数理解析研究所講究録
1016 巻 1997 年 42-57

ON CONTINUITY OF EXTREMAL DISTANCE IN CASE ISLANDS EXIST

神 直人 (学習院大・理) (NAONDO JIN)
大津賀 信 (MAKOTO OHTSUKA)

Part 1. Islanded curves

We shall treat the case when islands exist in a domain under consideration as in the title. Such a discussion was first made in [MR] on Riemann surfaces. Let K be a compact set consisting of mutually disjoint compact sets K_0 and K_1 in $\mathbb{R}^d$, and E be a relatively closed bounded subset of $\mathbb{R}^d \setminus K$ such that the closure of each component of E is disjoint from K and $\mathbb{R}^d \setminus (K \cup E)$ is a domain. Hence $K \cup E$ is a compact set and each component of $K \cup E$ is a subset of either K_0 or K_1 or E. We call each component of E an island.

We set $Z = \mathbb{R}^d \setminus (K \cup E)$. A sequence $\{Z_n\}_{n=0,1,\dots}$ of subdomains of $\mathbb{R}^d$ with the following properties will be called an exhaustion of Z: $Z = \cup_{n=0}^{\infty} Z_n$, $\mathbb{R}^d \setminus Z_0$ is a compact set, $\overline{Z}_n \subset Z_{n+1}$ for each $n \geq 0$, the boundary of each Z_n consists of finitely many polygonal surfaces, no component of $Z \setminus Z_n$ is compact in Z for each n. We note that the sequence $\{Z_n\}$ starts with Z_0 instead of Z_1, and that $\{\mathbb{R}^d \setminus Z_n\}$ is an approximation of $K \cup E$ from the outside. In the following Figure 1 an exhaustion $\{Z_n\}$ is obtained by a dyadic division of $\mathbb{R}^d$ and taking a suitable subsequence. In addition to the above properties of $\{Z_n\}$ we assume for every n that the boundary of each component of $\mathbb{R}^d \setminus Z_n$ is a (connected polygonal) surface and the interior of the component is a domain.

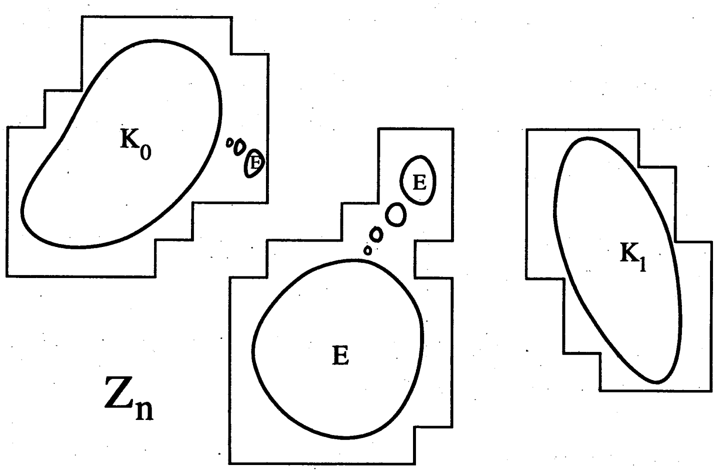

FIGURE 1.

We consider a new space Ξ whose elements consist of the points of $\mathbb{R}^d \setminus (K \cup E)$ and the components of $K \cup E$ so that each component of $K \cup E$ is regarded as a point in Ξ. We shall use the notation p to denote the mapping of $\mathbb{R}^d$ onto Ξ, and call this mapping the projection. Let us introduce a topology on Ξ by taking a base of neighborhood system for each element ξ of Ξ as follows. For simplicity we will call it a base at ξ. In case ξ is a point x of $\mathbb{R}^d \setminus (K \cup E)$ we take an open ball in $\mathbb{R}^d \setminus (K \cup E)$, centered at x, as an element of a base at ξ. In case ξ is the projection of a component κ of K_0 or K_1 or E, considering an exhaustion $\{Z_n\}$ of Z, we take the projection of the component of $\mathbb{R}^d \setminus Z_n$ which contains κ as an element of a base at ξ. In what follows we often identify points of $\mathbb{R}^d \setminus (K \cup E)$ in Ξ with those in $\mathbb{R}^d$. The space Ξ coincides with the Kerékjártó-Stoïlow compactification except that the latter includes the point at infinity but Ξ does not.

Now we consider a bounded open set G whose closure meets every component of $K \cup E$. We shall denote by $\Gamma_\Xi(K_0, K_1, E, G)$ the family of curves γ in Ξ which connect $p(K_0)$ and $p(K_1)$ such that every $\gamma \setminus p(K \cup E)$ is contained in G and the component curves of every $\gamma \setminus p(K \cup E)$ are locally rectifiable. We call $p^{-1}(\gamma)$ an islanded curve or simply i-curve connecting K_0 and K_1 through E in G, although K and E may not be contained in G. We shall use the notation $\Gamma(K_0, K_1, E, G)$ for the family of i-curves. We write it simply Γ too. See Figure 2.

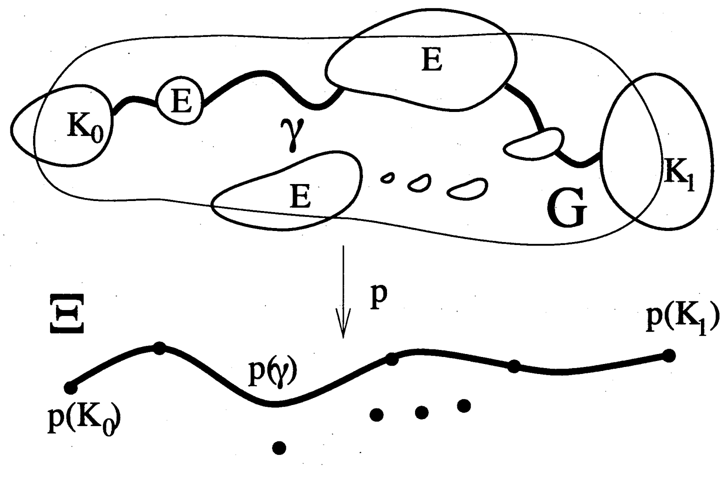

FIGURE 2. $p(K_0)$ etc may consist of more than one point

Let c be an i-curve in $\mathbb{R}^d$. We define the length s of any component of $c \setminus (K \cup E)$ in the ordinary manner. We shall associate a measure with c as follows: Let l be a component of $c \setminus (K \cup E)$ and $x(s)$ be a representation of l in terms of arc length. Define a measure μ_l in $\mathbb{R}^d$ by means of the equality $\mu_l(B) = \int_{s \in x^{-1}(B \cap l)} ds$ for Borel sets B in $\mathbb{R}^d$, and define $\mu_c(B) = \sum_n \mu_{l_n}(B)$, where $\{l_n\}$ are the components of $c \setminus (K \cup E)$. For an i-curve and a non-negative Borel measurable function ρ in $\mathbb{R}^d$ we shall write $\int_{c \setminus (K \cup E)} \rho ds$ or simply $\int_c \rho ds$ instead of $\int_{\mathbb{R}^d} \rho d\mu_c$.

We assume $\Gamma \neq \emptyset$ and call a function ρ in $\mathbb{R}^d$ Γ-admissible or simply Γ-ad. if it is non-negative Borel measurable and $\int_c \rho ds \geq 1$ for every $c \in \Gamma$. Given a weight ω and

$p, 1 \leq p < \infty$, we define the weighted modulus $M_p(\Gamma; \omega) = M_p(K_0, K_1, E, G; \omega)$ by

$$\inf_\rho \left\{ \int_{\mathbb{R}^d} \rho^p \omega dx; \rho \text{ is } \Gamma\text{-ad.} \right\};$$

in case $\Gamma = \emptyset$ we define $M_p(\Gamma; \omega)$ to be zero. The weighted extremal length $\lambda_p(\Gamma; \omega) = \lambda_p(K_0, K_1, E, G; \omega)$ is defined to be $1/M_p(K_0, K_1, E, G; \omega)$. We call them the weighted modulus and extremal length of a condenser (K_0, K_1, E, G) too. We shall also use the terminology such as (p, ω)-a.e. i-curve and (p, ω)-exc. family of i-curves. We shall call an i-curve c rectifiable if $\int_c ds = \int_{c \setminus (K \cup E)} ds$ is finite. Given a family of i-curves, (p, ω)-a.e. i-curve is rectifiable as in the case of ordinary curves.

Part 2. Main result

Let $\{Z_n\}_{n=0,1,\ldots}$ be an exhaustion of Z as above. By taking $\{Z_n\}_{n=n_0,n_0+1,\ldots}$ as $\{Z_n\}_{n=0,1,\ldots}$ for a large n_0 if necessary, we may assume from the beginning that no component of $Z \setminus Z_0$ contains both some points of K_0 and some points of K_1. Let $K_0^{(n)}$ (resp. $K_1^{(n)}$) be the union of the components of $\mathbb{R}^d \setminus Z_n$ each of which is not disjoint from K_0 (resp. K_1). Set $K^{(n)} = K_0^{(n)} \cup K_1^{(n)}$ and $E^{(n)} = \mathbb{R}^d \setminus (Z_n \cup K^{(n)})$. Like Ξ we consider the space Ξ_n for each $n \geq 0$ which consists of the points of $\mathbb{R}^d \setminus (K^{(n)} \cup E^{(n)})$ and the components of $K^{(n)} \cup E^{(n)}$. As above we introduce a topology on Ξ_n and use the terminology "projection" and the notation p_n for the mapping of $\mathbb{R}^d$ onto Ξ_n. We define $\Gamma(K_0^{(n)}, K_1^{(n)}, E^{(n)}, G)$ like $\Gamma = \Gamma(K_0, K_1, E, G)$ and write it as Γ_n for simplicity.

We define also $M_p(\Gamma_n; \omega) = M_p(K_0^{(n)}, K_1^{(n)}, E^{(n)}, G; \omega)$ and $\lambda_p(\Gamma_n; \omega) = \lambda_p(K_0^{(n)}, K_1^{(n)}, E^{(n)}, G; \omega) = 1/M_p(\Gamma_n; \omega)$. We shall use the simple notation M, λ, $M^{(n)}$ and $\lambda^{(n)}$. We call λ (resp. $\lambda^{(n)}$) the extremal distance between K_0 (resp. $K_0^{(n)}$) and K_1 (resp. $K_1^{(n)}$) through E (resp. $E^{(n)}$).

Our main result is the following theorem. It will be proved at the end of the paper.

Theorem. *As $n \to \infty$ $\lambda^{(n)}$ tends to λ.*

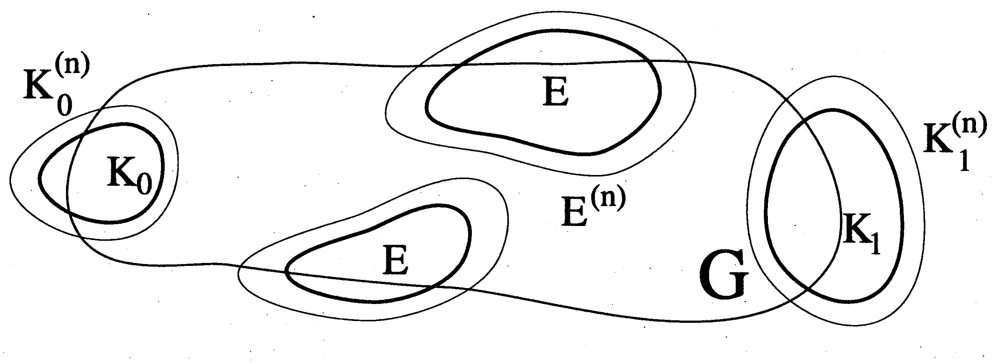

FIGURE 3. $\lambda^{(n)} = \lambda_p(K_0^{(n)}, K_1^{(n)}, E^{(n)}, G; \omega) \to \lambda = \lambda_p(K_0, K_1, E, G; \omega)$

Part 3. Lemma and part of its proof

We begin with a lemma.

Lemma. *Let $\omega \in A_p$, $0 < \varepsilon < 1$, $G, K_0, K_1, K, E, Z_n, K_0^{(n)}, K_1^{(n)}, K^{(n)}, E^{(n)}, \Gamma, \Gamma_n$ be as above and $0 < a < \infty$. Let $\rho \in L^{p,\omega}(\mathbb{R}^d)$ be a positive lower semicontinuous function which is continuous in $G \setminus (K \cup E)$. Then for each $\varepsilon > 0$, we can find a Borel measurable function $\rho' \geq \rho$ which have the following properties:*

1) $\displaystyle\int_{\mathbb{R}^d} \rho'^p \omega dx \leq \int_{\mathbb{R}^d} \rho^p \omega dx + \varepsilon.$

2) Suppose there exists a sequence of i-curves $c_n \in \Gamma_n, n = 1, 2, \ldots$, satisfying $\int_{c_n} \rho' ds \leq a$. Then there exists $\tilde{c} \in \Gamma$ such that $\int_{\tilde{c}} \rho ds \leq a + \varepsilon$.

Proof. Set $W_0 = Z_0$, $W_n = Z_n \setminus Z_{n-1}$ and $d_n = \operatorname{dist}(\partial Z_{n-1}, \partial Z_n)$ for $n \geq 1$. We define $\varepsilon_n, G_k, B_n, B$ as in the case when there are no islands. That is, we choose a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers decreasing to zero such that

$$(1) \qquad 2^p \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon, \qquad a\varepsilon_n < d_n \inf_{W_n \cap G} \rho \qquad \text{for } n = 1, 2, \ldots.$$

Let $\{G_k\}_{k=1,2,\ldots}$ be an exhaustion of G. Namely, $\overline{G}_k \subset G_{k+1}$ for each k and $\cup_k G_k = G$. We choose k_1 so that $\int_{B_1} \rho^p \omega dx < \varepsilon_1^{p+1}$, where $B_1 = (G \setminus G_{k_1}) \cap W_1$. For $n \geq 2$ we choose $k_1 < \cdots < k_n < \cdots$ so that $\int_{B_n} \rho^p \omega dx < \varepsilon_n^{p+1}$ for each n, where $B_n = (G \setminus G_{k_n}) \cap W_n$. We set $B = \cup_{n=1}^{\infty} B_n$.

Set

$$\rho'(x) = \begin{cases} \left(1 + \dfrac{1}{\varepsilon_n}\right) \rho(x) & \text{for } x \in B_n, n = 1, 2, \ldots, \\ \rho(x) & \text{for } x \in \mathbb{R}^d \setminus B. \end{cases}$$

Clearly ρ' is Borel measurable in $\mathbb{R}^d$. We see that

$$\int_B \rho'^p \omega dx = \sum_{n=1}^{\infty} \left(1 + \frac{1}{\varepsilon_n}\right)^p \int_{B_n} \rho^p \omega dx \leq \sum_{n=1}^{\infty} \frac{2^p}{\varepsilon_n^p} \cdot \varepsilon_n^{p+1} = 2^p \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$$

by (1). Hence

$$\int_G \rho'^p \omega dx = \int_{G \setminus B} \rho^p \omega dx + \int_B \rho'^p \omega dx < \int_G \rho^p \omega dx + \varepsilon.$$

Thus ρ' has the property 1).

We shall show that ρ' has the property 2). Let $c_n \in \Gamma_n$ for $n = 1, 2, \ldots$ and assume $\int_{c_n} \rho' ds \leq a$ for all n. Suppose there is an arc $c_{n,m} \subset c_n$ which connects ∂Z_{m-1} and ∂Z_m in $W_m \cap B$ for an $m, 1 \leq m < n$. Then

$$\int_{c_{n,m}} \rho ds = \frac{1}{1 + 1/\varepsilon_m} \int_{c_{n,m}} \rho' ds \leq \varepsilon_m \int_{c_n} \rho' ds \leq a\varepsilon_m.$$

Hence

$$d_m \inf_{W_m \cap G} \rho \leq \inf_{W_m \cap G} \rho \int_{c_{n,m}} ds \leq \int_{c_{n,m}} \rho ds \leq a\varepsilon_m.$$

This contradicts (1). Thus it is inferred that for any n and $m, 1 \leq m \leq n$, there is no arc on c_n which connects ∂Z_{m-1} and ∂Z_m entirely in B_m. See Figure 4.

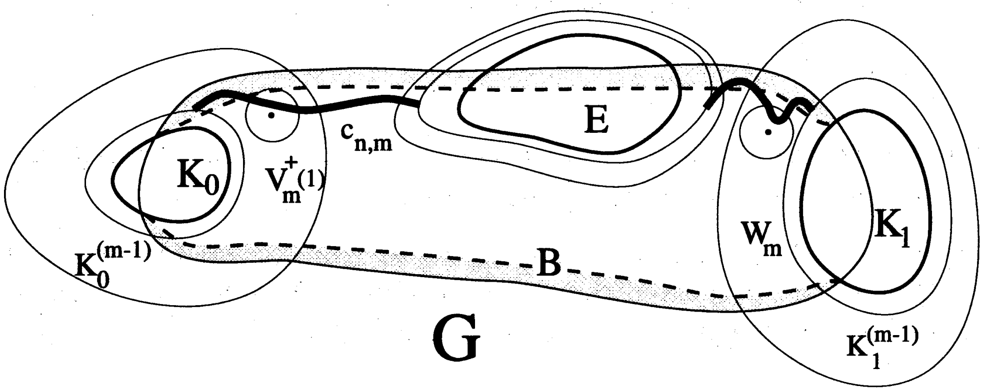

FIGURE 4.

Since $\rho' \geq \rho$ has a positive lower bound on $\overline{G}$ and $\int_{c_n} \rho' ds \leq a < \infty$ for each n, each c_n is rectifiable. This is not the end of the proof. There is a considerable way to go before the completion of the proof of the lemma.

Part 4. Contour sequence and graph

Still let $c_n \in \Gamma_n$ as in 2) in the statement of the lemma. We parametrize $p_n(c_n)$ as $\gamma_n = \{\eta_n(t); 0 \leq t \leq 1\}$. We apply this parametrization to c_n too. Then we have $c_n = \{p_n^{-1}(\eta_n(t)); 0 \leq t \leq 1\}$. The part of c_n outside $K^{(n)} \cup E^{(n)}$ consists of ordinary curves.

We follow the discussion at lines 17-25 of p.253 in [MR]. Fix $n \geq 1$ for a moment. Sometimes we shall call t "time" instead of parameter. For $m, 0 \leq m \leq n$, let $t_{n,m}(1)$ be the largest value of t for which $\eta_n(t) \in p_n(K_0^{(m)})$. We denote by $\alpha_m(1)$ the component of $\partial K_0^{(m)}$ on which $\eta_n(t_{n,m}(1))$ lies. Let $t'_{n,m}(1)$ be the smallest value of t for which $\eta_n(t) \in p_n(E^{(m)})$ and $t > t_{n,m}(1)$. Then $\eta_n(t'_{n,m}(1))$ lies on some component of $\partial E^{(m)}$ or $\partial K_1^{(m)}$. We denote the component by $\alpha_m(2)$. If $\eta_n(t'_{n,m}(1)) \in \partial K_1^{(m)}$, then the i-curve $c_{n,m}$ consisting of the component of $K_0^{(m)}$ bounded by $\alpha_m(1)$, the arc $\{\eta_n(t); t_{n,m}(1) < t < t'_{n,m}(1)\}$ and the component of $K_1^{(m)}$ bounded by $\alpha_m(2)$ is an element of Γ_m. If we want to regard this as a curve in Ξ_m, then we define $\eta_n^{(m)}(t) = p_m(\alpha_m(1))$ for $0 \leq t \leq t_{n,m}(1)$, $\eta_n^{(m)}(t) = \eta_n(t)$ for $t_{n,m}(1) < t < t'_{n,m}(1)$ and $\eta_n^{(m)}(t) = p_m(\alpha_m(2))$ for $t'_{n,m}(1) \leq t \leq 1$. Thus we obtain a curve $\gamma_{n,m}$ in Ξ_m which is represented by $\{\eta_n^{(m)}(t); 0 \leq t \leq 1\}$. See Figure 5 for an example of $\gamma_{n,m}$. In the subsequent figures we do not draw the open set G but it is supposed to exist.

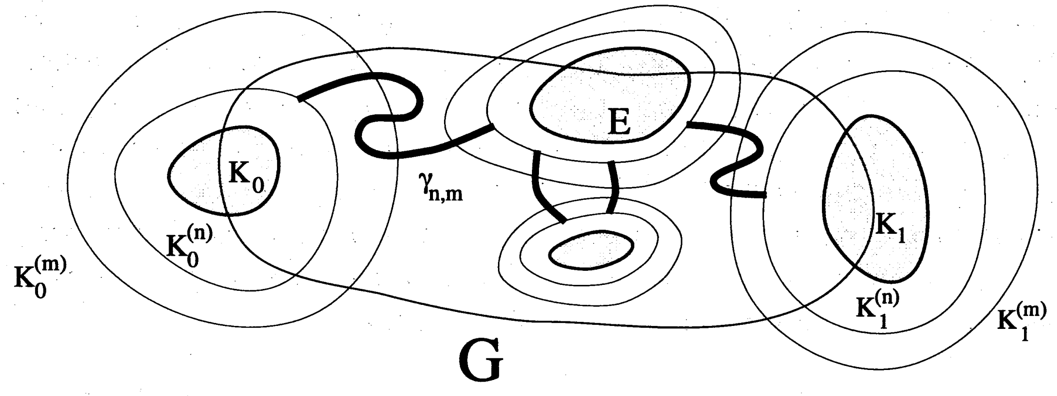

FIGURE 5.

If $\eta_n(t'_{n,m}(1)) \in \partial E^{(m)}$, then let $t_{n,m}(2)$ be the largest value of t for which $\eta_n(t) \in p_n(\alpha_m(2))$. It may happen that $t'_{n,m}(1) = t_{n,m}(2)$; in this case γ_n does not "cross" $\alpha_m(2)$ but only touches it. We continue this process, and obtain a sequence of "stopping times" $t_{n,m}(1) < t'_{n,m}(1) \leq t_{n,m}(2) \leq \cdots < t'_{n,m}(q_m)$, a sequence of stopping points $\eta_n(t_{n,m}(1)), ..., \eta_n(t'_{n,m}(q_m))$ and a sequence $\alpha_m(1)(\subset \partial K_0^{(m)}), \alpha_m(2), ..., \alpha_m(q_m+1)(\subset \partial K_1^{(m)})$ of distinct contours such that $\eta_n(t_{n,m}(1)) \in p_n(\alpha_m(1))$, $\eta_n(t'_{n,m}(1)) \in p_n(\alpha_m(2))$, $\eta_n(t_{n,m}(2)) \in p_n(\alpha_m(2)), ..., \eta_n(t'_{n,m}(j-1)) \in p_n(\alpha_m(j))$, $\eta_n(t_{n,m}(j)) \in p_n(\alpha_m(j)), ...,$ $\eta_n(t'_{n,m}(q_m)) \in p_n(\alpha_m(q_m+1))$. We call the sequence $\{\alpha_m(1), ..., \alpha_m(q_m+1)\}$ a (n,m)-contour sequence or simply a contour sequence. See Figure 6 for different contour sequences.

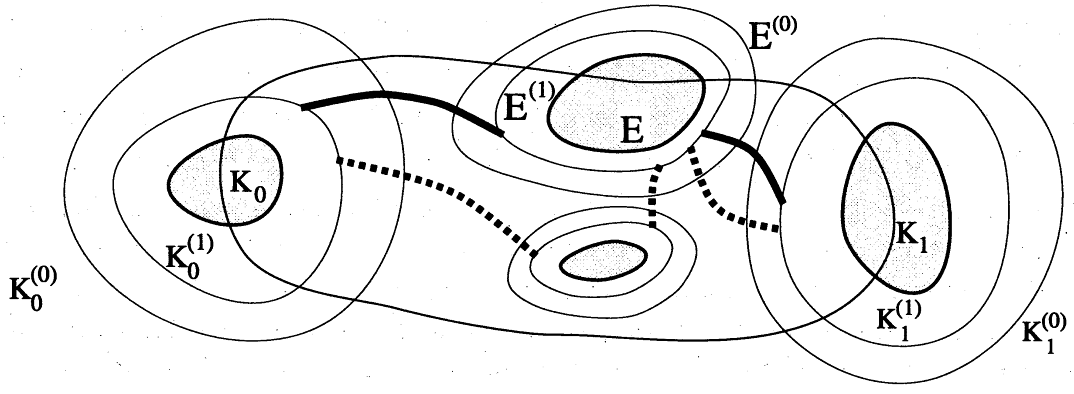

FIGURE 6. Example of different contour sequences in $Z_0 \cap G$

Define $\gamma_{n,m}$ to be the restriction of γ_n to

$$T_{n,m} = (t_{n,m}(1), t'_{n,m}(1)) \cup (t_{n,m}(2), t'_{n,m}(2)) \cup \cdots \cup (t_{n,m}(q_m), t'_{n,m}(q_m)).$$

We denote the components of $\gamma_{n,m}$ by $l_{n,m}(1), ..., l_{n,m}(q_m)$. When we want to regard $\gamma_{n,m}$ as part of a curve in Ξ_m also in case $q_m \geq 2$, besides $\eta_n^{(m)}(t) = \eta_n(t)$ for $t \in T_{m,n}$

we define $\eta_n^{(m)}(t)$ by $p_m(\alpha_m(1))$ for $0 \le t \le t_{n,m}(1)$, by the projection of the component of $E^{(m)}$ bounded by $\alpha_m(2)$ for $t'_{n,m}(1) \le t \le t_{n,m}(2), \dots$, by the projection of the component of $E^{(m)}$ bounded by $\alpha_m(q_m - 1)$ for $t'_{n,m}(q_{m-1}) \le t \le t_{n,m}(q_m)$ and finally by $p_m(\alpha_m(2))$ for $t'_{n,m}(q_m) \le t \le 1$. Now we have a curve in Ξ_m which we still denote by $\gamma_{n,m}$. We shall call such a modification of γ_n the m-shortening of γ_n. We perform the n-shortening of γ_n and denote the resulting curve in Ξ_n by $\gamma_n^{(n)}$. Next we perform the $(n-1)$-shortening of $\gamma_n^{(n)}$ and denote the resulting curve in Ξ_{n-1} by $\gamma_n^{(n-1)}$. We continue this process until we obtain $\gamma_n^{(0)}$. We note that

$$(2) \qquad \gamma_n^{(n)} \setminus (K^{(n)} \cup E^{(n)}) \supset \gamma_n^{(n-1)} \setminus (K^{(n-1)} \cup E^{(n-1)}) \supset \gamma_n^{(0)} \setminus (K^{(0)} \cup E^{(0)}).$$

We shall take a procedure similar to that at lines 9-15 of p.254 in [MR]. We consider $n = 1, 2, \dots$ Since there are only finitely many contour sequences on ∂Z_0, we select a subsequence $\{\gamma_{n_{0,k}}\}_{k=1,2,\dots}$ of $\{\gamma_n\}$ so that all the shortened curves $\{\gamma_{n_{0,k}}^{(0)}\}$ in Z_0 have the same contour sequence on ∂Z_0. Next we select a subsequence $\{\gamma_{n_{1,k}}\}_{k=1,2,\dots}$ of $\{\gamma_{n_{0,k}}\}_{k=1,2,\dots}$ so that all the shortened curves $\{\gamma_{n_{1,k}}^{(1)}\}$ in Z_1 have the same contour sequence on ∂Z_1. We continue this process and take a diagonal sequence $\{\gamma_{n_{k,k}}\}$. For each $k \ge 1$ we consider the k-shortening of $\gamma_{n_{k,k}}$ and replace the curves $\{c_k \in \Gamma_k\}$ originally given in 2) of the Lemma by the p_k^{-1}-images of the shortened curves of $\{\gamma_{n_{k,k}}\}$. We shall again use the notation $\gamma_1, \gamma_2, \dots$for these shortened curves. We note that the p_n^{-1}-images of the new $\{\gamma_n\}$ still satisfy the condition in 2) as to the ρ'-length. We emphasize for the new $\{\gamma_n\}$ that $\gamma_n, \gamma_{n+1}, \dots$ have the same contour sequence on ∂Z_n for each n. We represent each γ_n by $\eta_n(t), 0 \le t \le 1$, as a curve in Ξ_n as before.

We shall need a further study of contour sequences. Let $1 \le m \le n$. We denote the components of $\gamma_n^{(m)}$ in Z_m by $l_n^{(m)}(1), \dots, l_n^{(m)}(q_m)$. By (2) each $l_n^{(m-1)}(i)$ is contained in some $l_n^{(m)}(j)$. For $n' > n \ge m \ge 1$ we shall say that $\gamma_{n'}^{(m)}$ and $\gamma_n^{(m)}$ have the same m-contour graph if for each $j, 1 \le j \le q_m$, $l_{n'}^{(m)}(j)$ and $l_n^{(m)}(j)$ contain the same number of $l_{n'}^{(m-1)}(i)$'s and $l_n^{(m-1)}(i)$'s respectively; the number of $l_n^{(m)}(i)$ could be zero. In case $m = 0$ we shall say that $\gamma_{n'}^{(0)}$ and $\gamma_n^{(0)}$ have the same 0-contour graph if their contour sequences are same. Since there are only finitely many different contour graphs for each m, we may assume that the m-contour graph is same for all $\{\gamma_n^{(m)}\}, n \ge m$, for each $m \ge 1$ by choosing a subsequence of $\{\gamma_n\}$ if necessary. See two examples in Figure 7 which have the same contour sequences but the contour graph is different.

The following diagram shows the difference of the two contour graphs.

Example 1

$$\alpha \vdash \cdots\cdots \dashv \beta \qquad \beta \vdash \cdots\cdots\cdots \dashv \gamma \qquad \gamma \vdash \cdots\cdots\cdots \dashv \delta$$
$$\varepsilon \vdash \cdots\cdots\cdots\cdots \dashv \zeta \quad \zeta \vdash \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \dashv \eta$$

Example 2

$$\alpha \vdash \cdot\cdot \dashv \beta \qquad \beta \vdash \cdot\cdot \dashv \gamma \qquad\qquad \gamma \vdash \cdots\cdots \dashv \delta$$
$$\varepsilon \vdash \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \dashv \zeta \qquad \zeta \vdash \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \dashv \eta$$

In the next part we shall see the necessity of the notion of contour graph.

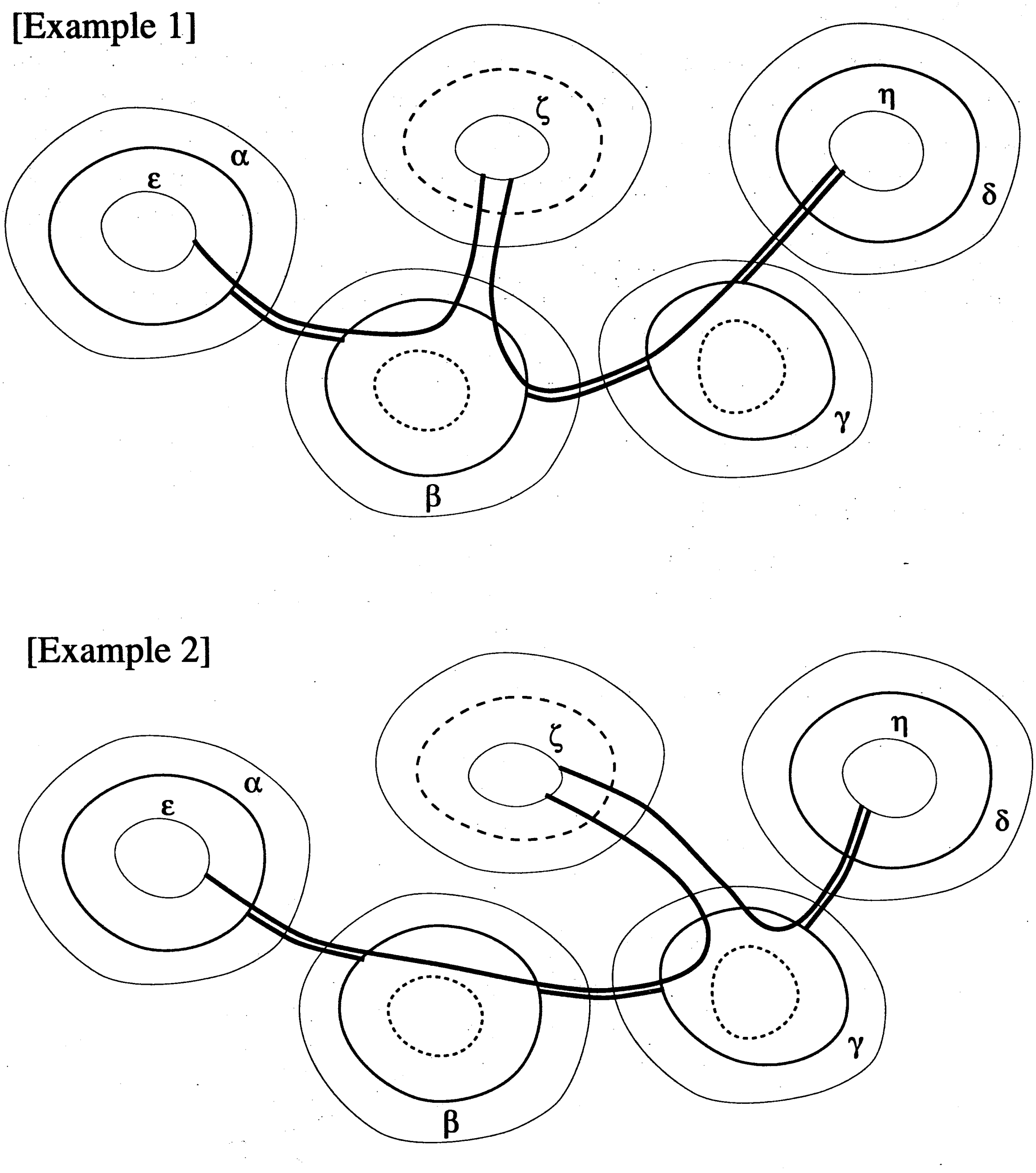

FIGURE 7.

Part 5. Relay posts

As in the preceding part, denote by $\gamma_n^{(m)}$ the i-curve obtained by the m-shortening of γ_n for $1 \le m \le n$. We are concerned with $\{\gamma_n^{(m)}\}$ for $n \ge 2$ and $2 \le m \le n$. We fix n for the moment. Let $(\alpha_m(j), \alpha_m(j+1)), 1 \le j \le q_1$, be a contour pair at the m-th step, and $l_n^{(m)}(j)$ be the arc on $\gamma_n^{(m)}$ which connects the above contour pair. We shall take replay posts on such arcs. How to take them depends on the contour graph. There are different relations of $l_n^{(m-1)}(j')$ to $l_n^{(m)}(j)$ as follows.

Type 1) There is no arc over $l_n^{(m)}(j)$. This type is shown in the following diagram.

$$\alpha_m(j) \quad \vdash \cdots\cdots\cdots\cdots \dashv \quad \alpha_m(j+1)$$
$$l_n^{(m)}(j)$$

We take no point on $l_n^{(m)}(j)$.

Type 2) There is just one arc over $l_n^{(m)}(j)$.

$$\alpha_{m-1}(j') \quad \vdash \cdots\cdots\cdots \dashv \quad \alpha_{m-1}(j'+1)$$
$$\alpha_m(j) \quad \vdash \cdots\cdots\cdots\cdots\cdots \dashv \quad \alpha_m(j+1)$$
$$l_n^{(m)}(j)$$

We divide the present type into various cases.

Case (i). First we draw a figure.

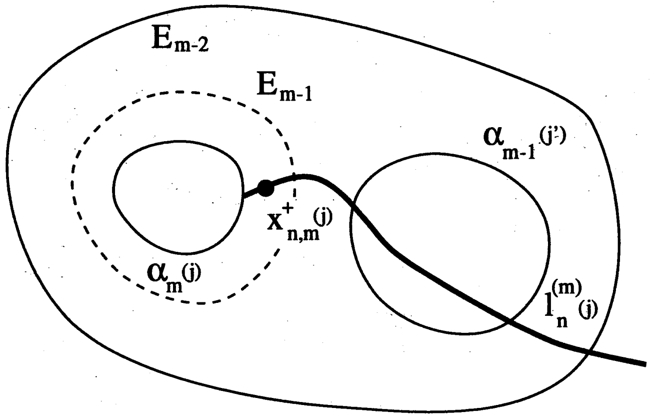

FIGURE 8.

Assume first that $m \ge 2$ and there is a component E_{m-2} of $\mathbb{R}^d \setminus Z_{m-2}$ which contains both $\alpha_m(j)$ and $\alpha_{m-1}(j')$. Then take the component E_{m-1} of $\mathbb{R}^d \setminus Z_{m-1}$ which contains $\alpha_m(j)$. This ∂E_{m-1} may coincide with $\alpha_{m-1}(j')$. We shall denote the compact domain bounded by a contour, say $\alpha_m(j)$, by $A_m(j)$. Since $l_n^{(m)}(j)$ connects $\alpha_m(j)$ and ∂E_{m-1}, there exists a point $x_{n,m}^+(j)$ on $l_n^{(m)}(j) \cap ((E_{m-1} \setminus A_m(j)) \setminus B)$.

If $m = 1$, then let E_0 be the component of $\mathbb{R}^d \setminus Z_0$ which contains $\alpha_1(j)$. This E_0 may coincide with $\alpha_0(j')$. As in the case $m \ge 2$ one can take $x_{n,1}^+(j)$.

Case (ii). In this case we always assume that $m \geq 2$ and that $\alpha_m(j)$ and $\alpha_{m-1}(j')$ are contained in different components of $\mathbb{R}^d \setminus Z_{m-2}$. Let E_{m-1} be the component of $\mathbb{R}^d \setminus Z_{m-1}$ which contains $\alpha_m(j)$. We draw a figure.

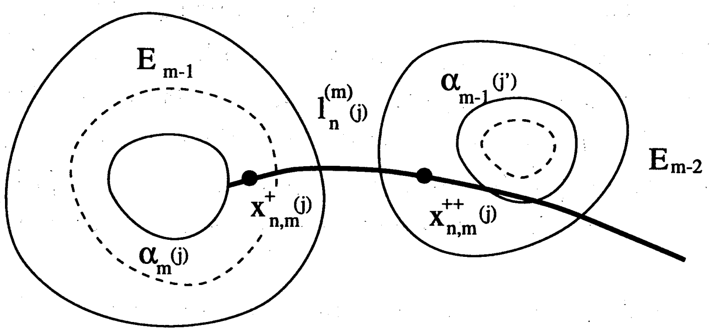

FIGURE 9.

As in Case (i) take a point $x^+_{n,m}(j)$ on $l_n^{(m)}(j) \cap ((E_{m-1} \setminus A_m(j)) \setminus B)$. Next let E_{m-2} be the component of $\mathbb{R}^d \setminus Z_{m-2}$ which contains $\alpha_{m-1}(j')$. Then, after leaving ∂E_{m-1}, $l_n^{(m)}(j)$ meets ∂E_{m-2} before meeting $\alpha_{m-1}(j')$. Thus $l_n^{(m)}(j)$ contains an arc connecting ∂E_{m-2} and $\alpha_{m-1}(j')$ in $E_{m-2} \setminus A_{m-1}(j')$. Since this arc is not entirely contained in B as observed in Part 3, there exists a point $x^{++}_{n,m}(j)$ on $l_n^{(m)}(j) \cap ((E_{m-2} \setminus A_{m-1}(j')) \setminus B)$.

Case (iii). In cases (iii) and (iv) we let $\alpha_m(j+1)$ play the main role instead of $\alpha_m(j)$ in (i) and (ii). First assume $m \geq 2$. If $\alpha_m(j+1)$ and $\alpha_{m-1}(j'+1)$ are contained in the same component of $\mathbb{R}^d \setminus Z_{m-2}$, let E_{m-1} be the component of $\mathbb{R}^d \setminus Z_{m-1}$ which contains $\alpha_m(j+1)$ and take a point $x^-_{n,m}(j+1)$ on $l_n^{(m)}(j) \cap ((E_{m-1} \setminus A_m(j+1)) \setminus B)$ as in Case (ii). We handle the case $m = 1$ as in (i).

Case (iv). Assume $m \geq 2$. If $\alpha_m(j+1)$ and $\alpha_{m-1}(j'+1)$ are contained in different components of $\mathbb{R}^d \setminus Z_{m-2}$ we take points $x^-_{n,m}(j+1)$ and $x^{--}_{n,m}(j+1)$ as in Case (ii).

Above we have taken points $x^+_{n,m}(j), x^-_{n,m}(j+1)$ and possibly one or both of $x^{++}_{n,m}(j)$, $x^{--}_{n,m}(j+1)$. We shall show them on a contour graph in case all four of them are taken.

$$
\begin{array}{ccccccc}
 & & \alpha_{m-1}(j') & \vdash \cdots\cdots\cdots \dashv & \alpha_{m-1}(j'+1) & & \\
\alpha_m(j) \vdash \cdots\cdots & \times \cdots\cdots\cdots & \times \cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \times \cdots\cdots\cdots & \times \cdots\cdots & \dashv \alpha_m(j+1) \\
 & x^+_{n,m}(j) & x^{++}_{n,m}(j) & & x^{--}_{n,m}(j+1) & x^-_{n,m}(j+1) &
\end{array}
$$

Type (3). There are many arcs over $l_n^{(m)}(j)$ as the following diagram shows. We take $x^+_{n,m}(j)$ and possibly $x^{++}_{n,m}(j)$ on $l_n^{(m)}(j)$ as in Type (2), and take $x^{--}_{n,m}(j+1)$ (possibly) and $x^-_{n,m}(j+1)$ also on $l_n^{(m)}(j)$. These are shown in the diagram given below.

$$
\begin{array}{ccccccc}
 & & \alpha_{m-1}(j') \vdash \cdots \dashv & \vdash \cdots \dashv \quad \cdots \quad \vdash \cdots\cdots \dashv & \alpha_{m-1}(j'+l) & & \\
\alpha_m(j) \vdash \cdots & \times \cdots\cdots & \times \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots \times & \cdots\cdots \times \cdots & \dashv \alpha_m(j+1) \\
 & x^+_{n,m}(j) & (x^{++}_{n,m}(j)) & & (x^{--}_{n,m}(j+1)) & x^-_{n,m}(j+1) &
\end{array}
$$

The parentheses for $x^{++}_{n,m}(j)$ and $x^{--}_{n,m}(j)$ indicate that one or both of them might be non-existent.

Now we consider all the points taken above on $\{\gamma_n\}_{n\geq 1}$. We note that for any fixed $m \geq 1$ the number of such points and the existence or non-existence of points of the form $x^{++}_{n,m}$ or $x^{--}_{n,m}$ do not depend on $n \geq m$ because the contour graph is same once $m \geq 1$ is fixed. Since the points $\{x^+_{n,1}(j)\}_{n=1,2,\dots}$ lie in the compact set $\overline{W}_1 \setminus B$ for each j, there exists at least one accumulation point, say $x^+_1(j)$. For each j we can choose a subsequence $\{n_k\}$ of $\{n\}$ such that $\{x^+_{n_k,1}(j)\}$ converges to some point $x^+_1(j)$. We may assume that for each j, all $\{x^+_{n_k,1}(j)\}$ are included in some closed ball $V^+_1(j) \subset G \cap (\overline{W}_0 \cup \overline{W}_1 \cup \overline{W}_2)$ with center at $x^+_1(j)$ such that $\int_s \rho ds < \varepsilon/(r_1 2^3)$ for any segment s in $V^+_1(j)$; recall that ρ is continuous in $G \setminus (K \cup E)$. We denote the subsequence of $\{\gamma_n\}$ containing $\{x^+_{n_k,1}(j)\}$ by $\{\gamma^{(1)}_{n_k}\}$. We do the same with $\{x^-_{n,1}(j)\}, \{x^{++}_{n,1}(j')\}, \{x^{--}_{n,1}(j'')\}$. We may assume that the resulting subsequences are same as above. That is, it is still $\{\gamma^{(1)}_{n_k}\}$.

Next we are concerned with the sequences $\{x^+_{n,2}(j)\}$ for each $j, 1 \leq j \leq r_2$. They lie in the compact set $\overline{W}_2 \setminus B$ so that there are an accumulation point $x^+_2(j)$ and a subsequence $\{x^+_{n_k,2}(j)\}$ of $\{x^+_{n,2}(j)\}$ converging to $x^+_2(j)$. As above we may assume that for each $j, 1 \leq j \leq r_2$, all $\{x^+_{n_k,2}(j)\}$ are included in some closed ball $V^+_2(j) \subset G \cap (\overline{W}_1 \cup \overline{W}_2 \cup \overline{W}_3)$ with center at $x^+_2(j)$ in such a way that $\int_s \rho ds < \varepsilon/(r_2 2^4)$ for any segment s in $V^+_2(j)$. We denote the subsequence of $\{\gamma^{(1)}_{n_k}\}$ containing $\{x^+_{n_k,2}(j)\}$ by $\{\gamma^{(2)}_{n_k}\}$. We do the same with $\{x^-_{n,2}(j)\}, \{x^{++}_{n,2}(j')\}, \{x^{--}_{n,1}(j'')\}$. We continue such procedure and take the diagonal sequence $\{\gamma^{(k)}_{n_k}\}$. We consider the k-shortening of $\gamma^{(k)}_{n_k}$ for each k. We shall write $\{\gamma_n\}$ for the sequence of the shorten curves. For this new sequence we have a subsequence of $\{x^+_n(j)\}_{n\geq 1}, \{x^-_n(j)\}_{n\geq 1}, \{x^{++}_n(j')\}_{n \geq 1}, \{x^{--}_n(j'')\}_{n\geq 1}$. We shall still use the same notation as above for curves and small balls.

We shall modify γ_n in $V^+_m(j), 1 \leq m \leq n, 1 \leq j \leq r_m$, which has center at $x^+_m(j)$. In $V^+_m(j)$ we replace $\gamma_n \cap V^+_m(j)$ by two radii of $V^+_m(j)$ terminating at $\gamma_n \cap \partial V^+_m(j)$. See Figure 10 for this modification.

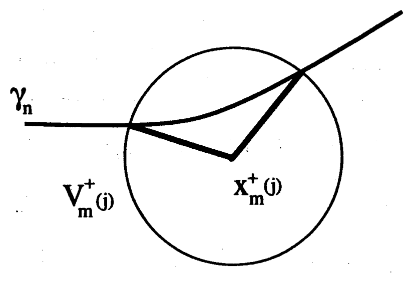

FIGURE 10.

Then $\int \rho ds$ along these radii is less than $\varepsilon/(r_m 2^{m+1})$. We do the same with other balls, and denote the resulting i-curve in Γ_n by γ_n. Recalling that we assumed $\int_{\gamma_n} \rho' ds \leq a$ after showing the property 1) in the statement of the lemma, we obtain

$$(3) \qquad \int_{\gamma'_n} \rho ds < a + \frac{\varepsilon}{2} \qquad \text{for any } n.$$

We shall call any one of $\{x^+_k(j)\}, \{x^-_k(j+1)\}, \{x^{++}_k(j')\}, \{x^{--}_k(j+1)\}$ a relay post. We place them in the order of increasing parameter on each of γ'_n, and two of them adjacent if they lie on a connected component of some $\gamma'_n \cap (G \setminus (K \cup E))$ and there is no relay post between them. One can put the order to any two relay posts according to their order on γ'_n which contains them; this order does not depend on the choice of γ'_n. Thus we see that the totality of relay posts forms an ordered set S. We write $x < y$ for two relay posts x and y if the parameters corresponding to x is smaller than that to y. Then x is said to be smaller than y. The notation $y < x$ and the terminology that y is smaller than x will have similar meanings.

Part 6. Defining i-curve $\tilde{c}$ desired in 2) of the lemma

We define a distance $d(x, x')$ between two points x and x' in $G \setminus (K \cup E)$ by $\inf \int_\gamma \rho ds$ for an arc γ connecting x and x' in $G \setminus (K \cup E)$; if x and x' do not belong to the same component of $G \setminus (K \cup E)$, then $d(x, x')$ is set to be ∞. We connect every couple of adjacent relay posts by a curve in $G \setminus (K \cup E)$ so that $\int \rho ds$ along this curve is very close to the d-distance between the relay posts. To be precise, the totality of adjacent couples being countable, we enumerate them as $a_1, a_2, \ldots$ without paying any attention to their order, and set $A_1 = \{a_1\}, A_2 = \{a_1, a_2\}, \ldots$ For an integer $q > 0$ let n_q be the smallest number such that γ'_{n_q} contains A_q. Naturally n_q increases with q. We connect the points x_j and y_j of a_j by a curve C_j in $G \setminus (K \cup E)$ so that $\int_{C_j} \rho ds < d(x_j, y_j) + \varepsilon/2^{1+j}$. Then using (3), for any q we have

$$\sum_{j=1}^{q} \int_{C_j} \rho ds < \sum_{j=1}^{q} d(x_j, y_j) + \frac{\varepsilon}{2} \leq \int_{\gamma'_{n_q}} \rho ds + \frac{\varepsilon}{2} < a + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = a + \varepsilon.$$

By letting $q \to \infty$ we obtain

$$\sum_{j=1}^{\infty} \int_{C_j} \rho ds \leq a + \varepsilon. \tag{5}$$

We shall call any one of the above arcs C_j a C-arc. Its direction is determined by the order of its end points. Thus we obtain a countably infinite number of directed curves in $G \setminus (E \cup K)$ each of which consists of some of C-arcs. Each curve is rectifiable on account of (5) and hence has two end points on $K \cup E$. We shall call any one of these curves a C-curve. Finally we shall form a curve of Γ by means of C-curves.

We have already given the order to the set S of relay posts. This order gives the order of points of the union of C-curves. We let correspond C-curves to mutually disjoint open intervals on the interval $(0, 1)$ so that each C-curve is a continuous image of the corresponding open interval and the order is preserved. Subsequently we shall parametrize C-curves differently.

Recall that there is no relay posts on the 0-th step. We denote all the ordered relay posts of the first step, excepting all those of the form $x_1^{++}(j)$ and $x_1^{--}(j)$, by

$$x_1^+(1), x_1^-(2), x_1^+(2), \ldots, x_1^-(r_1).$$

If there is a relay post of the form $x_1^{++}(\cdot)$ which is the right hand neighbor to $x_1^+(1)$, then it will be denoted by $x_1^{++}(1)$. We write similarly for all other relay posts of the form $x_1^{++}(\cdot)$ or $x_1^{--}(\cdot)$.

We take the following closed subintervals of $I = (0, 1)$:

$$I_{1,1} = \left[\frac{1}{2r_1+1}, \frac{2}{2r_1+1}\right], \cdots, I_{1,j} = \left[\frac{2j-1}{2r_1+1}, \frac{2j}{2r_1+1}\right], \cdots,$$
$$I_{1,r_1} = \left[\frac{2r_1-1}{2r_1+1}, \frac{2r_1}{2r_1+1}\right].$$

Set $I_1 = \cup_{j=1}^{r_1} I_{1,j}$. Then the length of $I \setminus I_1$ is equal to $1/2 + 1/(2(2r_1+1))$. We change the parameter of the C-arc connecting $x_1^+(j)$ and $x_1^-(j+1)$ so that this is expressed as a continuous image of $I_{1,j}$. Similarly we change the parameter for the union of the C-arc connecting $x_1^+(j)$ and $x_1^{++}(j)$ and the C-arc connecting $x_1^{++}(j)$ and x_{j+1}^- in case $x_1^{++}(j)$ exists but not $x_1^{--}(j+1)$. We do the same in case only $x_1^{--}(j+1)$ exists or both $x_1^{++}(j)$ and $x_1^{--}(j+1)$ exist. Thus we have a continuous mapping of I_1 into Ξ.

In the second step we denote the relay posts by $x_2^+(1), x_2^-(2), x_2^+(2), \ldots, x_2^+(r_2 - 1), x_2^-(r_2)$ as in the first step. In case $x_2^{++}(\cdot)$ and/or $x_2^{--}(\cdot)$ exist we treat them also as in the first step. As to the relation between the set of relay posts of the first step and of the second step there are the following two cases:

1) Between $x_2^+(j)$ and $x_2^-(j+1)$ there exist some relay posts of the first step.

2) Between them there exists no relay post of the first step.

In the first case 1) we take a closed interval $I_{2,j}$ whose end points correspond to $x_2^+(j)$ and $x_2^-(j+1)$ and whose interior includes the intervals taken for the relay posts of the first step lying between $x_2^+(j)$ and $x_2^-(j+1)$. In the second case 2) we take $I_{2,j}$ disjoint to I_1. We may assume that all the different intervals in the second step are mutually disjoint and that, denoting their union by I_2, the length of $I \setminus I_2$ is less than $1/4$. For each j we map $I_{2,j} \setminus I_{1,j'}$ as in the first step. We treat the second case 2) properly and obtain a continuous mapping of I_2 into Ξ.

We continue this process and obtain an increasing sequence $\{I_n\}$ of finite union of closed intervals. Setting $J = \cup_{n=1}^{\infty} I_n$, we shall denote the continuous mapping of J into Ξ by $\zeta(t)$. The set $I \setminus J$ is a totally disconnected compact set and its linear measure is zero. What is left is to show that the mapping $\zeta(t)$ of J can be extended continuously to I. For that purpose let $t_0 \in I \setminus J$. For given n let I_{n,j_n} (resp. I_{n,j_n+1}) be the closed interval nearest to t_0 on the left (resp. right). We note that the right (resp. left) end point of I_{n,j_n} (resp. I_{n,j_n+1}) is $x_n^-(j_n+1)$ (resp. $x_n^+(j_n+1)$). We shall call $\{I_{n,j_n}, I_{n,j_n+1}\}$ the defining sequence of t_0. See the following diagram:

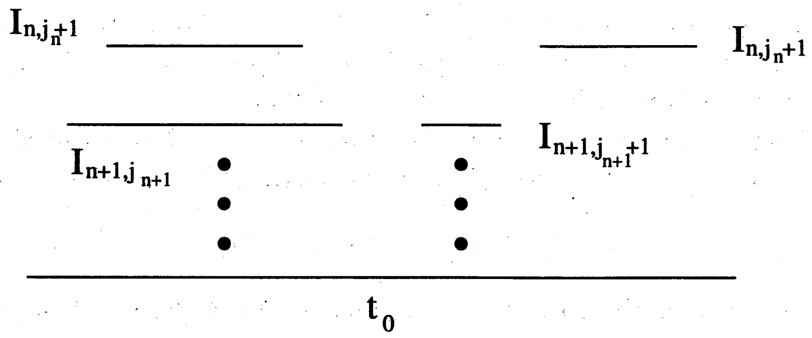

FIGURE 11.

Now fix N for the moment. As in the beginning of the proof of our main lemma we set $d_N = \text{dist } (\partial Z_{N-1}, \partial Z_N) > 0$, and by the assumption in the lemma $\inf_{W_N \cap G} \rho$ is positive. Hence there exists $\delta_N > 0$ such that $\int_\gamma \rho ds \geq \delta_N$ for every arc γ connecting ∂Z_{N-1} and ∂Z_N in $W_n \cap G$. For simplicity set $\tilde{C} = \cup_{j=1}^{\infty} C_j$. Since $\int_{\tilde{C}} \rho ds < a + \varepsilon$, there exists n_0 such that

$$\int_{\tilde{C} \setminus \zeta(I_{n_0})} \rho ds < \delta_N. \tag{6}$$

As stated above t_0 is determined by $\{(I_{n,j_n}, I_{n,j_n+1})\}$. The relay posts $x_n^{\pm}(j_n + 1)$ are illustrated in the following diagram:

$$\begin{array}{llllllll}
\vdash \cdot\cdot \dashv \ \alpha_{n-1}(j) & & & & & & & \alpha_{n-1}(j) \ \vdash \cdot\cdot \dashv \\
\cdots\cdots\cdots \times \cdot\cdot \dashv & \alpha_n(j') & \alpha_n(j') \vdash \cdots \dashv & \cdots\cdots \vdash \cdots \dashv & \alpha_n(j'') & \alpha_n(j'') & \vdash \cdot\cdot \times \cdots\cdots\cdots & \\
\quad x_n^-(j_n+1) & & & & & & & x_n^+(j_n+1)
\end{array}$$

Now we shall show that $\tilde{C}$ can be extended to be a continuous curve in Ξ by proving that for any point $t_0 \in (0,1)$ the $\lim_{t \in J} \zeta(t)$ exists. Let $\{I_{n,j_n}, I_{n,j_n+1}\}$ be the defining sequence of t_0. As stated already the right end point of I_{n,j_n} corresponds to the relay post $x_n^-(j_n+1)$ and the left end point of I_{n,j_n+1} corresponds to the relay post $x_n^+(j_n+1)$. We recall that in general, if $\alpha_n(j)$ is a contour, then $A_n(j)$ denotes the compact domain bounded by $\alpha_n(j)$. Also we note that there is an arc which is a subarc of the curve $\gamma'_{k,n}$ obtained by the n-shortening of $\gamma'_k, k \geq n$, which connects contours $\alpha_n(j'-1)$ and $\alpha_n(j')$ of the contour sequence of ∂Z_n and which contains $x_n^-(j_n+1)$. Let $\alpha_{n-1}(j)$ be the contour of the contour sequence of ∂Z_{n-1} which lies on the upper left to $\alpha_n(j')$ on the contour graph, and $\alpha_n(j'')$ be the contour of the sequence of ∂Z_n which lies on the lower right to $\alpha_{n-1}(j)$.

We will show that if $n > \max\ (n_0, N+1)$, then $\alpha_n(j'), \alpha_n(j'')$ and $\alpha_{n-1}(j)$ are contained in the same component of $\mathbb{R}^d \setminus Z_{N-1}$. Suppose $\alpha_{n-1}(j)$ and $\alpha_n(j')$ are not contained in the same component of $\mathbb{R}^d \setminus Z_{N-1}$. Then $\alpha_{n-1}(j)$ and $\alpha_n(j')$ are not contained in the same component of $\mathbb{R}^d \setminus Z_{n-2}$. From the definition of the relay posts there is a relay post $x_n^{--}(j_n+1)$ in the component of $\overline{W}_{n-1}$ whose boundary contains $\alpha_{n-1}(j)$. We know that $x_n^-(j_n+1)$ is contained in the component of $\overline{W}_n$ whose boundary contains $\alpha_n(j)$. Since $x_n^-(j_n+1)$ and $x_n^{--}(j_n+1)$ are not in the same component of $\mathbb{R}^d \setminus Z_{N-1}$ and they are in G, the C-arc which connects $x_n^{--}(j_n+1)$ and $x_n^-(j_n+1)$ contains an arc connecting ∂Z_{N-1} and ∂Z_N in $G \cap W_N$. Hence $\int_{\zeta(I_n \setminus I_{n-1})} \rho ds \geq \delta_N$. But this is contradictory to the inequality $\int_{\zeta(I_n \setminus I_{n-1})} \rho ds < \delta_N$ by (6). Therefore $\alpha_{n-1}(j)$ and $\alpha_n(j')$ are contained in the same component of $\mathbb{R}^d \setminus Z_{N-1}$. We see similarly that $\alpha_{n-1}(j)$ and $\alpha_n(j'')$ are contained in the same component of $\mathbb{R}^d \setminus Z_{n-1}$. It follows that the relay posts $x_n^-(j_n+1), x_n^+(j_n+1)$ and, if exist, $x_n^{--}(j_n+1)$, $x_n^{++}(j_n+1)$ are in the same component of $\mathbb{R}^d \setminus Z_{N-1}$.

Denote by $I_{n+1,a}$ and $I_{n+1,b}$ the components of I_{n+1} which contain I_{n,j_n} and I_{n,j_n+1} respectively. Then the intervals $I_{n+1,j}$, $a < j < b$, are placed between $I_{n+1,a}$ and $I_{n+1,b}$ and $a \leq j_{n+1} < j_{n+1} + 1 \leq b$ holds, where j_{n+1} appears, for instance, in $I_{n+1,j_{n+1}}$ in the diagram given below. The right end point of $I_{n+1,a}$ corresponds to the relay post $x_{n+1}^-(a+1)$. See the following diagram:

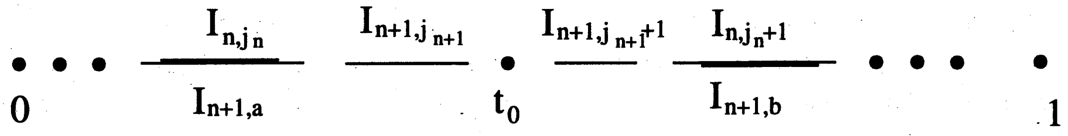

FIGURE 12.

Since $\tilde{C}|_{I_{n+1,a}\setminus I_{n,j_n}}$ contains an arc starting from $x_n^-(j_n+1)$ and terminating at $x_{n+1}^-(a+1)$ and

$$\int_{\tilde{C}|_{I_{n+1,a}\setminus I_{n,j_n}}} \rho ds \le \int_{\tilde{C}\setminus\zeta(I_{n_0})} \rho ds < \delta_N$$

by (6), $x_{n+1}^-(a+1)$ and $x_n^-(j_n+1)$ are contained in the same component of $\mathbb{R}^d \setminus Z_{N-1}$. It was shown above that $x_n^-(j_n+1)$ and $x_n^+(j_n+1)$ are in the same component of $\mathbb{R}^d \setminus Z_{N-1}$. Similarly it follows that $x_{n+1}^-(a+1)$ and $x_{n+1}^+(a+1)$ are contained in the same component of $\mathbb{R}^d \setminus Z_{N-1}$. Since the left (resp. right) end point of $I_{n+1,a+1}$ corresponds to the relay post $x_{n+1}^+(a+1)$ (resp. $x_{n+1}^-(a+2)$), $\tilde{C}|_{I_{n+1,a+1}}$ is an arc starting from $x_{n+1}^+(a+1)$ and terminating at $x_{n+1}^-(a+2)$. Hence $x_{n+1}^+(a+1)$ and $x_{n+1}^-(a+2)$ are contained in the same component of $\mathbb{R}^d \setminus Z_{N-1}$, because

$$\tilde{C}\big|_{I_{n+1,a+1}} \subset \zeta(I_{n+1}\setminus I_n) \subset \tilde{C}\setminus\zeta(I_{n_0}).$$

In the same way we conclude that the relay posts $x_{n+1}^-(j)$ and $x_{n+1}^+(j)$ and, if exist, $x_{n+1}^{--}(j)$ and $x_{n+1}^{++}(j)$, $a+1 \le j \le b$, are contained in the same component of $\mathbb{R}^d \setminus Z_{N-1}$. By repeating such process we conclude that all relay posts between $x_n^-(j_n+1)$ and $x_n^+(j_n+1)$ are contained in the same component of $\mathbb{R}^d \setminus Z_{N-1}$. By the arbitrariness of N we infer that $\zeta(t)$ is continuous at t_0.

Similarly we can show that $\zeta(t)$ is continuous at $\zeta = 0, 1$. It is now proved that $\tilde{C}$ can be extended to be an i-curve $\tilde{c}$ of $\Gamma = \Gamma(K_0, K_1, E, G)$. Our lemma is now completely proved.

Part 7. Proof of the Theorem

We begin with the case when $\Gamma \neq \emptyset$. Given $\varepsilon, 0 < \varepsilon < 1/2$, as in the case when there is no island, we can find a lower semicontinuous Γ-ad. function ρ in $\mathbb{R}^d$ which is continuous in $G\setminus(K\cup E)$ and which satisfies $\int_{\mathbb{R}^d}\rho^p\omega dx < M+\varepsilon$. Moreover, we may assume that ρ is positive in $\mathbb{R}^d$. Take ρ' as in the lemma and suppose there exist $\{n_k\}$ and $c_k \in \Gamma_{n_k}$ such that $\int_{c_k}\rho' ds \le 1-2\varepsilon$ for $k = 1, 2, \ldots.$ By the lemma we can find an i-curve $\tilde{c} \in \Gamma$ such that $\int_{\tilde{c}}\rho ds \le 1-2\varepsilon+\varepsilon = 1-\varepsilon$. Since $\tilde{c}\in\Gamma$ and ρ is Γ-ad., $\int_{\tilde{c}}\rho ds \ge 1$. This is a contradiction. Hence there exists n_0 such that $\int_c \rho' ds > 1-2\varepsilon$ for all $c \in \Gamma_n$ if $n \ge n_0$. Using the property 1) in the lemma, we have $\int_{\mathbb{R}^d}\rho'^p\omega dx \le \int_{\mathbb{R}^d}\rho^p\omega + \varepsilon < M+2\varepsilon$ and

$$M \le M^{(n)} \le \frac{1}{(1-2\varepsilon)^p}\int_{\mathbb{R}^d}\rho'^p\omega dx \le \frac{M+2\varepsilon}{(1-2\varepsilon)^p}.$$

The arbitrariness of ε yields the equality $M = \lim_{n\to\infty} M^{(n)}$.

Next we consider the case when $\Gamma = \emptyset$. Let ρ be a positive lower semicontinuous function in $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} \rho^p \omega dx < \varepsilon$. Let ρ' be a function obtained in the Lemma for the above ε and ρ. Then it is a measurable function satisfying $\rho' \geq \rho$ and $\int_{\mathbb{R}^d} \rho'^p \omega dx \leq \int_{\mathbb{R}^d} \rho^p \omega dx + \varepsilon < 2\varepsilon$. Since $\Gamma = \emptyset$, there is no curve in G connecting K_0 and K_1. It follows that for every sequence $\{\gamma_k\}$ of curves in G which have end points tending to K_0 and K_1 respectively, $\int_{\gamma_k} \rho ds \geq 1$ only with a finite number of exceptional k's. Suppose there exists an infinite sequence $\{n_j\}$ such that ρ' is not Γ_{n_j}-ad. Then for each n_j there is $\gamma_j \in \Gamma_{n_j}$ such that $\int_{\gamma_j} \rho' ds < 1$. This is impossible because all γ_{n_j} are contained in G and their two end points tend to K_0 and K_1 respectively. Therefore there exists n_0 such that ρ' is Γ_n-ad. for all $n \geq n_0$ so that $M^{(n)} \leq \int_G \rho'^p \omega dx < 2\varepsilon$. This implies that $M^{(n)} \to 0$ as $n \to \infty$. Thus the assertion in the theorem is true in the present case. The proof of the theorem is now completely proved.

Comments. The proof of our lemma in the case when no islands exist will appear in [AO] as Lemma 6.1 (Shlyk-Ohtsuka). [Sh, p.91, Theorem 1.3] gives a proof of our Theorem in the non-weighted case but it does not seem to be easy to understand it.

References

[AO] H. Aikawa and M. Ohtsuka, *Extremal length of vector measures* (to appear).

[MR] A. Marden and B. Rodin, *Extremal and conjugate extremal distance on open Riemann surfaces with applications to circular-radial slit mappings*, Acta Math. **115** (1996), 237-269.

[Sh] V. Shlyk, *Normal domains and removable singularities*, Izv. Ross. Akad. Nauk Cer. Mat. **57** (1993), no. 4; English transl. in, Russian Acad. Sci. Izv. Math. **43** (1994), 83-104.

Postscript. The second author gave a talk at the Colloquium. However, the proof of the "Claim" in Part 5 of the distributed abstract was found to be incomplete. So considerable parts of Parts 5 and 6 were rewritten following the idea proposed by the first author, and thus the present report is presented as a joint work.