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INTRODUCTION

Let \( \mathfrak{g}(A) \) be the complex contragredient Lie algebra associated to a symmetrizable real square matrix \( A = (a_{ij})_{i,j \in I} \) indexed by a finite set \( I \) (see [K1] and [KK] for details). In [K2], Kac introduced a complex associative algebra \( \hat{U}_F(\mathfrak{g}(A)) \), which can be thought of as a certain completion of the universal enveloping algebra \( U(\mathfrak{g}(A)) \) of the contragredient Lie algebra \( \mathfrak{g}(A) \). In it he showed that there exists an isomorphism \( H \) (called the Harish-Chandra homomorphism) between the center \( Z_F \) of the algebra \( \hat{U}_F(\mathfrak{g}(A)) \) and the algebra \( \mathcal{F} \) of complex-valued functions on the set \( \mathfrak{h}^* \setminus L \), where \( L \) is the union of certain infinitely many affine hyperplanes in the algebraic dual \( \mathfrak{h}^* \) of the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g}(A) \).

Moreover, he studied the "holomorphicity" of the elements of the algebra \( Z_F \) as "vector-valued" functions on the interior \( K \) of the complexified Tits cone \( X_C \) in the case where \( \mathfrak{g}(A) \) is the symmetrizable Kac-Moody algebra (i.e., the matrix \( A = (a_{ij})_{i,j \in I} \) is a symmetrizable generalized Cartan matrix).

In this paper, we generalize his results in [K2] to the case where \( \mathfrak{g}(A) \) is the symmetrizable generalized Kac-Moody algebra (i.e., the complex contragredient Lie algebra associated to a certain symmetrizable real matrix \( A = (a_{ij})_{i,j \in I} \), called a GGCM).
1. HARISH-CHANDRA HOMOMORPHISM

In this section we briefly review the setting and some results in [K2], which are valid for arbitrary symmetrizable contragredient Lie algebras over \( \mathbb{C} \), hence for symmetrizable generalized Kac-Moody algebras over \( \mathbb{C} \).

1.1. A completion of the universal enveloping algebra. Let \( g(A) \) be the symmetrizable generalized Kac-Moody algebra (GKM algebra for short) over \( \mathbb{C} \). Then the Lie algebra \( g(A) \) is nothing but the contragredient Lie algebra associated to a symmetrizable real matrix \( A = (a_{ij})_{i,j \in I} \) (called a GGCM) indexed by a finite set \( I \) satisfying the following conditions:

(C1) either \( a_{ii} = 2 \) or \( a_{ii} \leq 0 \) for \( i \in I \);

(C2) \( a_{ij} \leq 0 \) if \( i \neq j \), and \( a_{ij} \in \mathbb{Z} \) for \( j \neq i \) if \( a_{ii} = 2 \);

(C3) \( a_{ij} = 0 \iff a_{ji} = 0 \).

Note that this definition of GKM algebras is due to Kac (see [K1, Chap. 11]), and slightly different from the original one by Borcherds in [B1]). From now on we follow the notation of [K1], and freely use results in it (see also our previous papers [N1] – [N3]).

Let \( \mathfrak{h} \) be the Cartan subalgebra of the GKM algebra \( g(A) \). Then, since we have been assuming that the GGCM \( A = (a_{ij})_{i,j \in I} \) is symmetrizable, there exists a nondegenerate symmetric \( \mathbb{C} \)-bilinear form \( (\cdot|\cdot) \) on the dual \( \mathfrak{h}^* \) of \( \mathfrak{h} \), which is invariant under the action of the Weyl group \( W \). (Here recall that the Weyl group \( W \) of the GKM algebra \( g(A) \) is by definition the subgroup of \( GL(\mathfrak{h}^*) \) generated by the fundamental reflections \( r_i \) with \( a_{ii} = 2 \).)

Now, for \( \alpha \in Q = \sum_{i \in I} \mathbb{Z} \alpha_i \), we define the affine linear function \( T_\alpha(\cdot) \) on \( \mathfrak{h}^* \) by:

\[
T_\alpha(\lambda) = 2(\lambda + \rho|\alpha) - (\alpha|\alpha) \quad (\lambda \in \mathfrak{h}^*),
\]

where \( \rho \in \mathfrak{h}^* \) is a fixed element of \( \mathfrak{h}^* \) such that...
$2(\rho|\alpha_i) = (\alpha_i|\alpha_i)$ for $i \in I$. Then we put

$$L := \bigcup_{\gamma \in Q} \{ \lambda \in \mathfrak{h}^* | T_{n\beta}(\lambda + \gamma) = 0 \}.$$ 

Let $\mathcal{F}$ be the algebra of $\mathbb{C}$-valued functions defined on $\mathfrak{h}^* \setminus L$. Because the set $\mathfrak{h}^* \setminus L$ is dense in $\mathfrak{h}^*$ in the usual metric topology, there exists a canonical embedding $\iota: S(\mathfrak{h}) \rightarrow \mathcal{F}$, where $S(\mathfrak{h})$ is viewed as the algebra of polynomial functions on $\mathfrak{h}^*$. Here we define the action $\pi$ of the universal enveloping algebra $U(\mathfrak{g}(A))$ of the GKM algebra $\mathfrak{g}(A)$ on the algebra $\mathcal{F}$ by:

$$\pi(e_{\beta})\varphi(\cdot) = \varphi(\cdot + \beta)$$

for $\varphi(\cdot), e_{\beta} \in U(\mathfrak{g}(A))$, where $h(e_{\beta}) = \beta(h)e_{\beta} (\beta \in Q, h \in \mathfrak{h})$. By using the action $\pi$ of $U(\mathfrak{g}(A))$ on $\mathcal{F}$, we can define the structure of an associative algebra on the vector space $U(\mathfrak{g}(A)) \otimes_{\mathbb{C}} \mathcal{F}$ by:

$$(e_{\alpha} \otimes \varphi(\cdot))(e_{\beta} \otimes \psi(\cdot)) := e_{\alpha}e_{\beta} \otimes (\pi(e_{\beta})\varphi(\cdot)\psi(\cdot)),$$

for $\varphi(\cdot), \psi(\cdot) \in \mathcal{F}$ and $e_{\alpha}, e_{\beta} \in U(\mathfrak{g}(A))$ with $\alpha, \beta \in Q$. Let $U_{\mathcal{F}}(\mathfrak{g}(A))$ be the quotient algebra of this associative algebra $U(\mathfrak{g}(A)) \otimes_{\mathbb{C}} \mathcal{F}$ by the two-sided ideal generated by the elements $f \otimes 1 - 1 \otimes i(f)$ for $f \in S(\mathfrak{h})$. Then the associative algebra $U_{\mathcal{F}}(\mathfrak{g}(A))$ is generated by the algebra $\mathcal{F}$ and $U(\mathfrak{g}(A))$, and the following relation holds in it:

$$\varphi(\cdot)e_{\beta} - e_{\beta}\varphi(\cdot) = e_{\beta}(\varphi(\cdot + \beta) - \varphi(\cdot)),$$

where $\varphi(\cdot) \in \mathcal{F}$ and $e_{\beta} \in U(\mathfrak{g}(A))$, with $\beta \in Q$. Moreover this algebra $U_{\mathcal{F}}(\mathfrak{g}(A))$ decomposes into the tensor product of vector spaces as:

$$U_{\mathcal{F}}(\mathfrak{g}(A)) = U(n_{-}) \otimes_{\mathbb{C}} \mathcal{F} \otimes_{\mathbb{C}} U(n_{+}),$$

and canonically contains the algebra $U(\mathfrak{g}(A)) = U(n_{-}) \otimes_{\mathbb{C}} S(\mathfrak{h}) \otimes_{\mathbb{C}} U(n_{+})$.

By putting $\deg(e_i) = 1$ and $\deg(f_i) = -1$ for $i \in I$, and $\deg(\mathcal{F}) = 0$, we have a $\mathbb{Z}$-gradation of $U_{\mathcal{F}}(\mathfrak{g}(A))$ as:

$$U_{\mathcal{F}}(\mathfrak{g}(A)) = \bigoplus_{j \in \mathbb{Z}} U_{\mathcal{F}}(\mathfrak{g}(A))_j, \quad U_{\mathcal{F}}(\mathfrak{g}(A))_j := \bigoplus_{m-k=j} U_{-k}(n_{-}) \otimes_{\mathbb{C}} \mathcal{F} \otimes_{\mathbb{C}} U_m(n_{+}),$$
so that we can "complete" it in a canonical way as:

\[ \hat{U}_F(\mathfrak{g}(A)) := \bigoplus_{j \in \mathbb{Z}} \hat{U}(\mathfrak{g}(A))_j := \prod_{m-k=j, k,m \geq 0} U_{-k}(\mathfrak{n}_-) \otimes_{\mathbb{C}} \mathcal{F} \otimes \mathbb{C} U_m(\mathfrak{n}_+), \]

where \( U_m(\mathfrak{n}_+) \) (resp. \( U_{-k}(\mathfrak{n}_-) \)) is the subspace of \( U(\mathfrak{n}_+) \) (resp. \( U(\mathfrak{n}_-) \)) of degree \( m \) (resp. \( -k \)). Note that the multiplication in \( \hat{U}_F(\mathfrak{g}(A)) \) extends to \( \hat{U}_\mathcal{F}(\mathfrak{g}(A)) \), so that \( \hat{U}_\mathcal{F}(\mathfrak{g}(A)) \) is an associative algebra containing \( \hat{U}_F(\mathfrak{g}(A)) \).

Moreover, if \( V(\Lambda) \) is a highest weight \( \mathfrak{g}(A) \)-module with highest weight \( \Lambda \in \mathfrak{h}^* \setminus L \), then the action of \( U(\mathfrak{g}(A)) \) on \( V(\Lambda) \) can be extended to the action of the algebra \( \hat{U}_\mathcal{F}(\mathfrak{g}(A)) \), while the algebra \( \mathcal{F} \) acts on \( V(\Lambda) \) by:

\[ \varphi(\cdot)(v_\tau) = \varphi(\tau)v_\tau, \]

where \( \varphi(\cdot) \in \mathcal{F} \) and \( v_\tau \in V(\Lambda)_\tau \) is a weight vector of weight \( \tau \in \mathfrak{h}^* \).

1.2. Harish-Chandra homomorphism. We denote by \( Z_\mathcal{F} \) the center of the associative algebra \( \hat{U}_\mathcal{F}(\mathfrak{g}(A)) \).

Now we prepare some notation. Let \( \tilde{\Delta}_+ \) be the multiset in which every positive root \( \alpha \in \Delta_+ \) appears with its multiplicity. For \( \beta \in Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \), denote by \( \text{Par} \beta \) the set of maps \( k: \tilde{\Delta}_+ \rightarrow \mathbb{Z}_{\geq 0} \) such that \( \beta = \sum_{\alpha \in \tilde{\Delta}_+} k(\alpha) \alpha \), and put \( \text{Par} := \bigcup_{\beta \in Q_+} \text{Par} \beta \).

For each \( \beta \in Q_+ \), we can choose a basis \( \{F^k\}_{k \in \text{Par} \beta} \) of the vector space \( U(\mathfrak{n}_-)_{-\beta} \) consisting of elements of the form \( F^k = \prod_{\alpha \in \tilde{\Delta}_+} f_\alpha^{k(\alpha)} \) (finite product) for \( k = (k(\alpha))_{\alpha \in \tilde{\Delta}_+} \in \text{Par} \beta \), where \( f_\alpha \in \mathfrak{g}_- \alpha \) is a root vector for a root \( \alpha \in \tilde{\Delta}_+ \) such that \( \mathfrak{g}_- \alpha = \oplus \mathbb{C} f_\alpha \). Then elements of \( \hat{U}_\mathcal{F}(\mathfrak{g}(A)) \) are expressed in the form

\[ \sum_{k,m \in \text{Par}} F^k \varphi_{k,m}(F^m) \] (infinite sum),

with \( \varphi_{k,m} \in \mathcal{F} \) and \( |\deg(F^m) - \deg(F^k)| < \text{constant} \).

In [K2], Kac proved the following theorem. (Here we also record the full proof by Kac for later use.)
Theorem 1 ([K2, Theorem 1]). Let \( \varphi \in \mathcal{F} \) be a function on \( \mathfrak{h}^* \setminus L \). Then there exists a unique element \( z_\varphi = \sum_{\beta \in Q_+} \sum_{k,m \in \text{Par}_\beta} F^k \varphi_{k,m} \sigma(F^m) \) in \( Z_\mathcal{F} \) with \( \varphi_{k,m} \in \mathcal{F} \) such that \( \varphi_{0,0} = \varphi \). Here \( \sigma \) is the involutive anti-automorphism of \( U(\mathfrak{g}(A)) \) determined by \( \sigma(e_i) = f_i, \sigma(f_i) = e_i \) for \( i \in I \), and \( \sigma(h) = h \) for \( h \in \mathfrak{h} \).

Proof. First we note that an element \( x \in \hat{U}_\mathcal{F}(\mathfrak{g}(A)) \) is zero if and only if it acts as a zero operator on each Verma module \( M(\Lambda) \) with highest weight \( \Lambda \in \mathfrak{h}^* \setminus L \) (cf. the proof of Proposition 1 below). So the element \( z_\varphi \in \hat{U}_\mathcal{F}(\mathfrak{g}(A)) \) of the form \( z_\varphi = \sum_{\beta \in Q_+} \sum_{k,m \in \text{Par}_\beta} F^k \varphi_{k,m} \sigma(F^m) \) with \( \varphi_{k,m} \in \mathcal{F} \) is in the center \( Z_\mathcal{F} \) if \( z_\varphi \) acts as the scalar \( \varphi_{0,0}(\Lambda) \) on each Verma module \( M(\Lambda) \) with highest weight \( \Lambda \in \mathfrak{h}^* \setminus L \). Therefore, we will choose \( \varphi_{k,m} \in \mathcal{F} \) with \( k,m \in \text{Par} \) by induction on \( \beta \in Q_+ \) in such a way that \( z_\varphi \) acts as the scalar \( \varphi_{0,0}(\Lambda) = \varphi(\Lambda) \) on the weight space \( M(\Lambda)_{\Lambda-\beta} \) for each \( \beta \in Q_+ \). Here we use a partial ordering \( \leq \) on \( \mathfrak{h}^* \) defined by: \( \lambda \leq \mu \iff \mu - \lambda \in Q_+ \).

We denote by \( G_\gamma^\beta(\Lambda) \) the matrix of the operator \( \sum_{k,m \in \text{Par}_\gamma} F^k \varphi_{k,m} \sigma(F^m) \) on \( M(\Lambda)_{\Lambda-\beta} \) in the basis \( \{F^s(v_\Lambda)\}_{s \in \text{Par}_\beta} \) for \( \beta, \gamma \in Q_+ \), where \( v_\Lambda \in M(\Lambda) \) is a highest weight vector of weight \( \Lambda \in \mathfrak{h}^* \setminus L \). Let us fix \( \beta \in Q_+ \). Assume that we have already chosen the functions \( \varphi_{k,m} \) with \( k,m \in \text{Par} \) for \( \gamma < \beta \), so that we know the matrices \( G_\gamma^\beta(\Lambda) \) for \( \gamma < \beta \) and \( \Lambda \in \mathfrak{h}^* \setminus L \). For the matrix \( G_\gamma^\beta(\Lambda) \), we have that

\[
G_\gamma^\beta(\Lambda) = \Phi_\beta(\Lambda) B_\beta^\Lambda, \quad \Phi_\beta(\Lambda) := (\varphi_{k,m}(\Lambda))_{k,m \in \text{Par}_\beta}, \quad B_\beta^\Lambda := (B_\beta^k(F^k, F^m))_{k,m \in \text{Par}_\beta}.
\]

Here \( B_\beta^\Lambda(F^k, F^m) \in \mathbb{C} \) is determined by \( \sigma(F^k)F^m(v_\Lambda) = B_\beta^k(F^k, F^m)v_\Lambda \). Moreover, the condition that \( z_\varphi \) acts on \( M(\Lambda)_{\Lambda-\beta} \) as the scalar \( \varphi(\Lambda) \) can be written as:

\[
(\ast) \quad \Phi_\beta(\Lambda) B_\beta^\Lambda + \sum_{\gamma \leq \beta} G_\gamma^\beta(\Lambda) = \varphi(\Lambda) \text{Id},
\]

since \( G_\gamma^\beta(\Lambda) = 0 \) for \( \gamma \nleq \beta \). Here we recall from [KK, Theorem 1] that the determinant \( \det B_\beta^\Lambda \) can be written as:

\[
\det B_\beta^\Lambda = \prod_{\alpha \in \Delta_+} \prod_{n=1}^{\infty} T_{n\alpha}(\Lambda)^{\#(\text{Par}(\beta-n\alpha))},
\]
up to a nonzero constant factor independent of $\Lambda$. Because $\Lambda \in \mathfrak{h}^* \setminus L$, we have $\det B_\beta^\Lambda \neq 0$, so that $\varphi_{k,m}(\Lambda)$ for $\Lambda \in \mathfrak{h}^* \setminus L$, $k, m \in \text{Par } \beta$ is determined. \qed

Conversely we have the following proposition.

**Proposition 1.** An element $x \in \hat{U}_F(\mathfrak{g}(A))$ lies in the center $Z_F$ only if it is of the form

$$x = \sum_{\beta \in Q_+} \sum_{k, m \in \text{Par } \beta} F^k \varphi_{k,m} \sigma(F^m)$$

for some $\varphi_{k,m} \in \mathcal{F}$.

**Proof.** Let $x = \sum_{k, m \in \text{Par } \beta} F^k \varphi_{k,m} \sigma(F^m)$ with $\varphi_{k,m} \in \mathcal{F}$ and $|\deg(F^m) - \deg(F^k)| < \text{constant}$ be an element of the center $Z_F$. It is clear that, for a highest weight vector $v_\Lambda$ of the Verma module $M(\Lambda)$ with highest weight $\Lambda \in \mathfrak{h}^* \setminus L$, we have $x(v_\Lambda) \in \mathbb{C} v_\Lambda$.

So $x$ acts as a scalar on each Verma module $M(\Lambda)$ with highest weight $\Lambda \in \mathfrak{h}^* \setminus L$. Note that, in the summation above for the expression of $x$, $m$ is an element of the set $\text{Par } = \bigcup_{\beta \in Q_+} \text{Par } \beta$. We will show by induction on $\beta$ that if $m \in \text{Par } \beta$, then $\varphi_{k,m} = 0$ for $k \notin \text{Par } \beta$.

Let us fix $\beta \in Q_+$ and $\Lambda \in \mathfrak{h}^* \setminus L$. The element $x$ acts as a scalar (independent of $\beta$) on the weight space $M(\Lambda)_{\Lambda - \beta}$. Now fix an arbitrary $m_0 \in \text{Par } \beta$. Because the matrix $B_\beta^\Lambda = (B_\beta^\Lambda(F^k, F^m))_{k, m \in \text{Par } \beta}$ is nonsingular for $\Lambda \in \mathfrak{h}^* \setminus L$, we can choose an element $v \in M(\Lambda)_{\Lambda - \beta}$ such that $\sigma(F^{m_0})(v) = cv_\Lambda$ for some nonzero $c \in \mathbb{C}$, and $\sigma(F^m)(v) = 0$ for any $m \neq m_0 \in \text{Par } \beta$. Then we have

$$M(\Lambda)_{\Lambda - \beta} \ni \mathbb{C} v \ni x(v) = \sum_{k \in \text{Par } \gamma < \beta} \sum_{m \in \text{Par } \gamma} F^k \varphi_{k,m} \sigma(F^m)(v) + \sum_{k \in \text{Par } \gamma} c \varphi_{k,m_0}(\Lambda) F^k(v_\Lambda),$$

where $F^k \varphi_{k,m} \sigma(F^m)(v) \in M(\Lambda)_{\Lambda - \beta}$ for $m \in \text{Par } \gamma$ with $\gamma < \beta$ by the inductive assumption. Therefore, we deduce that $\varphi_{k,m_0}(\Lambda) = 0$ for any $k \notin \text{Par } \beta$ since the vectors $\{F^k(v_\Lambda)\}_{k \in \text{Par } \gamma}$ are linearly independent. This means that $\varphi_{k,m_0} = 0$ as an element of $\mathcal{F}$ for $k \notin \text{Par } \beta$. \qed

From Theorem 1 and Proposition 1, we see that there exists an algebra isomorphism $H : Z_F \to \mathcal{F}$ defined by $z_\varphi \mapsto \varphi = \varphi_{0,0}$. we call this isomorphism $H$ the Harish-Chandra homomorphism.
2. Holomorphicity of the Functions $\varphi_{k,m}$

2.1. The Tits cone of GKM algebras. From now on, we assume that the GKM algebra $g(A)$ over $\mathbb{C}$ is the complexification of the GKM algebra $g(A)_{\mathbb{R}}$ over $\mathbb{R}$ (i.e., $g(A) = \mathbb{C} \otimes_{\mathbb{R}} g(A)_{\mathbb{R}}$). So the Cartan subalgebra $\mathfrak{h}$ over $\mathbb{C}$ is also the complexification of the Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ (i.e., $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}$), and the set of simple roots $\Pi = \{\alpha_i\}_{i \in I}$ is a linearly independent subset of the algebraic dual $\mathfrak{h}_{\mathbb{R}}^*$ of $\mathfrak{h}_{\mathbb{R}}$ over $\mathbb{R}$. Further there exists a nondegenerate $W$-invariant symmetric $\mathbb{R}$-bilinear form $(\cdot | \cdot)$ on $\mathfrak{h}_{\mathbb{R}}^*$, whose complexification on $\mathfrak{h}^*$ is also denoted by $(\cdot | \cdot)$.

Here we define the fundamental chamber $C$ and the Tits cone $X$ of the GKM algebra $g(A)$. We put

$$C := \{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* | (\lambda | \alpha_i) \geq 0 \text{ for } i \in I \},$$

and then $X := W \cdot C = \bigcup_{w \in W} w \cdot C$. We denote by $X^\circ$ (resp. $X^-$) the interior (resp. the closure) of $X$ in the usual metric topology of $\mathfrak{h}_{\mathbb{R}}^*$.

Remark 1. In [B3] and [K1], the fundamental chamber was defined to be the set

$$C^{re} := \{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* | (\lambda | \alpha_i) \geq 0 \text{ for } i \in I \text{ with } a_{ii} = 2 \},$$

and the the Tits cone was defined to be $X^{re} := W \cdot C^{re}$. However this definition is not appropriate for our purpose here.

The proof of the following lemma is almost the same as in the case of Kac-Moody algebras (see [K1, Chap. 3] and [W, Chap. 4]).

Lemma 1. (1) The fundamental chamber $C$ is a fundamental domain for the action of $W$ on $X$, i.e., any orbit $W \cdot \lambda$ of $\lambda \in X$ intersects $C$ in exactly one point. Moreover, $W$ operates simply transitively on chambers.

(2) $X = \{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* | (\lambda | \alpha) < 0 \text{ for only a finite number of } \alpha \in \Delta_+ \}$. In particular, $X$ is a convex cone.

(3) $X^\circ = \{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* | (\lambda | \alpha) \leq 0 \text{ for only a finite number of } \alpha \in \Delta_+ \}$. 
Here we prepare some more notation for GKM algebras. Let $\Pi^r := \{\alpha_i \in \Pi \mid a_{ii} = 2\}$ be the set of real simple roots, and $\Pi^{im} := \{\alpha_i \in \Pi \mid a_{ii} \leq 0\}$ the set of imaginary simple roots, $\Delta^r := W \cdot \Pi^r$ the set of real roots, and $\Delta^{im} := \Delta \setminus \Delta^r$ the set of imaginary roots. We know from [K1, Chap. 11] that $\Delta^{im} \cap \Delta_+ = W \cdot N$, where

$$N = \{\alpha \in \mathbb{Q}_+ \setminus \{0\} \mid (\alpha|\alpha_i) \leq 0 \text{ for } i \text{ with } a_{ii} = 2, \text{ and } \text{supp}(\alpha) \text{ is connected}\} \setminus \bigcup_{j \geq 2} j \cdot \Pi^{im}.$$

In particular, the set $\Delta^{im}_+ := \Delta_+ \cap \Delta^{im}$ is $W$-stable.

Now we have the following lemma.

**Lemma 2.** (1) $X^- \subset \{\lambda \in \mathfrak{h}_R^* \mid (\lambda|\alpha) \geq 0 \text{ for all } \alpha \in \Delta^{im}_+\}$.

(2) $X^0 \subset \{\lambda \in \mathfrak{h}_R^* \mid (\lambda|\alpha) > 0 \text{ for all } \alpha \in \Delta^{im}_+\}$.

**Proof.** (1) Let $X' := \{\lambda \in \mathfrak{h}_R^* \mid (\lambda|\alpha) \geq 0 \text{ for all } \alpha \in \Delta^{im}_+\}$. Then it is clear that the set $X'$ is a $W$-stable closed subset of $\mathfrak{h}_R^*$ since $\Delta^{im}_+$ is $W$-stable. Because $C \subset X'$ from the definition, we have $X \subset X'$, so that $X^- \subset X'$.

(2) Put $l := \dim_R \mathfrak{h}_R^*$, and take a basis $\{v_i\}_{i=1}^l$ of $\mathfrak{h}_R^*$. Let $\lambda \in X^0$. Then there exists $\epsilon > 0$ such that $\lambda \pm \epsilon v_i \in X$ for $1 \leq i \leq l$. For any $\alpha \in \Delta^{im}_+$, there exists some $v_i$ such that $(v_i|\alpha) \neq 0$. If $(v_i|\alpha) > 0$, we have $(\lambda|\alpha) \geq \epsilon (v_i|\alpha) > 0$ since $(\lambda - \epsilon v_i|\alpha) \geq 0$ by (1). If $(v_i|\alpha) < 0$, we have $(\lambda|\alpha) \leq -\epsilon (v_i|\alpha) > 0$ since $(\lambda + \epsilon v_i|\alpha) \geq 0$. □

Let $X_C := X + \sqrt{-1} \mathfrak{h}_R^* = \{x + \sqrt{-1} y \mid x \in X, y \in \mathfrak{h}_R^*\}$ be the complexified Tits cone, and denote by $K$ the interior of $X_C$ in the usual metric topology of $\mathfrak{h}^*$. It is obvious that $K = X^0 + \sqrt{-1} \mathfrak{h}_R^*$.

From the lemmas above, we get the following lemma which will be used later.

**Lemma 3.** (1) Let $\alpha \in \Delta^{im}_+$ and $n \in \mathbb{Z}_{\geq 1}$. Then the affine hyperplane $T_{n\alpha}(\cdot) = 0$ does not intersect the domain $-\rho + K$.

(2) Let $\alpha \in \Delta^r_+$ and $n \in \mathbb{Z}_{\geq 1}$. If $\lambda \in -\rho + K$ and $T_{n\alpha}(\lambda) = 0$, then $\lambda - n\alpha \in -\rho + K$.

**Proof.** (1) Let $\lambda \in -\rho + K$, and suppose that $2(\lambda + \rho|\alpha) = n(\alpha|\alpha)$. Obviously we may assume that $\lambda \in -\rho + X^0$. We show that $(\alpha|\alpha) \leq 0$. Because $\Delta^{im}_+ = W \cdot N$, we may
assume that $\alpha = \sum_{i \in I} k_i \alpha_i \in N \subset Q_+$. Then we have $(\alpha | \alpha) = \sum_{i \in I} k_i (\alpha_i | \alpha_i) \leq 0$, since $(\alpha | \alpha_i) \leq 0$ for $\alpha_i \in \Pi^{re}$ by the definition of $N$ and $(\alpha_j | \alpha_i) \leq 0$ ($j \in I$) for $\alpha_i \in \Pi^{im}$. Now the equality above contradicts part (2) of Lemma 2.

(2) Because $\alpha \in \Delta^{re} = W \cdot \Pi^{re}$, we can write $\alpha = w \cdot \alpha_i$ for some $w \in W$ and $\alpha_i \in \Pi^{re}$. In particular $(\alpha | \alpha) = (\alpha_i | \alpha_i) > 0$. Here note that the reflection $r_\alpha$ of $\mathfrak{h}^*$ with respect to $\alpha$ is defined by $r_\alpha(\lambda) := \lambda - (2(\lambda | \alpha)/(|\alpha| \alpha)) \alpha$ for $\lambda \in \mathfrak{h}^*$ and can be written as $r_\alpha = wr_i w^{-1}$, so that $r_\alpha \in W$. Now we have $r_\alpha(\lambda + \rho) = \lambda + \rho - (2(\lambda + \rho) | \alpha)/(|\alpha| \alpha)) \alpha = \lambda + \rho - n\alpha$ by the assumption. Since $K$ is $W$-stable, we deduce that $\lambda - n\alpha \in -\rho + K$. ∎

2.2 Holomorphicity of the functions $\varphi_{k,m}$ on the domain $-\rho + K$. We first recall the following elementary lemma in [K2].

**Lemma 4 ([K2, Lemma 2])**. Let $B = (b_{ij})$ and $C = (c_{ij})$ be two $N \times N$-matrices, where $b_{ij}$ and $c_{ij}$ are holomorphic functions in the variables $z_1, \ldots, z_N$ on some neighborhood $U$ of the origin $0$. Put $V := U \cap \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid z_1 = 0\}$. Suppose that $B$ is invertible on $U \setminus V$ and that on $V$ one has:

(a) $\det B$ has zero of multiplicity $s \in \mathbb{Z}_{\geq 1}$;

(b) $\dim (\ker B) = s$;

(c) $\ker B \subset \ker C$.

Here $\ker B = \{x \in \mathbb{C}^N \mid Bx = 0\}$ (which, in general, depends on $(z_1, \ldots, z_N) \in \mathbb{C}^N$). Then the entries of the matrix $CB^{-1}$ can be extended to holomorphic functions on $U$.

We remark that the classification theorem ([K1, Theorem 4.3]) holds also in the case of indecomposable GGCMs:

(1) GGCMs of finite type are exactly GCMs of finite type;

(2) GGCMs of affine type are GCMs of affine type plus the zero $1 \times 1$ matrix.

(3) If $A = (a_{ij})_{i,j \in I}$ is a GGCM of indefinite type, then there exists a positive imaginary root $\alpha = \sum_{i \in I} k_i \alpha_i$ such that $k_i > 0$ and $(\alpha | \alpha_i) < 0$ for all $i \in I$ for the GKM algebra $g(A)$ (cf. the proof of [K1, Theorem 5.6 c]).
From now on we assume that the GGCM \( A = (a_{ij})_{i,j \in \mathcal{I}} \) is indecomposable, hence is either a GCM of finite type, a GCM of affine type, the zero \( 1 \times 1 \) matrix, or a GGCM (possibly GCM) of indefinite type.

Here we recall the following well-known facts about the (ordinary) Kac-Moody algebras \( g(A) \) associated to a GCM \( A = (a_{ij})_{i,j \in \mathcal{I}} \):

1. if \( A \) is a GCM of finite type, then \( X = \mathfrak{h}_R^* \);
2. if \( A \) is a GCM of affine type, then \( X^o = \{ \lambda \in \mathfrak{h}_R^* \mid (\lambda|\delta) > 0 \} \), where \( \delta \) is the unique (up to a constant factor) element of \( Q \) such that \( (\delta|\alpha_i) = 0 \) for all \( i \in \mathcal{I} \). In particular, we have \( K-Q_+ = K \) in both of these cases.

In addition, if \( g(A) \) is the GKM algebra associated to a GGCM \( A = (a_{ij})_{i,j \in \mathcal{I}} \) such that \( a_{ii} \leq 0 \) for all \( i \in \mathcal{I} \), then obviously we have \( X - \beta \subset X \) for \( \beta \in Q_+ \) since \( X = C \), \( W = \{ 1 \} \), and \( Q_+ = \sum_{\alpha_i \in \Pi^m} \mathbb{Z}_{\geq 0} \alpha_i \). Hence we have \( K - \beta \subset K \) for \( \beta \in Q_+ \), so that \( K - Q_+ = K \) in this case (including the case where \( A \) is the zero \( 1 \times 1 \) matrix).

We are now in a position to state our main theorem (compare with [K2, Theorem 2]).

**Theorem 2.** Let \( \varphi \in \mathcal{F} \) be a function that can be extended to a holomorphic function on the domain \( -\rho + K \), and \( z_\varphi = \sum_{\beta \in Q_+} \sum_{k,m \in \text{Par} \beta} F^k \varphi_{k,m} \sigma(F^m) \) be the (unique) element of the center \( Z_\mathcal{F} \) such that \( H(z_\varphi) = \varphi \).

1. If all the functions \( \varphi_{k,m} \) can be extended to holomorphic functions on the domain \( -\rho + K - Q_+ = \cup_{\beta \in Q_+} (-\rho + K - \beta) \), then we have for \( \alpha \in \Delta_+^e \) and \( n \in \mathbb{Z}_{\geq 1} \),

\[
T_{n\alpha}(\lambda) = 0 \text{ with } \lambda \in -\rho + K \text{ implies } \varphi(\lambda) = \varphi(\lambda - n\alpha).
\]

2. Let the function \( \varphi \) satisfy the condition that for \( \alpha \in \Delta_+^e \) and \( n \in \mathbb{Z}_{\geq 1} \),

\[
T_{n\alpha}(\lambda) = 0 \text{ with } \lambda \in -\rho + K \text{ implies } \varphi(\lambda) = \varphi(\lambda - n\alpha).
\]

Then, for each \( \beta \in Q_+ \), there exists a nonempty domain \( M_{\beta} \subset K \) such that the functions \( \varphi_{k,m} \in \mathcal{F} \) with \( k, m \in \text{Par} \beta \) can be extended to holomorphic functions on the domain
$-\rho + M_\beta$. If the GGCM $A$ is of finite or affine type, then we can take $M_\beta = K$ for all $\beta \in Q_+$. In the case of indefinite type, as $M_\beta$, we can take a domain of the form $\mu_\beta + K \subset K$ for some $\mu_\beta \in V := K \cap (-\sum_{\alpha_i \in \Pi^+, R > 0} \alpha_i)$.

**Proof.** (1) First note that if the GGCM $A$ is not of indefinite type, then we have $K - Q_+ = K$ from the remarks above. Second we remark that even in the case of indefinite type, the set $-\rho + K - Q_+$ is really a connected open set in $\mathfrak{h}^*$. In fact it is obvious that $-\rho + K - Q_+$ is an open set since it is the union of open sets $-\rho + K - \beta$ ($\beta \in Q_+$). The connectedness of $-\rho + K - Q_+$ follows from the connectedness of $K$ itself and the fact that $K \cap (K - \beta) \neq \emptyset$ for any $\beta \in Q_+$. The latter fact is because $K$ is an open convex cone in $\mathfrak{h}^* = h^*_R + \sqrt{-1} h^*_I$.

Let $\lambda \in -\rho + K$. We will show that the element $z_{\varphi} \in \mathbb{U} (\mathfrak{g}(A))$ can act on the Verma module $M(\lambda)$ with highest weight $\lambda$ as the scalar $\varphi(\lambda)$, or equivalently, that $z_{\varphi}$ acts as the scalar $\varphi(\lambda)$ on each weight space $M(\lambda)_{\lambda-\beta}$ for $\beta \in Q_+$. It clearly suffices to show that the equation $(*)$ (is well-defined and) holds for this $\lambda \in -\rho + K$ (see the proof of Theorem 1).

Here the entries of the matrix $\Phi_\beta(\cdot) = (\varphi_{k,m}(\cdot))_{k,m \in \text{Par } \beta}$ are holomorphic on $-\rho + K$ by assumption, so are the entries of the matrix $G_{\beta}^\gamma(\cdot) = \Phi_\beta(\cdot) B_\beta$. Moreover we show that for any $\gamma < \beta$, the entries of the matrix $G_{\gamma}^\beta(\cdot)$ are holomorphic on $-\rho + K$ above. Let $\lambda \in -\rho + K$, $v \in M(\lambda)_{\lambda-\beta}$, and $s, t \in \text{Par } \gamma$. Then we have $\sigma(F^t)v \in M(\lambda)_{\lambda-(\beta-\gamma)}$, so that $F^s \varphi_{s,t}(\cdot) \sigma(F^t)v = \varphi_{s,t}(\lambda - (\beta-\gamma))F^s \sigma(F^t)v$, where $\lambda - (\beta-\gamma) \in -\rho + K - Q_+$. Because the functions $\varphi_{s,t}(\cdot)$ are holomorphic on $-\rho + K - Q_+$ by assumption, the entries of the matrix $G_{\gamma}^\beta(\cdot)$ are holomorphic at any $\lambda \in -\rho + K$.

On the other hand, for each $\lambda \in \mathfrak{h}^* \setminus L$, the equation $(*)$ holds by (the proof of) Theorem 1. Since the set $\mathfrak{h}^* \setminus L$ is dense in $\mathfrak{h}^*$, we can take a sequence $\{\lambda_m\}_{m=1}^\infty$ in $(\mathfrak{h}^* \setminus L) \cap (-\rho + K)$ such that $\lim_{m \to \infty} \lambda_m = \lambda$ for each $\lambda \in -\rho + K$. Because all the entries of the matrices $G_{\beta}^\gamma(\cdot)$, $G_{\gamma}^\beta(\cdot)$ are holomorphic at $\lambda \in -\rho + K$, by taking the limit as $m \to \infty$, we have the equation $(*)$ for this $\lambda \in -\rho + K$. 


Now let $\Lambda \in -\rho + K$ be such that $T_{n\alpha}(\Lambda) = 0$ for some $\alpha \in \Delta^{\text{re}}_{+}$ and $n \in \mathbb{Z}_{\geq 1}$. Then we have an embedding $M(\Lambda - n\alpha) \hookrightarrow M(\Lambda)$ by [KK, Prop. 4.1 (b)]. The element $z_{\varphi}$ obviously acts on the highest weight vector $v_{\Lambda - n\alpha} \neq 0 \in M(\Lambda - n\alpha)$ as the scalar $\varphi(\Lambda - n\alpha)$. Thus we have the equality $\varphi(\Lambda) = \varphi(\Lambda - n\alpha)$ for $\Lambda \in -\rho + K$ with $T_{n\alpha}(\Lambda) = 0$.

(2) First of all we remark that, in the case of indefinite type, $V \neq \emptyset$ since there exists $\alpha = \sum_{i \in I} k_{i}\alpha_{i} \in \Delta^{\text{re}}_{+}$ such that $k_{i} > 0$ and $(\alpha|\alpha_{i}) < 0$ for all $i \in I$ (see the comment above for the classification theorem of GGCMs).

Now we will take domain $M_{\beta}$ by induction on $\beta \in Q_{+}$. We first take $M_{0} = K$. Note that $K - \alpha_{j} \subset K$ for $\alpha_{j} \in \Pi^{\text{re}}_{+}$ by part (3) of Lemma 1. Let us take $\beta \neq 0 \in Q_{+}$. Suppose that we have already taken domains $M_{\gamma} = \mu_{\gamma} + K \subset K$ with $\mu_{\gamma} \in V = K \cap (-\sum_{\alpha_{i} \in \Pi_{\text{re}}^{+}, \Re > 0} \alpha_{i})$ such that $M_{\gamma} - \alpha_{j} \subset M_{\gamma}$ ($\alpha_{j} \in \Pi^{\text{re}}_{+}$) for $Q_{+} \ni \gamma < \beta$. Put

$$M_{\beta}' := \bigcap_{\gamma < \beta} \bigcap_{\eta \in \sum_{\alpha_{i} \in \Pi_{\text{re}}^{+}} \mathbb{Z}_{\geq 0} \alpha_{i}} (M_{\gamma} + \eta).$$

For $\alpha_{j} \in \Pi^{\text{re}}_{+}$, we have $M_{\beta}' - \alpha_{j} \subset M_{\beta}'$ since $M_{\gamma} - \alpha_{j} \subset M_{\gamma}$ for $\gamma < \beta$ by the inductive assumption. For $\eta \in \sum_{\alpha_{i} \in \Pi_{\text{re}}^{+}} \mathbb{Z}_{\geq 0} \alpha_{i}$ with $\eta \leq \beta$, we obviously have $M_{\beta}' - \eta \subset M_{\gamma}$ for any $\gamma < \beta$. Hence we have $M_{\beta}' - \eta \subset M_{\gamma}$ for any $Q_{+} \ni \gamma < \beta$ and $Q_{+} \ni \eta \leq \beta$. We write $M_{\beta}' = \bigcap_{i=1}^{m} (v_{i} + K)$ for $v_{i} \in \sum_{\alpha_{i} \in \Pi_{\text{re}}^{+}} \mathbb{R} \alpha_{i}$. Because the set $V = K \cap (-\sum_{\alpha_{i} \in \Pi_{\text{re}}^{+}} \Re > 0 \alpha_{i})$ is an open convex cone in $\sum_{\alpha_{i} \in \Pi_{\text{re}}^{+}} \mathbb{R} \alpha_{i}$, we can write $v_{i} = x_{i} - y_{i}$ with $x_{i}, y_{i} \in V$ for each $i$, since $V - V = \sum_{\alpha_{i} \in \Pi_{\text{re}}^{+}} \mathbb{R} \alpha_{i}$. Then we have

$$M_{\beta}' = \bigcap_{i=1}^{m} (v_{i} + K) \supset \bigcap_{i=1}^{m} (x_{i} + K) \supset K + \sum_{i=1}^{m} x_{i},$$

since $K \supset V$ is a convex set. So we put $\mu_{\beta} := \sum_{i=1}^{m} x_{i} \in V$, and $M_{\beta} := \mu_{\beta} + K \subset K$. It is obvious that the set $M_{\beta}$ is really a nonempty open connected set in $\mathfrak{h}^{*}$.

We proceed by induction on $\beta \in Q_{+}$. Let us fix $\beta \in Q_{+}$ and show that the functions $\varphi_{k,m} \in \mathcal{F}$ with $k, m \in \text{Par} \beta$ can be extended to holomorphic functions on the domain $-\rho + M_{\beta}$. We have $M_{\beta} - \eta \subset M_{\gamma}$ for any $\gamma < \beta$ and $\eta \leq \beta$. Therefore the entries of the
matrices $G^\beta_\gamma(\cdot)$ for $\gamma < \beta$ are holomorphic on $-\rho + M_\beta$, since the functions $\varphi_{s,t}(\cdot)$ with $s,t \in \text{Par}\gamma$ are holomorphic on $-\rho + M_\gamma$ for $\gamma < \beta$. Hence, by the equation (*) in the proof of Theorem 1, we have only to show that the functions $\varphi_{k,m}$ with $k,m \in \text{Par}\beta$ can be holomorphically extended on $-\rho + M_\beta$ across the finitely many affine hyperplanes $T_{n\alpha}(\cdot) = 0$ for $\alpha \in \Delta_+$, $n \in \mathbb{Z}_{\geq 1}$ with $n\alpha \leq \beta$. Furthermore, by part (1) of Lemma 3, we may assume that $\alpha \in \Delta_+^{re}$.

Let us fix arbitrary $\alpha \in \Delta_+^{re}$ and $n \in \mathbb{Z}_{\geq 1}$ with $n\alpha \leq \beta$, and consider the set 

\{$\Lambda \in -\rho + M_\beta \mid T_{n\alpha}(\Lambda) = 0$\}. We now want to apply Lemma 4 to the case where $B = B_\beta^\Lambda$ and $C = \varphi(\Lambda)I_N - \sum_{\gamma < \beta} G^\beta_\gamma(\Lambda)$ with $N = \dim_{\mathbb{C}} M(\Lambda)_{\beta-\gamma}$ and $s = \#(\text{Par}(\beta - n\alpha))$ (remark that $\dim_{\mathbb{C}} g_\alpha = 1$ for $\alpha \in \Delta_+^{re} = W \cdot \Pi^{re}$). So we will show that for any $\Lambda \in -\rho + M_\beta$ with $T_{n\alpha}(\Lambda) = 0$, we have

$$\varphi(\Lambda)I_N = \sum_{\gamma < \beta} G^\beta_\gamma(\Lambda).$$

Because the entries of the matrices $G^\beta_\gamma(\cdot)$ with $\gamma < \beta$ are holomorphic on $-\rho + M_\beta \subset -\rho + K$, we may assume that $T_{m\alpha'}(\Lambda) \neq 0$ for all $\alpha' \neq \alpha \in \Delta_+$ and $m \in \mathbb{Z}_{\geq 1}$ (recall that $\mathfrak{h}^* \setminus L$ is dense in $\mathfrak{h}^*$). Then, by [KK, Prop. 4.1 (b) and the formula (4.2) on p. 106], we can deduce that the kernel $J(\Lambda)$ of the contravariant bilinear form $B^\Lambda(\cdot, \cdot)$ on the Verma module $M(\Lambda)$ is isomorphic to $M(\Lambda - n\alpha)$, where $B^\Lambda(F^kv_\Lambda, F^mv_\Lambda) = \delta_{\beta,\gamma} B^\beta_\gamma(F^k, F^m)$ for $k \in \text{Par}\beta$, $m \in \text{Par}\gamma$. Let $R := M(\Lambda)_{\beta-\gamma} \cap J(\Lambda) \cong M(\Lambda - n\alpha)_{(\Lambda - n\alpha)(\beta - n\alpha)}$. Since $J(\Lambda)$ is the kernel of the contravariant bilinear form $B^\Lambda(\cdot, \cdot)$ on $M(\Lambda)$, the matrix of the operator $z_\varphi$ on $R$ is $\sum_{\gamma < \beta} G^\beta_\gamma(\Lambda)$. We will show that the operator acts as the scalar $\varphi(\Lambda - n\alpha)$ on $R$. As in the proof of part (1), it suffices to show that the following equation (is well-defined and) holds for this $\Lambda \in -\rho + M_\beta$:

\[(**)
\phi_{\beta-n\alpha}(\Lambda - n\alpha)B^\Lambda_{\beta-n\alpha} + \sum_{\gamma < \beta-n\alpha} G^\beta_{\gamma-n\alpha}(\Lambda - n\alpha) = \varphi(\Lambda - n\alpha)\text{Id.}\]

(Note that $(\Lambda - n\alpha) - (\beta - n\alpha) = \Lambda - \beta$. Here we have $F^s\varphi_{s,t}(F^t)v = \varphi_{s,t}(\lambda - (\beta - n\alpha) + \gamma)F^s\sigma(F^t)v$ for $v \in M(\Lambda)_{\lambda-(\beta-n\alpha)}$ with $\lambda \in -\rho - n\alpha + M_\beta$ and $s,t \in \text{Par}\gamma$.)
with $\gamma \leq \beta - n\alpha$. So, for each $\gamma \leq \beta - n\alpha < \beta$, the entries of the matrix $G_{\gamma}^{\beta-n\alpha}(\cdot)$ (including $\Phi_{\beta-n\alpha}(\cdot)$) are holomorphic on $-\rho - n\alpha + M\beta$ by the inductive assumption, since $\lambda \in -\rho - n\alpha + M\beta$ implies $\lambda - (\beta - n\alpha) + \gamma = \lambda + n\alpha - (\beta - \gamma) \in -\rho + M\gamma$. On the other hand, for each $\lambda \in \mathfrak{h}^* \setminus L$, the equation $(\star\star)$ with $\Lambda$ replaced with $\lambda$ holds by (the proof of) Theorem 1. Hence, by taking the limit, we have the equation $(\star\star)$ for $\Lambda$ above. Thus the operator $z \varphi$ acts on $R \cong M(\Lambda - n\alpha)_{\Lambda \beta} -$ as the scalar $\varphi(\Lambda - n\alpha)$.

Due to Lemma 4 above, we deduce that the functions $\varphi_{k,m}$ with $k, m \in \text{Par} \beta$ have a removable singularity at any $\Lambda \in \{\Lambda \in -\rho + M\beta \mid T_{n\alpha}(\Lambda) = 0, \text{ and } T_{m\alpha'}(\Lambda) \neq 0 \text{ for } \alpha' \neq \alpha \in \triangle_{+}^{re}, m \in \mathbb{Z} \geq 1 \text{ with } m\alpha' \leq \beta\}$. Then we quote the theorem (cf. [GR, Theorem 1.8]) which asserts that a function of at least two complex variables can be holomorphically extended across the intersection of finitely many (but at least two) affine hyperplanes. Therefore we have proved that the functions $\varphi_{k,m}$ with $k, m \in \text{Par} \beta$ can be extended to holomorphic functions on $-\rho + M\beta$.

**Remark 2.** Let $f \in S(\mathfrak{h})$ be $W$-invariant. Then the function $\varphi(\cdot) \in \mathcal{F}$ defined by $\varphi(\lambda) := f(\lambda + \rho)$ $(\lambda \in \mathfrak{h}^*)$ satisfies the conditions of Theorem 2 (see the proof of part (2) of Lemma 3).

Finally we consider the domain $-\rho + K - Q_+$ in part (1) and the domain $-\rho + \cap_{\beta \in Q} M\beta$ in part (2) of Theorem 2 above in the case of indefinite type.

We prepare the following lemma, which can be proved almost in the same way as in the case of Kac-Moody algebras (cf. the proof of [K1, Proposition 5.8 c])).

**Lemma 5.** Let $g(A)$ be the GKM algebra associated to a GGCM of indefinite type. Then we have

$$X^- = \{\lambda \in \mathfrak{h}_R^* \mid (\lambda | \alpha) \geq 0 \text{ for all } \alpha \in \triangle_{+}^{im}\}.$$  

We now have the following proposition.

**Proposition 2.** Let $g(A)$ be the GKM algebra associated to a GGCM $A = (a_{ij})_{i,j \in I}$ of indefinite type with $a_{ii} = 2$ for some $i \in I$. Then we have $K \subsetneq K - Q_+$, and $\cap_{\beta \in Q_+} M\beta = \emptyset$. 
Proof. We first show that there exists a positive imaginary root \( \alpha \in \Delta_{+}^{im} \) and a real simple root \( \alpha_{i_{0}} \in \Pi^{re} \) such that \( (\alpha|\alpha_{i}) > 0 \) for all \( i \in I \). Take \( i_{0} \in I \) with \( a_{i_{0}i_{0}} = 2 \), and put \( \alpha := r_{i_{0}}(\alpha') \). We have \( (\alpha|\alpha_{i_{0}}) = (r_{i_{0}}(\alpha')|\alpha_{i_{0}}) = -(\alpha'|\alpha_{i_{0}}) > 0 \), and \( \alpha \in \Delta_{+}^{im} \) since the set \( \Delta_{+}^{im} \) is W-stable.

If \( K - \alpha_{i_{0}} \subset K \) for this \( \alpha_{i_{0}} \), then we obviously have \( X^{o} - \alpha_{i_{0}} \subset X^{o} \) since \( K = X^{o} + \sqrt{-1} h_{R}^{*} \). Then we have \( X^{-} - \alpha_{i_{0}} \subset X^{-} \) since \( (X^{o})^{-} = X^{-} \) from the convexity of the set \( X \). Because \( 0 \in X^{-} \), we get \( -\alpha_{i_{0}} \in X^{-} \), so that \( (\alpha_{i_{0}}|\alpha) \geq 0 \) by Lemma 5. This is a contradiction. Hence we have \( K - \alpha_{i_{0}} \not\subset K \), so that \( K \not\subset K - Q_{+} \).

Let \( x \in \cap_{\beta \in Q_{\rho}} M_{\beta} \). Then we have \( x - \beta \in M_{\beta} - \beta \subset K \) for all \( \beta \in Q_{+} \). Because \( K \ni x \) is an open convex cone, we can easily deduce that \( K - \beta \subset K \) for all \( \beta \in Q_{+} \), which contradicts the fact that \( K \not\subset K - Q_{+} \) just proved above. \( \square \)

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