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<tr>
<td>引用</td>
<td>数理解析研究所講究録 (1997), 1021: 63-69</td>
</tr>
<tr>
<td>発行年月</td>
<td>1997-12</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/61683">http://hdl.handle.net/2433/61683</a></td>
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<td>タイプ</td>
<td>Departmental Bulletin Paper</td>
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<td>テキストバージョン</td>
<td>publisher</td>
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Crispness and Representation Theorem in Dedekind Categories

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1 Introduction

Since Zadeh's invention the concept of fuzzy sets has been extensively investigated in mathematics, science and engineering. The notion of fuzzy relations is also a basic one in processing fuzzy information in relational structures, see e.g. Pedrycz [15]. Goguen [5] generalized the concepts of fuzzy sets and relations taking values from partially ordered sets. Fuzzy relational equations were initiated and applied to medical models of diagnosis by Sanchez [17].

On the other hand, the theory of relations, namely relational calculus, has a long history, see [13, 18, 19] for more details. Almost all modern formalizations of relation algebras are affected by the work of Tarski [20]. Mac Lane [12] and Puppe [16] exposed a categorical basis for the calculus of additive relations. Freyd and Scedrov [2] developed and summarized categorical relational calculus, which they called allegories. Concerning applications to the relational theory of graphs and programs, Schmidt and Ströhlein [18] gave a simple proof of a representation theorem for Boolean relation algebras satisfying the Tarski rule and the point axiom. They also wrote an excellent text book [19] on relations and graphs with many useful examples from computer science. In relational calculus one calculates with relations in an element-free style, which makes relational calculus a very useful framework for the study of mathematics [8] and theoretical computer science [1, 7, 11] and also a useful tool for applications. Some element-free formalizations of fuzzy relations and proofs of representation theorems were provided in [3, 9, 10].

In this paper we consider Dedekind categories named by Olivier and Serrato [14]. One of the aim of this paper is to study notions of crispness and scalar relations in Dedekind categories. A notion of crispness was introduced in [10] under the assumption that Dedekind categories have unit objects which are an abstraction of singleton (or one-point) sets. To capture the notion of crispness without such assumption, we use a notion of scalar relations. The notion of scalar relations in homogeneous relation algebras was introduced in [4]. The other aim of this paper is to prove a representation theorem for Dedekind categories. Such a theorem for Dedekind categories with a unit object satisfying strict point axiom was also proved in [10]. This paper is organized as follows:

In section 2 we first state the definition of complete Dedekind categories [14] as a categorical structure formed by $L$-relations [5] with sup-inf composition. Also we define a preoder among objects of Dedekind categories which compares the lattice structures on objects in a sense. Section 3 studies notions of scalars and crispness for Dedekind categories. The scalars on an object form a distributive lattice, which would be seen as the underlying lattice structure. In section 4 we recall the definition of $L$-relations, due to Goguen [5], and illustrate a few

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relationships between crispness and lattice structures of scalars. In section 5 we show a representation theorem for connected Dedekind categories satisfying the strict point axiom without the assumption of existence of unit objects, and it is proved that the representation function is a bijection preserving all operations of Dedekind categories.

2 Dedekind Categories

In this section we recall the fundamentals on relation categories, which we will call Dedekind categories following Olivier and Serrato [14].

Throughout this paper, a morphism $\alpha$ from an object $X$ into an object $Y$ in a Dedekind category (which will be defined below) will be denoted by a half arrow $\alpha : X \rightarrow Y$, and the composite of a morphism $\alpha : X \rightarrow Y$ followed by a morphism $\beta : Y \rightarrow Z$ will be written as $\alpha \beta : X \rightarrow Z$.

Definition 2.1 A Dedekind category $\mathcal{D}$ is a category satisfying the following:

D1. [Complete Distributive Lattice] For all pairs of objects $X$ and $Y$ the hom-set $\mathcal{D}(X, Y)$ consisting of all morphisms of $X$ into $Y$ is a complete distributive lattice with the least morphism $0_{XY}$ and the greatest morphism $\nabla_{XY}$.

D2. [Involution] An involution $\dagger : \mathcal{D} \rightarrow \mathcal{D}$ is a monotone contravariant functor. That is, for all morphisms $\alpha, \alpha' : X \rightarrow Y, \beta : Y \rightarrow Z$,

(a) $(\alpha \beta)^{\dagger} = \beta^{\dagger} \alpha^{\dagger}$, (b) $\alpha^{\dagger \dagger} = \alpha$, (c) If $\alpha \subseteq \alpha'$, then $\alpha^{\dagger} \subseteq \alpha'^{\dagger}$.

D3. [Dedekind Formula] For all morphisms $\alpha : X \rightarrow Y, \beta : Y \rightarrow Z$ and $\gamma : X \rightarrow Z$ the Dedekind formula $\alpha \beta \cap \gamma \subseteq \alpha (\beta \cap \alpha^{\dagger} \gamma)$ holds.

D4. [Residues] For all morphisms $\beta : Y \rightarrow Z$ and $\gamma : X \rightarrow Z$ the residue (or division, weakest precondition) $\gamma \div \beta : X \rightarrow Y$ is a morphism such that $\alpha \beta \subseteq \gamma$ if and only if $\alpha \subseteq \gamma \div \beta$ for all morphisms $\alpha : X \rightarrow Y$.

Note that complete distributive lattices are equivalent to complete Brouwerian lattices or complete Heyting algebras.

Throughout this section, all discussions will assume a fixed complete Dedekind category $\mathcal{D}$. We denote the identity morphism on an object $X$ of $\mathcal{D}$ by $\text{id}_X$. The greatest morphism $\nabla_{XY}$ is called the universal morphism and the least morphism $0_{XY}$ the zero morphism. A morphism is nonzero if it is not equal to the zero morphism. An object $X$ is called empty if $\nabla_{XX} = 0_{XX}$, and nonempty if $\nabla_{XX} \neq 0_{XX}$.

Proposition 2.2 Let $\alpha, \alpha' : X \rightarrow Y$ and $\beta, \beta' : Y \rightarrow Z$ be morphisms in $\mathcal{D}$.

(a) $\nabla_{XY} \nabla_{YY} = \nabla_{XY} \nabla_{YY} = \nabla_{XY}$.

(b) If $\alpha \cup \alpha' = \nabla_{XY}, \alpha \cap \alpha' = 0_{XY}$ and $\nabla_{XX} \alpha = \alpha$, then $\nabla_{XX} \alpha' = \alpha'$.

(c) If $u \subseteq \text{id}_X$ and $v \subseteq \text{id}_X$, then $u^{\dagger} = uu = u$ and $uv = u \cap v$.

(d) If $u \subseteq \text{id}_X$ and $v \subseteq \text{id}_Y$, then $u \alpha = \alpha \cap u \nabla_{XY}$ and $\alpha v = \alpha \cap \nabla_{XY} v$.

The statement (a) in the last proposition indicates that if $\nabla_{XY} \neq 0_{XY}$, then both of $X$ and $Y$ are nonempty.

Proposition 2.3 Let $\alpha : X \rightarrow Y$ be a morphism such that $\nabla_{XX} \alpha = \alpha$. Then the following three conditions are equivalent: (a) $\text{id}_X \subseteq \alpha^{\dagger}$, (b) $\nabla_{XX} = \alpha^{\dagger}$, (c) $\nabla_{XX} = \alpha \nabla_{YX}$.
A binary relation $\prec$ among objects of $\mathcal{D}$ is defined as follows: For two objects $X$ and $Y$ a relation $X \prec Y$ holds if and only if $\nabla_{XX} = \nabla_{XY} \nabla_{YX}$. Then $\prec$ is a preorder, that is, reflexive and transitive. For $\nabla_{XX} = \nabla_{XX} \nabla_{XX}$ and $\nabla_{XX} = \nabla_{XY} \nabla_{YX}$ and $\nabla_{YY} = \nabla_{YZ} \nabla_{ZY}$, then $\nabla_{XX} = \nabla_{XY} \nabla_{YY} \nabla_{YX} = \nabla_{XY} \nabla_{YZ} \nabla_{ZY} \nabla_{YX} \subseteq \nabla_{XZ} \nabla_{ZX}$. Hence its symmetric closure $X \sim Y$, which means $X \prec Y$ and $Y \prec X$, is an equivalence relation.

**Proposition 2.4** Assume that $X \prec Y$. If $u \nabla_{XY} \subseteq v \nabla_{XY}$ for $u, v : X \rightarrow X$ such that $u \subseteq \text{id}_X$ and $u \subseteq \text{id}_X$, then $u \subseteq v$.

**Definition 2.5** A Dedekind category $\mathcal{D}$ is connected if all pairs of objects of $\mathcal{D}$ are equivalent, that is, if $X \sim Y$ for all objects $X$ and $Y$ of $\mathcal{D}$.

3 Scalars and Crispness

We now introduce the two notions of scalars and s-crisp relations to define a concept of points with a separation property that two different points does not meet.

**Definition 3.1** A scalar $k$ on $X$ is a morphism $k : X \rightarrow X$ of $\mathcal{D}$ such that $k \subseteq \text{id}_X$ and $k \nabla_{XX} = \nabla_{XX} k$.

A scalar $k$ on $X$ commutes with all morphisms $\alpha : X \rightarrow X$, that is, $k \alpha = \alpha k$, because

$$k \alpha = \alpha \cap k \nabla_{XX} = \alpha \cap \nabla_{XX} k = \alpha k.$$ 

It is trivial that the zero morphism $0_{XX} : X \rightarrow X$ and the identity morphism $\text{id}_X : X \rightarrow X$ are scalars on $X$. The set of all scalars on $X$ is denoted by $\mathcal{F}(X)$. It is clear that $\mathcal{F}(X)$ is a complete distributive lattice for all objects $X$.

**Lemma 3.2** For a morphism $\xi : X \rightarrow Y$ and an object $W$ define a morphism

$$\phi_{XYW}(\xi) = \nabla_{WX} \xi \nabla_{YW} \cap \text{id}_W : W \rightarrow W.$$ 

Then

(a) $\phi_{XYW}(\xi) \nabla_{WZ} = \nabla_{WX} \xi \nabla_{YZ}$ and $\nabla_{ZW} \phi_{XYW}(\xi) = \nabla_{ZX} \xi \nabla_{YW}$ for each object $Z$,

(b) $\phi_{XYW}(\xi)$ is a scalar on $W$,

(c) $\phi_{XXW} \phi_{XYX}(\xi) = \phi_{YYW} \phi_{YXY}(\xi) = \phi_{XYW}(\xi)$,

(d) If $\nabla_{XY} = \nabla_{XW} \nabla_{WY}$, then $\xi \subseteq \nabla_{XW} \phi_{XYW}(\xi) \nabla_{WY}$,

(e) If $\nabla_{XY} = \nabla_{XW} \nabla_{WY}$, an identity $\phi_{XYW}(\xi) = 0_{WW}$ is equivalent to $\xi = 0_{XY}$.

From the above Lemma 3.2(b) one have a function $\phi_{XYW} : \mathcal{D}(X, Y) \rightarrow \mathcal{F}(W)$. Note that if $W = X$ or $W = Y$, then $\nabla_{XY} = \nabla_{XW} \nabla_{WY}$.

**Proposition 3.3** (a) If $X \prec Y$, then $\phi_{YYX}(\phi_{XXY}(k)) = k$ for all scalars $k \in \mathcal{F}(X)$,

(b) If $X \sim Y$, then $\mathcal{F}(X)$ is isomorphic to $\mathcal{F}(Y)$ as lattices.

(c) $\phi_{ZZX}(k) \alpha = \alpha \phi_{ZZY}(k)$ for all scalars $k$ on $Z$ and all morphisms $\alpha : X \rightarrow Y$. 
(d) For every nonzero morphism $\xi : X \to Y$ in $D$ there is a nonzero scalar $k \in \mathcal{F}(X)$ such that $\nabla_{XX} \xi \nabla_{YY} = k \nabla_{XY}$.

**Definition 3.4** A morphism $\alpha : X \to Y$ is s-crisp if $k \alpha \subseteq \alpha$ implies $\tau \subseteq \alpha$ for all nonzero scalars $k : X \to X$ and all morphisms $\tau : X \to Y$.

It is trivial from the above definition that all universal morphism $\nabla_{XY}$ is s-crisp.

**Proposition 3.5** If the identity morphism $\text{id}_Y$ is s-crisp, then so are all total functions $f : X \to Y$.

**Proof.** Let $f : X \to Y$ be a total function. Assume that $k \tau \subseteq f$ for a nonzero scalar $k$ on $X$ and a morphism $\tau : X \to Y$. First note that $k \tau = \tau \phi_{XY}(k)$ by 3.3(c). Then we have

$$\phi_{XY}(k) \tau f = (\tau \phi_{XY}(k)) f = (k \tau) f \subseteq f \tau f \subseteq \text{id}_Y$$

and so $\tau f \subseteq \text{id}_Y$ from the assumption. Therefore $\tau f \subseteq \tau f f \subseteq f$, which completes the proof. $\square$

**Lemma 3.6** A morphism $\alpha : X \to Y$ is s-crisp if and only if a relatively pseudo-complement $\alpha' \Rightarrow \alpha$ is s-crisp for all morphisms $\alpha' : X \to Y$.

**Proof.** First assume that $\alpha : X \to Y$ is s-crisp and $k \tau \subseteq \alpha' \Rightarrow \alpha$ for a nonzero scalar $k$ and morphisms $\tau, \alpha' : X \to Y$. Then we have

$$k(\tau \cap \alpha') = k \tau \cap \alpha' \subseteq \alpha$$

and so $\tau \cap \alpha' \subseteq \alpha$, since $\alpha : X \to Y$ is s-crisp. Therefore $\tau \subseteq \alpha' \Rightarrow \alpha$. Conversely if $\alpha' \Rightarrow \alpha$ is s-crisp for all morphisms $\alpha' : X \to Y$, then $\alpha = \nabla_{XY} \Rightarrow \alpha$ is s-crisp. This completes the proof. $\square$

**Theorem 3.7** The following three statements are equivalent:

(a) If $k \not= 0_{XX}$ and $k \cap k' = 0_{XX}$ for scalars $k, k' \in \mathcal{F}(X)$, then $k' = 0_{XX}$,

(b) The zero morphism $0_{XY}$ is s-crisp for all objects $Y$, (that is, if $k \tau = 0_{XY}$ for a nonzero scalar $k$ on $X$ and a morphism $\tau : X \to Y$, then $\tau = 0_{XY}$),

(c) For every morphism $\alpha : X \to Y$ its pseudo-complement $-\alpha : X \to Y$ is s-crisp for all objects $Y$,

(d) Every complemented morphism $\alpha : X \to Y$ is s-crisp for all objects $Y$.

4 **$L$-Relations**

Let $L$ be a complete distributive lattice (or, a complete Heyting algebra) with the least element 0 and the greatest element 1. The supremum (the least upper bound) and the infimum (the greatest lower bound) of a family $\{k_\lambda\}$ of elements in $L$ will be denoted by $\vee_\lambda k_\lambda$ and $\wedge_\lambda k_\lambda$, respectively. For two elements $a, b \in L$ the relative pseudo-complement of $a$ relative to $b$ will be written as $a \Rightarrow b$. Now recall some fundamentals on $L$-relations [5].
Let $X$ and $Y$ be sets. An $L$-relation $R$ from $X$ into $Y$, written $R : X \rightarrow Y$, is a function $R : X \times Y \rightarrow L$. The set of all $L$-relations from $X$ into $Y$ will be denoted by $L - \text{Rel}(X)$. An $L$-relation $R$ is contained in an $L$-relation $S$, written $R \subseteq S$, if $R(x, y) \leq S(x, y)$ for all $(x, y) \in X \times Y$. The zero relation $O_{XY}$ and the universal relation $\nabla_{XY}$ are $L$-relations with $O_{XY}(x, y) = 0$ and $\nabla_{XY}(x, y) = 1$ for all $(x, y) \in X \times Y$, respectively. It is trivial that $\subseteq$ is a partial order, and $O_{XY} \subseteq R \subseteq \nabla_{XY}$ for all fuzzy relations $R$. For a family $\{R_{\lambda}\}_{\lambda}$ of fuzzy relations we define fuzzy relations $\cup_{\lambda} R_{\lambda}$ and $\cap_{\lambda} R_{\lambda}$ as follows:

$$(\cup_{\lambda} R_{\lambda})(x, y) = \vee_{\lambda} R_{\lambda}(x, y)$$

and

$$(\cap_{\lambda} R_{\lambda})(x, y) = \wedge_{\lambda} R_{\lambda}(x, y)$$

for all $x, y \in X$. It is obvious that $\cup_{\lambda} R_{\lambda}$ and $\cap_{\lambda} R_{\lambda}$ are the least upper bound and the greatest lower bound of a family $\{R_{\lambda}\}_{\lambda}$, respectively, with respect to the order $\subseteq$. The composite $RS = R; S : X \rightarrow Z$ of an $L$-relation $R : X \rightarrow Y$ followed by an $L$-relation $S : Y \rightarrow Z$ is defined by

$$(RS)(x, z) = \vee_{y \in Y}[R(x, y) \wedge S(y, z)]$$

for all $(x, z) \in X \times Z$. This composition of $L$-relations is called as sup-inf composition. The associativity $(RS)T = R(ST)$ holds for all $L$-relations $R$, $S$ and $T$. The identity relation $\text{id}_X$ of a set $X$ is an $L$-relation such that $\text{id}_X(x, x') = 1$ if $x = x'$ and $\text{id}_X(x, x') = 0$ otherwise. The unitary law $\text{id}_X R = R \text{id}_Y = R$ holds for all $R : X \rightarrow Y$. The inverse (or transpose) $R^t : Y \rightarrow X$ of an $L$-relation $R : X \rightarrow Y$ is defined by

$$R^t(y, x) = R(x, y)$$

for all $(y, x) \in Y \times X$. For $L$-relations $S : Y \rightarrow Z$ and $T : X \rightarrow Z$ the residue $T \div S : X \rightarrow Y$ is defined by

$$(T \div S)(x, y) = \wedge_{z \in Z}[S(y, z) \Rightarrow T(x, z)]$$

for all $(x, y) \in X \times Y$. The readers can easily see that $L$-relations and their operations defined above satisfy almost all axioms of Dedekind categories, except for D3(Dedekind formula) and D4(Residues), which will be proved in the following:

**Proposition 4.1** Let $R : X \rightarrow Y, S : Y \rightarrow Z$ and $T : X \rightarrow Z$ be $L$-relations. Then

(a) $RS \cap T \subseteq R(S \cap R^tT)$ (Dedekind formula),

(b) $RS \subseteq T$ if and only if $R \subseteq T \div S$.

In relational calculus ([2, 8, 19]) a function $R$ on $X$ is a relation satisfying the univalency $R^tR \subseteq I$ and the totality $I \subseteq RR^t$.

An $L$-relation $k : X \rightarrow X$ is a scalar on $X$ if and only if

$$\forall x, x' \in X : k(x, x) = k(x', x') \text{ and } x \neq x' \Rightarrow k(x, x') = 0.$$

An $L$-relation $R : X \rightarrow Y$ is 0-1 crisp ([5]) if $R(x, y) = 0$ or $R(x, y) = 1$ for all $(x, y) \in X \times Y$. Of course $O_{XY}$, $\nabla_{XY}$ and $\text{id}_X$ are 0-1 crisp. For a 0-1 crisp $L$-relation $R : X \rightarrow Y$ define an $L$-relation $\bar{R} : X \rightarrow Y$ by $\bar{R}(x, y) = 0$ if $R(x, y) = 1$ and $\bar{R}(x, y) = 1$ otherwise. Then $R \cup \bar{R} = \nabla_{XY}$ and $R \cap \bar{R} = O_{XY}$. This fact means that all 0-1 crisp $L$-relations are complemented.

**Proposition 4.2** All s-crisp $L$-relations are 0-1 crisp.
Proposition 4.3 For $L$-relations the following statements are equivalent:

C0. $\forall a, b \in L : a \land b = 0 \Rightarrow a = 0$ or $b = 0$.

K0. All $0$-$1$ crisp $L$-relations are $s$-crisp.

Proposition 4.4 For $L$-relations the following statements are equivalent:

C1. $\forall a, b \in L : a \land b = 0$ and $a \lor b = 1 \Rightarrow a = 0$ or $b = 0$.

K1. All complemented $L$-relations are $0$-$1$ crisp.

K2. All totally functional $L$-relations are $0$-$1$ crisp.

5 Representation Theorem

Definition 5.1 Let $D$ be a complete Dedekind category. A point $x$ of $X$ is an $s$-crisp morphism $x : X \to X$ such that $\nabla_{XX}x = x$, $x^t x \subseteq \text{id}_X$ and $\text{id}_X \subseteq xx^t$.

Proposition 5.2 Let $x$ and $x'$ be points of $X$. Then

(a) If $\nabla_{XX} \rho = \rho$ and $\rho \subseteq x$ for a morphism $\rho : X \to X$, then $\rho = kx$ for a unique scalar $k$ on $X$.

(b) If $x \neq x'$, then $x \cap x' = 0_{XX}$ and $xx'^t = 0_{XX}$.

Set $L = \mathcal{F}(W)$ for a fixed object $W$. Then $L$ is a complete distributive lattice. A function $\chi(\alpha) : \chi(X) \times \chi(Y) \to L$ assigning $\chi(\alpha)(x, y) = \phi_{XYW}(x\alpha y^t) \in L$ to a pair $(x, y)$ of points $x$ of $X$ and $y$ of $Y$, gives an $L$-relation of $\chi(X)$ into $\chi(Y)$. Thus we have a function $\chi : \mathcal{D}(X, Y) \to L-\text{Rel}(\chi(X), \chi(Y))$.

Proposition 5.3 If $D$ is a connected Dedekind category, then the function $\chi : \mathcal{D}(X, Y) \to L-\text{Rel}(\chi(X), \chi(Y))$ satisfies the following properties:

(a) $\chi(O_{xy}) = O_{\chi(X)\chi(Y)}$, $\chi(\nabla_{xy}) = \nabla_{\chi(X)\chi(Y)}$ and $\chi(\text{id}_X) = \text{id}_\chi(X)$;

(b) $\chi(\alpha \cup \alpha') = \chi(\alpha) \cup \chi(\alpha')$ and $\chi(\alpha \cap \alpha') = \chi(\alpha) \cap \chi(\alpha')$;

(c) $\chi(\alpha^t) = (\chi(\alpha))^t$;

(d) $\chi(\alpha) \chi(\beta) = \chi(\alpha (\cup_{y \in \chi(Y)} y^t \beta))$.

(e) The function $\chi : \mathcal{D}(X, Y) \to L - \text{Rel}(\chi(X), \chi(Y))$ is surjective.

Definition 5.4 A complete Dedekind category $\mathcal{D}$ satisfies the strict point axiom if and only if

$$\mathbb{L}_{x \in \chi(X)} x = \nabla_{XX}$$

for all objects $X$, where $\chi(X)$ denotes the set of all points of $X$.

Proposition 5.5 A complete Dedekind category $\mathcal{D}$ satisfies the strict point axiom if and only if the function $\chi : \mathcal{D}(X, X) \to L - \text{Rel}(\chi(X), \chi(X))$ is injective for all objects $X$.

Proposition 5.6 If a complete Dedekind category $\mathcal{D}$ satisfies the strict point axiom, then for all objects $X$ the identity morphism $\text{id}_X$ is complemented. Moreover, if the condition C1 is in addition valid in $\mathcal{D}$, then $\text{id}_X$ is $s$-crisp for all objects $X$.

Theorem 5.7 (Representation Theorem) Assume that $\mathcal{D}$ satisfies the strict point axiom. Then every morphism $\alpha : X \to Y$ has a unique representation

$$\alpha = \mathbb{L}_{x \in \chi(X)} \mathbb{L}_{y \in \chi(Y)} \chi(X)(\alpha)(x, y) z^t \nabla_{XY} y.$$
References