)$ is trivial and therefore it suffices to obtain the implication $(2) \rightarrow (3)$. 
Suppose (2). For any measure $\nu \in M_o$ and $c > 0$, we put
\[ E = E(G\nu(x) \geq c) = \{x \in X ; G\nu(x) \geq c\}. \]

Given a compact $K$ and an open $\omega$, we denote by $\gamma_{F\cap C\omega \cap K}$ (resp. $\gamma_{F\cap C\omega \cap E \cap K}$).

By (ii), we can find, for any $\varepsilon > 0$, an open set $\omega_o \in \Omega_o$ verifying
\[ (3.1) \quad \int d\gamma_{F\cap C\omega \cap E \cap K} < \varepsilon \quad \text{for any open } \omega \supset \omega_o. \]

Then we have, for $\forall \nu \in F_o(G)$,
\[ (3.2) \quad \int R_{F}(1)d\nu = \int G\nu d\gamma_{F\cap C\omega \cap K} = \int_E + \int_{CE}. \]

We shall estimate the last two integrals. By (3.1), there exists $M > 0$ such that
\[ (3.3) \quad \int_E \leq \int G\nu d\gamma_{F\cap C\omega \cap E \cap K} < M \cdot \varepsilon \quad \text{for any open } \omega \supset \omega_o. \]

On the other hand, the second integral is estimated as follows.
\[ (3.4) \quad \int_{CE} = \int_{CE} G\nu d\gamma_{F\cap C\omega \cap K} < c \cdot \text{cap}_G^i(F). \]

Let $K$ and $\omega$ tend to $X$ and we have
\[ (3.5) \quad \int R_{F}^{\delta}(1)d\nu \leq M \cdot \varepsilon + c \cdot \text{cap}_G^i(F). \]

Further letting $c$ and $\varepsilon$ tend to 0, we obtain
\[ (3.6) \quad \int R_{F}^{\delta}(1)d\nu = 0, \]

and hence (iii). This completes the proof. \[ \blacksquare \]
4. Thinness at infinity $\delta$ of a closed set with infinite capacity

For a closed set, the following characterizations of $G$-thinness at infinity $\delta$ have been already obtained (cf. [1], [2], [3] and [4]).

**Theorem 2.** Suppose that $G$ satisfies the complete maximum principle and that $G$ is non-degenerate, namely, the potentials $G\varepsilon_{x_1}(x)$ and $G\varepsilon_{x_2}(x)$ are not proportional if $x_1 \neq x_2$. Then for any closed set $F$, the following statements are equivalent:

1. $F$ is $G$-thin at infinity $\delta$.

2. On $F$, for $\forall \mu \in M_K$, $G\mu(x)$ converges to 0 at infinity $\delta$.

3. $G$ has the so-called dominated convergence property:

\[ \{\mu_n\}_{n=1}^{\infty} \subset M, \quad S\mu_n \subset F \text{ and } \mu_n \rightarrow \mu_0 \text{ vaguely as } n \rightarrow +\infty, \text{ and} \]

\[ \exists \nu \in M_o \text{ such that } G\mu_n(x) \leq G\nu(x) \text{ on } X \text{ for all } n. \]

\[ \Rightarrow \]

\[ G\mu_0(x) = \liminf_{n \rightarrow \infty} G\mu_n(x) \text{ } G\text{-n.e. on } X. \]

4. $G$ is strongly balayable, namely, for $\forall u \in S(G)$ dominated by a potential in $P_{M_o}(G)$ and for every closed set $F' \subset F$, there exists a positive measure $\mu'$ supported by $F'$ and verifying

\[ G\mu'(x) = u(x) \text{ } G\text{-n.e. on } F', \]

\[ G\mu'(x) \leq u(x) \text{ on } X. \]

By the same methods used in the proof of Theorem 2, we can also characterize the $G$-1-thinness at infinity $\delta$ of a closed set with infinite $G$-inner capacity.
**Theorem 3.** Suppose that $G$ satisfies the complete maximum principle and that $G$ is non-degenerate. Then, for any closed set $F$ in $X$, the following three statements are equivalent:

1. $F$ is $G$-1-thin at infinity $\delta$.
2. $G$ has the following dominated convergence property:
   \[ \{\mu\}_{n=1}^{\infty} \subset M, S\mu_n \subset F, \mu_n \rightharpoonup \mu_0 \text{ vaguely as } n \to +\infty, \]
   \[ \{G\mu_n(x)\}_{n=1}^{\infty} \text{ is uniformly bounded on } X \]
   \[ \Rightarrow \]
   \[ G\mu_0(x) = \liminf_{n \to +\infty} G\mu_n(x) \text{ G-n.e. on } X. \]
3. Every bounded $G$-superharmonic function can be balayaged on every closed set contained in $F$.

For the proof of Theorem 3, it suffices to prepare the following two lemmata.

**Lemma 2.** Suppose that $G$ satisfies the domination principle and that $G$ is non-degenerate. Then for any closed set $F$, the following two statements are equivalent:

1. $P_F(G) \subset S_0(F;G)$.
2. Every $G$-superharmonic function dominated by a potential in $P_F(G)$ can be balayaged on every closed set contained in $F$.

**Lemma 3.** Suppose that $G$ satisfies the complete maximum principle. Then for any closed set, the following two statements are equivalent:

1. $F$ is $G$-1-thin at infinity $\delta$.
2. (i) There exists an equilibrium measure of $F$, and
   (ii) $P_{M_0}(G) \subset S_0(F;G)$. 
REFERENCES


