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Minimization of M-convex Function

Akiyoshi SHIOURA*

Abstract

M-convex function, introduced by Murota (1995), is an extension of valuated matroid of Dress–Wenzel (1990) as well as a quantitative generalization of the set of integral points in an integral base polyhedron. In this paper, we study the minimization of an M-convex function. It is shown that any vector in the domain can be easily separated from a minimizer of the function. Based on this property, we develop a polynomial time algorithm. We also discuss the layer structure of an M-convex function and the minimization in each layer.

Keywords: matroid, base polyhedron, convex function, minimization.

1 Introduction

Recently, the concept of M-convex function was introduced by Murota [14, 15, 16] as an extension of valuated matroid due to Dress and Wenzel [4, 5] as well as a quantitative generalization of (the integral points of) the base polyhedron of an integral submodular system [7]. M-convexity is quite a natural concept appearing in many situations, and enjoys several nice properties which are sufficient to be regarded as convexity in combinatorial optimization. Let V be a finite set with cardinality n. A function $f : \mathbf{Z}^V \rightarrow \mathbf{R} \cup \{+\infty\}$ is said to be M-convex if it satisfies

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$f(x)+f(y) \geq f(x-\chi_u+\chi_v)+f(y+\chi_u-\chi_v),$$

where $\mathrm{dom}\, f=\{x \in \mathbf{Z}^V \mid f(x)<+\infty\}$, $\mathrm{supp}^+(x-y)=\{w \in V \mid x(w)>y(w)\}$, $\mathrm{supp}^-(x-y)=\{w \in V \mid x(w)<y(w)\}$, and $\chi_w \in \{0,1\}^V$ is the characteristic vector of $w \in V$. For an M-convex function f with $\mathrm{dom}\, f \subseteq \{0,1\}^V$, $-f$ is a valuated matroid in the sense of [4, 5]. The property (M-EXC) implies that $\mathrm{dom}\, f$ is a base polyhedron.

In this paper, we consider the problem of minimizing an M-convex function. While the concept of M-convexity is quite new and no efficient algorithm is known yet, several polynomial

*塩浦 昭義, Department of Mechanical Engineering, Sophia University, 7-1 Kioi-cho, Chiyoda-ku, Tokyo 102, Japan, shioura@keka.me.sophia.ac.jp.

time algorithms are proposed for special cases of M-convex function. It is well-known that a linear function can be easily minimized over a base polyhedron by a simple greedy algorithm (see [7]). A strongly-polynomial time algorithm was given by Fujishige [6] for a separable-convex quadratic function, and weakly-polynomial time algorithms were given by Groenevelt [9] and Hochbaum [10] for a general separable-convex function. It was reported that there is no strongly-polynomial time algorithm for a general separable-convex function [10].

The aim of this paper is to develop an efficient algorithm for minimizing an M-convex function. Since the local optimality is equal to the global optimality, an optimal solution can be found by a descent method, which does not necessarily terminate in polynomial time. Instead, we propose a different approach. Our approach is based on the property that any vector in the domain can be efficiently separated from a minimizer of the function, which is shown later. Each iteration finds a certain vector in the current domain, and divides the domain so that the vector and an optimal solution are separated. By the clever choice of the vector, the size of the domain reduces in a certain ratio iteratively, which leads to a weakly-polynomial time algorithm.

We also discuss the layer structure of an M-convex function and the minimization in each layer, where a layer is the restriction of the function to $\{x \in \mathbf{Z}^V \mid x(W) = k\}$ for $W \subseteq V, k \in \mathbf{Z}$. Recently, many researchers analyze set systems and functions with respect to the layer structure; for example, greedoid by Korte, Lovász, and Schrader [11], valuated bimatroid and valuation on independent sets by Murota [12, 13], well-layered map and rewarding map by Dress and Terhalle [1, 2, 3], M-convex function on generalized polymatroid by Murota and Shioura [17], and so on. We reveal that each layer has a nice structure such as M-convexity, and that the minimizers in consecutive layers are closely related. Exploiting these properties, we can solve the minimization problems in successive layers efficiently.

2 Minimization of an M-convex Function

2.1 Theorems

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be M-convex. The global minimality of an M-convex function is characterized by the local minimality.

Theorem 2.1 ([14, 16]) *For any $x \in \mathrm{dom} f$, $f(x) \leq f(y)$ $(\forall y \in \mathbf{Z}^V)$ if and only if $f(x) \leq f(x - \chi_u + \chi_v)$ $(\forall u, v \in V)$.* ∎

Any vector in $\mathrm{dom} f$ can be easily separated from some minimizer of f.

Theorem 2.2 *(i) For $x \in \mathrm{dom} f$ and $v \in V$, let $u \in V$ satisfy $f(x-\chi_u+\chi_v) = \min_{s \in V}\{f(x-\chi_s+\chi_v)\}$. Set $x' = x - \chi_u + \chi_v$. Then, there exists $x^* \in \arg\min f$ with $x^*(u) \leq x'(u)$.*
(ii) For $x \in \mathrm{dom} f$ and $u \in V$, let $v \in V$ satisfy $f(x - \chi_u + \chi_v) = \min_{t \in V}\{f(x - \chi_u + \chi_t)\}$. Set $x' = x - \chi_u + \chi_v$. Then, there exists $x^* \in \arg\min f$ with $x^*(v) \geq x'(v)$.

Proof. We prove the first claim only. Let $x^* \in \arg\min f$ with the minimum value of $x^*(u)$, and to the contrary suppose $x^*(u) > x'(u)$. By (M-EXC), there exists $w \in \mathrm{supp}^-(x^* - x')$ such that $f(x^*) + f(x') \geq f(x^* - \chi_u + \chi_w) + f(x + \chi_v - \chi_w)$. The assumptions for x^* and x' imply $x^* - \chi_u + \chi_w \in \arg\min f$, a contradiction. ∎

Corollary 2.3 *Let $x \in \mathrm{dom}\, f$ with $x \notin \arg\min f$ and $u, v \in V$ satisfy $f(x - \chi_u + \chi_v) = \min_{s,t \in V}\{f(x - \chi_s + \chi_t)\}$. Then, there exists $x^* \in \arg\min f$ with $x^*(u) \leq x(u) - 1$, $x^*(v) \geq x(v) + 1$.*

Let $B \subseteq \mathbf{Z}^V$ be a bounded base polyhedron, and $\rho_B : 2^V \to \mathbf{Z}$ the submodular function corresponding to B, i.e., $\rho_B(X) = \max_{y \in B}\{y(X)\}$ $(\forall X \subseteq V)$. For each $w \in V$, set $l_B(w) = \rho_B(V) - \rho_B(V - w) (= \min_{y \in B}\{y(w)\})$ and $u_B(w) = \rho_B(w) (= \max_{y \in B}\{y(w)\})$. Define

$$N_B = \{y \in B \mid l'_B(w) \leq y(w) \leq u'_B(w) \ (\forall w \in V)\},$$

where $l'_B(w) = \lfloor (1-1/n) l_B(w) + (1/n) u_B(w) \rfloor$, $u'_B(w) = \lceil (1/n) l_B(w) + (1-1/n) u_B(w) \rceil$ $(w \in V)$.

Theorem 2.4 *$N_B \neq \emptyset$ for any bounded base polyhedron B.*

Proof. We abbreviate the subscript B for notational simplicity. It suffices to show the following (see [7, Theorem 3.8]):

$$\text{(i) } l'(X) \leq \rho(X) \ (\forall X \subseteq V), \qquad \text{(ii) } u'(X) \geq \rho(V) - \rho(V - X) \ (\forall X \subseteq V).$$

Since (ii) can be shown similarly, we prove (i) only. Let $X \subseteq V$ with $|X| = k$. We claim

$$k\rho(X) + k \sum_{v \in X}\{\rho(V - v) - \rho(V)\} \geq \sum_{v \in X}\{\rho(v) + \rho(V - v) - \rho(V)\}. \tag{1}$$

Indeed, we have

$$\begin{aligned} \text{LHS} &= k\rho(X) + \sum_{v \in X} \sum_{w \in X - v} \{\rho(V - w) - \rho(V)\} + \sum_{v \in X}\{\rho(V - v) - \rho(V)\} \\ &\geq k\rho(X) + \sum_{v \in X}\{\rho(V - (X - v)) - \rho(V)\} + \sum_{v \in X}\{\rho(V - v) - \rho(V)\} \\ &= \sum_{v \in X}\{\rho(X) + \rho(V - (X - v)) - \rho(V)\} + \sum_{v \in X}\{\rho(V - v) - \rho(V)\} \ \geq \ \text{RHS}, \end{aligned}$$

where the inequalities are by the submodularity of ρ. Since the LHS is nonnegative, k in (1) can be replaced by $n (\geq k)$. Thus,

$$\rho(X) \geq (1 - 1/n) \sum_{v \in X}\{\rho(V) - \rho(V - v)\} + (1/n) \sum_{v \in X} \rho(v) \geq l'(X).$$

∎

We call N_B the *narrowed base polyhedron* of B. From its definition, N_B is a base polyhedron with the corresponding submodular function $\rho_N : 2^V \to \mathbf{Z}$ such that

$$\rho_N(X) = \min_{Y \subseteq V}\{\rho_B(Y) - l'_B(Y - X) + u'_B(X - Y)\} \qquad (\forall X \subseteq V). \tag{2}$$

Using the function ρ_N, an extreme point x of N_B is written as

$$x(v_i) = \rho_N(V_i) - \rho_N(V_{i-1}) \qquad (i = 1, 2, \cdots, n), \tag{3}$$

where $\{v_1, \cdots, v_n\}$ is any ordering of elements in V, $V_0 = \emptyset$, and $V_i = \{v_1, \cdots, v_i\}$ $(i = 1, \cdots, n)$.

2.2 Algorithms

In this section we assume that $\mathrm{dom}\, f$ is bounded for the finiteness of the algorithms. Theorem 2.1 immediately leads to the following algorithm.

Algorithm STEEPEST_DESCENT

Step 0: Let x be any vector in $\mathrm{dom}\, f$.

Step 1: If $f(x) = \min_{s,t \in V}\{f(x - \chi_s + \chi_t)\}$ then stop. x is a minimizer.

Step 2: Find $u, v \in V$ with $f(x - \chi_u + \chi_v) = \min_{s,t \in V}\{f(x - \chi_s + \chi_t)\}$.

Step 3: Set $x := x - \chi_u + \chi_v$. Go to Step 1. □

The algorithm STEEPEST_DESCENT always terminates since the function value of x decreases strictly in each iteration. However, there is no guarantee for the polynomiality of the number of iterations.

The second algorithm maintains a set $S(\subseteq \mathrm{dom}\, f)$ containing a minimizer, which is represented by two vectors $a, b \in \mathbf{Z}^V$ as $S = \{y \in \mathrm{dom}\, f \mid a(w) \leq y(w) \leq b(w)\ (\forall w \in V)\}$. We see from definition that S is a base polyhedron with the corresponding submodular function $\rho_S : 2^V \to \mathbf{Z}$ such that

$$\rho_S(X) = \min_{Y \subseteq V}\{\rho(Y) - a(Y - X) + b(X - Y)\} \qquad (\forall X \subseteq V). \tag{4}$$

The algorithm reduces S iteratively by exploiting Corollary 2.3 and finally finds a minimizer. We assume that the submodular function $\rho : 2^V \to \mathbf{Z}$ corresponding to $\mathrm{dom}\, f$ is also given.

Algorithm DOMAIN_REDUCTION

Step 0: Set $a(w) := \rho(V) - \rho(V - w)$, $b(w) := \rho(w)$ for any $w \in V$.

Step 1: Find a vector x in the narrowed base polyhedron of S.

Step 2: If $f(x) = \min_{s,t \in V}\{f(x - \chi_s + \chi_t)\}$ then stop. x is a minimizer.

Step 3: Find $u, v \in V$ with $f(x - \chi_u + \chi_v) = \min_{s,t \in V}\{f(x - \chi_s + \chi_t)\}$.

Step 4: Update $a(v)$ and $b(u)$ as $a(v) := x(v) + 1$ and $b(u) := x(u) - 1$. Go to Step 1. □

We analyze the number of iterations of the algorithm. Denote by S_i the set S in the i-th iteration, and let $l_i(w) = \min_{y \in S_i}\{y(w)\}$, $u_i(w) = \max_{y \in S_i}\{y(w)\}$ for each $w \in V$. It is clear that $u_i(w) - l_i(w)$ is monotonically nonincreasing w.r.t. i. Furthermore, we have the following:

Lemma 2.5 *$u_{i+1}(w) - l_{i+1}(w) < (1 - 1/n)\{u_i(w) - l_i(w)\}$ for $w \in \{u, v\}$, where $u, v \in V$ are the elements chosen in Step 3.*

Proof. We show the case $w = u$. Let x be the vector chosen in Step 1. Then,

$$\begin{aligned} u_{i+1}(u) - l_{i+1}(u) &\leq x(u) - 1 - l_i(u) \\ &\leq \lceil (1/n) l_i(u) + (1 - 1/n) u_i(u) \rceil - 1 - l_i(u) < (1 - 1/n)\{u_i(u) - l_i(u)\}. \end{aligned}$$

The proof for the case $w = v$ is similar and omitted. ∎

Let $L = \max_{w \in V}\{u_1(w) - l_1(w)\} (= \max_{w \in V}\{\rho(w) - \rho(V) + \rho(V - w)\})$.

Theorem 2.6 *The algorithm* DOMAIN_REDUCTION *terminates in $O(n^2 \ln L)$ iterations.*

Proof. Since the value $u_i(w) - l_i(w)$ $(w \in V)$ is a nonnegative integer, the algorithm stops if $u_i(w) - l_i(w) < 1$ for all $w \in V$. Let k be the minimum integer with $(1 - 1/n)^k\{u_1(w) - l_1(w)\} < 1$. If $u_1(w) \neq l_1(w)$ and $n \geq 2$ then $k = \lceil -\ln\{u_1(w) - l_1(w)\} / \ln(1 - 1/n) \rceil$, and a well-known inequality $\ln z \leq z - 1$ $(\forall z > 0)$ implies

$$-\ln\{u_1(w) - l_1(w)\} / \ln(1 - 1/n) \leq n \ln\{u_1(w) - l_1(w)\}.$$

Thus the claim follows. ∎

In each iteration, Step 1 finds a vector x in the narrowed base polyhedron of S by using the equations (2), (3), and (4), which can be done by minimizing certain submodular functions $O(n)$ times and using floor and ceiling operations $O(n)$ times. Note that a submodular function can be minimized in strongly-polynomial time [8], and floor and ceiling operations can be performed easily since n is the denominator of each value for which floor or ceiling is operated. The other steps require $O(n^2)$-time evaluation of f. Hence, the algorithm DOMAIN_REDUCTION terminates in weakly-polynomial time.

3 Greedily Solvable Layer Structure

Suppose we are given an M-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. We discuss the layer structure of f and the minimization problem in each layer, where a layer is the restriction of f to $\{x \in \mathbf{Z}^V \mid x(W) = k\}$ for $W \subseteq V$ and $k \in \mathbf{Z}$. For any $W \subseteq V$, set $\lambda^W = \inf\{x(W) \mid x \in \operatorname{dom} f\}$ and $\mu^W = \sup\{x(W) \mid x \in \operatorname{dom} f\}$. For any $W \subseteq V$ and $k \in \mathbf{Z}$, define a function $f_k^W : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ as $f_k^W(x) = f(x)$ if $x(W) = k$, and $= +\infty$ otherwise. The following properties reveal that certain layers have a nice structure.

Theorem 3.1 *If $|W| = 1$ or $|W| = |V| - 1$, f_k^W satisfies (M-EXC) for any k with $\lambda^W \leq k \leq \mu^W$.*

Proof. Assume $|W| = 1$ and denote by w the unique element in W. Let $x, y \in \mathrm{dom}\, f_k^W$ and $u \in \mathrm{supp}^+(x-y)$. Since $x, y \in \mathrm{dom}\, f$, (M-EXC) for f implies

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v) \tag{5}$$

for some $v \in \mathrm{supp}^-(x-y)$. Since $x(w) = y(w) = k$, we have $w \neq u, v$. Hence, $(x - \chi_u + \chi_v)(w) = (y + \chi_u - \chi_v)(w) = k$, which together with (5) implies (M-EXC) for f_k^W.

When $|W| = |V| - 1$, $x(V - W) = y(V - W)$ for any $x, y \in \mathrm{dom}\, f_k^W$ and therefore the proof is similar. ∎

Theorem 3.2 *f_k^W satisfies (M-EXC) if either $k = \lambda^W$ or $k = \mu^W$.*

Proof. Let $x, y \in \mathrm{dom}\, f_k^W$ and $u \in \mathrm{supp}^+(x-y)$. Since $x, y \in \mathrm{dom}\, f$, we can apply (M-EXC) and obtain $v \in \mathrm{supp}^-(x-y)$ with

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v). \tag{6}$$

Since $x(W) = y(W) = \lambda^W$ (or μ^W), $u \in W$ if and only if $v \in W$. Hence, $(x - \chi_u + \chi_v)(W) = (y + \chi_u - \chi_v)(W) = \lambda^W$ (or μ^W), which together with (6) implies (M-EXC) for f_k^W. ∎

Thus, we can find a minimizer of f_k^W efficiently by the algorithm DOMAIN_REDUCTION in Section 2 if either (i) $|W| = 1$ or $|W| = |V| - 1$, or (ii) $k = \lambda^W$ or $k = \mu^W$.

The converse of Theorem 3.1 partially holds. For any $x \in \mathbf{Z}^V$, define $||x|| = \sum\{|x(w)| \mid w \in V\}$.

Theorem 3.3 *Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $|V| \geq 5$. Suppose that $\mathrm{dom}\, f$ is a base polyhedron and that f_k^W satisfies (M-EXC) for any $W \subseteq V$ with $|W| = 1$ and $k \in \mathbf{Z}$ with $\mu^W \leq k \leq \lambda^W$. Then, f satisfies (M-EXC).*

Proof. It suffices to show the following [15, Theorem 3.1]:

$\forall x, y \in \mathrm{dom}\, f$ with $||x - y|| = 4$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

Let $x, y \in \mathrm{dom}\, f$ with $||x - y|| = 4$. Since $|V| \geq 5$, there exists some $w \in V$ with $x(w) = y(w)$. Hence, (M-EXC) for f_k^W with $W = \{w\}$ and $k = x(w)$ immediately implies the above property. ∎

The statement of the above theorem does not hold necessarily when $|V| = 4$. Let $f : \mathbf{Z}^4 \to \mathbf{R} \cup \{+\infty\}$ be defined by

$$\mathrm{dom}\, f = \{(0,0,0,0), (1,0,-1,0), (0,1,-1,0), (1,0,0,-1), (0,1,0,-1), (1,1,-1,-1)\},$$
$$f(0,0,0,0) = 0, \quad f(x) = 1 \text{ for } x \in \mathrm{dom}\, f - \{(0,0,0,0)\}.$$

The function f fulfills the assumptions of Theorem 3.3 except for $|V| \geq 5$. However, f is not M-convex since (M-EXC) does not hold for $x = (0,0,0,0)$ and $y = (1,1,-1,-1)$.

Next, we state two properties on the relationship between consecutive layers. For any $W \subseteq V$ and $k \in \mathbf{Z}$ with $\lambda^W \leq k \leq \mu^W$, define $\alpha_k^W = \inf\{f(x) \mid x(W) = k\}$. It is remarked that α_k^W may take the value $-\infty$. The next theorem shows the convexity of the sequence $\{\alpha_k^W\}$.

Theorem 3.4 *Let $W \subseteq V$. Then, $\alpha_{k-1}^W + \alpha_{k+1}^W \geq 2\alpha_k^W$ $(\lambda^W + 1 \leq \forall k \leq \mu^W - 1)$.*

Proof. Let $\{x_i\}_{i=1}^\infty$, $\{y_i\}_{i=1}^\infty$ be sequences of vectors in $\mathrm{dom}\, f$ such that $\lim_{i\to\infty} f(x_i) = \alpha_{k+1}^W$, $\lim_{i\to\infty} f(y_i) = \alpha_{k-1}^W$. For each $i = 1, 2, \cdots$, let $x_i', y_i' \in \mathrm{dom}\, f$ satisfy $x_i'(W) = k+1$, $y_i'(W) = k-1$, and $f(x_i') + f(y_i') \leq f(x_i) + f(y_i)$, and assume that x_i', y_i' have the minimum value of $||x_i' - y_i'||$ of all such vectors. Note that $\mathrm{supp}^+(x_i' - y_i') \cap W \neq \emptyset$ since $x_i'(W) > y_i'(W)$. Apply (M-EXC) to x_i', y_i', and some $u_i \in \mathrm{supp}^+(x_i' - y_i') \cap W$, we have

$$f(x_i') + f(y_i') \geq f(x_i' - \chi_{u_i} + \chi_{v_i}) + f(y_i' + \chi_{u_i} - \chi_{v_i})$$

for some $v_i \in \mathrm{supp}^-(x_i' - y_i')$. By the assumption for x_i', y_i', v_i must be in $V - W$, and therefore $(x_i' - \chi_{u_i} + \chi_{v_i})(W) = (y_i' + \chi_{u_i} - \chi_{v_i})(W) = k$. Hence,

$$\begin{aligned}
\alpha_{k+1}^W + \alpha_{k-1}^W &= \lim_{i\to\infty} \{f(x_i) + f(y_i)\} \\
&\geq \inf_i \{f(x_i' - \chi_{u_i} + \chi_{v_i}) + f(y_i' + \chi_{u_i} - \chi_{v_i})\} \geq 2\alpha_k^W.
\end{aligned}$$

∎

Corollary 3.5 *Let $W \subseteq V$. If $\alpha_j^W = -\infty$ for some j with $\lambda^W \leq j \leq \mu^W$, then $\alpha_k^W = -\infty$ for any k with $\min\{\lambda^W + 1, j\} \leq k \leq \min\{\mu^W - 1, j\}$.*

Proof. For any $k = j+1, j+2, \cdots, \mu^W - 1$, we can inductively show that $\alpha_k^W = -\infty$ by applying Theorem 3.4 since $\alpha_{k-1}^W = -\infty$ by the inductive assumption and $\alpha_{k+1}^W < \infty$. In the similar way we can also show $\alpha_k^W = -\infty$ for $k = j-1, j-2, \cdots, \lambda^W + 1$. ∎

For any $W \subseteq V$ and any $k \in \mathbf{Z}$ with $\lambda^W \leq k \leq \mu^W$, define $M_k^W = \{x \mid x(W) = k, f(x) = \alpha_k^W\}$ if α_k^W is finite.

Theorem 3.6 *Let $W \subseteq V$.*
(i) Let $k \in \mathbf{Z}$ be with $\lambda^W \leq k \leq \mu^W - 1$ and suppose both α_k^W and α_{k+1}^W are finite. Then, for any $x_k \in M_k^W$, there exists $x_{k+1} \in M_{k+1}^W$ with $||x_{k+1} - x_k|| = 2$.
(ii) Let $k \in \mathbf{Z}$ be with $\lambda^W + 1 \leq k \leq \mu^W$ and suppose both α_k^W and α_{k-1}^W are finite. Then, for any $x_k \in M_k^W$, there exists $x_{k-1} \in M_{k-1}^W$ with $||x_{k-1} - x_k|| = 2$.

Proof. We show (i) only, since the proof of (ii) is similar. Let $x_k \in M_k^W$. Suppose $y \in M_{k+1}^W$ minimizes the value $||y - x_k|| (\geq 2)$ of all vectors in M_{k+1}^W, and to the contrary assume $||y - x_k|| \geq 4$. Note that $\mathrm{supp}^+(y - x_k) \cap W \neq \emptyset$. For $u \in \mathrm{supp}^+(y - x_k) \cap W$, (M-EXC) yields

$$f(y) + f(x_k) \geq f(y - \chi_u + \chi_v) + f(x_k + \chi_u - \chi_v)$$

for some $v \in \mathrm{supp}^-(y-x_k)$. We have $x_k+\chi_u-\chi_v \in M^W_{k+1}$ if $v \in V-W$, and $y-\chi_u+\chi_v \in M_{k+1}$ if $v \in W$. In either case, it is a contradiction since $||(x_k+\chi_u-\chi_v)-x_k|| = 2$ and $||(y-\chi_u+\chi_v)-x_k|| = ||y-x_k||-2$. ∎

Using this property, we can find a minimizer in each layer by the next algorithm if $\mathrm{dom}\, f$ is bounded.

Algorithm AUGMENT

STEP 0: Find any $x_{\lambda^W} \in M_{\lambda^W}$. Set $k = \lambda^W$.

STEP 1: If $k = \mu^W$ then stop.

STEP 2: Find $u_k \in W$ and $v_k \in V-W$ with $f(x_k+\chi_{u_k}-\chi_{v_k}) = \min_{u\in W, v\in V-W}\{f(x_k+\chi_u-\chi_v)\}$.

STEP 3: Set $x_{k+1} = x_k+\chi_{u_k}-\chi_{v_k}$, $k = k+1$. Go to STEP 1.

The algorithm DOMAIN_REDUCTION in Section 2 can be used in STEP 0 by Theorem 3.2. The algorithm REDUCE, which iteratively reduces k, can be constructed similarly. These algorithms work well if we can find an vector $x_{\lambda^W} \in M^W_{\lambda^W}$ or $x_{\mu^W} \in M_{\mu^W}$ efficiently, in particular if $|\{x \in \mathrm{dom}\, f \mid x(W) = \lambda^W\}| = 1$ or $|\{x \in \mathrm{dom}\, f \mid x(W) = \mu^W\}| = 1$.

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