Semigroup semantics for orthomodular logic

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Abstract

Quantum logic is usually considered as a logic which is based on orthomodular lattices. Here we introduce a different type of semantics, in which we use particular semigroups, and show that these two ways of interpretation of formulas are equivalent.

0 Basic notions

First we will give here some basic notions. The language of our logics consists of:

(i) a countable collection \( \{ p_i \mid i < \omega \} \) of propositional variables,
(ii) the connectives \( \neg \) and \( \wedge \) of negation and conjunction,
(iii) parentheses (and).

The set \( \Phi \) of formulas is defined in the usual way. That is, \( \Phi \) is the minimum set which satisfies the following three conditions:

(i) for every \( i < \omega \), \( p_i \in \Phi \),
(ii) if \( \alpha \in \Phi \), then \( (\neg\alpha) \in \Phi \),
(iii) if \( \alpha, \beta \in \Phi \), then \( (\alpha \wedge \beta) \in \Phi \).

The letters \( \alpha, \beta \), etc. are used as metavariables ranging over \( \Phi \). Parentheses may be omitted by the convention that \( \neg \) binds strongly than \( \wedge \). The disjunction \( \alpha \vee \beta \) of \( \alpha \) and \( \beta \) can be introduced as the abbreviation of \( \neg(\neg\alpha \wedge \neg\beta) \).

Definition 0.1 (Orthomodular lattice) An orthomodular lattice \( A \) is a structure \( \langle A, \leq, \cap, \cup, \perp, 1, 0 \rangle \), which satisfies the following conditions:

(i) \( \langle A, \leq, \cap, \cup, 1, 0 \rangle \) is a lattice with 1(maximum) and 0(minimum). We denote, for any \( x, y \in A \), \( x \cap y := \inf \{x, y\}, \ x \cup y := \sup \{x, y\} \).
(ii) The unary operation \( \perp \) (orthocomplement) satisfies the following conditions, (a), (b) and (c): for any \( x, y \in A \),

(a) \( x \cap x^\perp = 0 \)
(b) \( x^{\perp\perp} = x \)
(c) \( x \leq y \implies y^\perp \leq x^\perp \)
(d) \( x \leq y \) implies \( y = x \lor (x^\perp \land y) \)

It is easy to see that \( x \lor y = (x^\perp \land y^\perp)^\perp \) holds in any orthomodular lattice.
Definition 0.2 (Valuation) A valuation is a function \( v \), which associates with any formula \( \alpha \in \Phi \) an element \( v(\alpha) \) in an orthomodular lattice \( A \), and satisfies the following conditions:

for any formula \( \alpha, \beta \),

(i) \( v(\neg \alpha) = (v(\alpha))^\perp \)

(ii) \( v(\alpha \land \beta) = v(\alpha) \cap v(\beta) \)

We call this \( v \) an orthomodular valuation.

It is easy to see that for any valuation \( v \) and for any formula \( \alpha \), the value \( v(\alpha) \) is uniquely determined by the values \( v(p_i) \) for propositional variables \( p_i \) appearing in \( \alpha \).

Definition 0.3 (Orthomodular logic) The orthomodular logic \( L \) is the set of pairs of formulas \( (\alpha, \beta) \) satisfying the following conditions: for any orthomodular lattice \( A \) and for any orthomodular valuation \( v \) from \( \Phi \) to \( A \), \( v(\alpha) \leq v(\beta) \). We denote \( \alpha \vdash_{L} \beta \) in place of \( (\alpha, \beta) \in L \).

R.I. Goldblatt proposed his "quantum model" for orthomodular logic in 1974[1].

Definition 0.4 (Quantum frame and quantum model) \( \mathcal{F} = (X, \bot, \xi) \) is a quantum frame if it satisfies the following conditions (i), (ii) and (iii).

(i) \( X \) is a nonempty set.

(ii) \( \bot \) is an irreflexive and symmetric binary relation. (orthogonality relation)

- For \( P \subseteq X \), \( x \bot P \) means that \( x \perp y \) for all \( y \in P \).
- \( P (\subseteq X) \) is \( \bot \)-closed iff the following condition holds:

\[
\forall x \in X(x \not\in P), \exists y \in X [y \perp P \text{ and not}(y \perp x)]
\]

- \( P (\subseteq X) \) is \( \bot \)-closed in \( Q (Q \subseteq X) \) iff the following condition holds:

\[
\forall x \in Q(x \not\in P), \exists y \in Q [y \perp P \text{ and not}(y \perp x)]
\]

(iii) \( \xi \) is a nonempty collection of \( \bot \)-closed subsets of \( X \), such that

(a) \( \xi \) is closed under set-inclusion and the following operation \( \dagger \).

\[
P(=\{X\} \cup_{X} \perp P) \]

(b) For any \( P, Q \) in \( \xi \), if \( P \subseteq Q \) then \( P \) is \( \bot \)-closed in \( Q \).

\( Q = (X, \bot, \xi, V) \) is a quantum model if it satisfies the following:

(i) \( \mathcal{F} = (X, \bot, \xi) \) is a quantum frame.

(ii) \( V \) is a function assigning to each propositional variables \( p \), a member \( V(p) \) of \( \xi \).
The notion of truth in quantum models is defined inductively as follows: the symbol \( \mathcal{Q} \models_x \alpha \) is read as “formula \( \alpha \) is true at \( x \) in \( Q \).

(i) \( \mathcal{Q} \models_x p_i \) iff \( p_i \in V(p_i) \),
(ii) \( \mathcal{Q} \models_x \alpha \land \beta \) iff \( \mathcal{Q} \models_x \alpha \) and \( \mathcal{Q} \models_x \beta \),
(iii) \( \mathcal{Q} \models_x \neg \alpha \) iff for any \( y \in X \), \( \mathcal{Q} \models_y \alpha \Rightarrow x \perp y \).

\[ \bullet \] \( \alpha \) implies \( \beta \) in a model \( \mathcal{Q} \) iff for all \( x \) in the model \( \mathcal{Q} \), either \( \mathcal{Q} \models_x \alpha \) does not hold, or \( \mathcal{Q} \models_x \beta \) holds.

Using his quantum models, Goldblatt showed the following completeness theorem.

Theorem 0.5 (Completeness Theorem) For given formulas \( \alpha \) and \( \beta \), the statements (P) and (Q) are mutually equivalent, that is

(P): for any orthomodular lattice \( A \) and any valuation \( v : \Phi \to A \), \( v(\alpha) \leq v(\beta) \) holds.

(Q): for any quantum model \( Q \), \( Q \models \alpha \Rightarrow \beta \) holds.

In study of orthomodular lattice, D.J.Foulis [2] found in 1960 the following representation theorem for orthomodular lattices with a particular kind of semigroups.

Theorem 0.6 (Foulis's representation theorem) Let \( A \) be an orthomodular lattice. Then \( \mathcal{G}(A) = (G(A), \cdot, \ast) \) is a Rickart * semigroup and \( A \) is isomorphic to \( P_c(G(A)) \). \( \square \)

We will give another type of models for orthomodular logic using this representation theorem.

1 Rickart * semigroups

Now we introduce a special type of semigroups called Rickart * semigroups and lead some properties of them.

Definition 1.1 (Rickart * semigroups) A Rickart * semigroup is a structure \( \mathcal{G} = (G, \cdot, \ast) \) which satisfies the following conditions (i), (ii), (iii) and (iv).

(i) \( \langle G, \cdot \rangle \) is a semigroup, that is,
   (a) \( \cdot \) is a binary operation on \( G \).
   (b) For any \( x, y, z \in G \), \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \).

(ii) There exists the unique element 0 (zero element) in \( G \) such that \( 0 \cdot x = x \cdot 0 = 0 \) holds for any \( x \in G \).

(iii) \( \ast \) is a unary operation on \( G \), which satisfies the following:
    For any \( x, y \in G \),
    (a): \( (x^\ast)^\ast = x \).
    (b): \( (x \cdot y)^\ast = y^\ast \cdot x^\ast \).
Before introducing the condition (iv), it is necessary to introduce some other notions.

- An element $e \in G$ is called a projection iff it satisfies $e^* = e \cdot e = e$. We denote the set of all projections in $G$ by $P(G)$.
- For an element $x \in G$, the set $\{ x \}^{(r)} := \{ y \in G \mid x \cdot y = 0 \}$ is called the right annihilator for $x$.

By using these two notions, we formulate the condition (iv) as follows:

(iv) For any $x \in G$, there exists a projection $e$ such that the right annihilator for $x$ can be expressed as: $\{ x \}^{(r)} = e \cdot G = \{ e \cdot y \mid y \in G \}$. We call this $e$ a right annihilating projection for $x$.

Lemma 1.2 (Properties of $P(G)$) Let $\mathcal{G} = (G, \cdot, *)$ be a Rickart * semigroup.

(i) For any $x \in G$, the right annihilating projection for $x$ is uniquely determined. Hereafter, this will be written as $x^r$.
(ii) There is the unit element in $G$, that is, an element 1 satisfying that for any $x \in G$, $x \cdot 1 = 1 \cdot x = x$.
(iii) Both 0 and 1 are projections.
(iv) For any $e, f \in P(G)$, the following three conditions are equivalent.
   (a) $e \cdot f = e$.
   (b) $f \cdot e = e$.
   (c) $e \cdot G \subseteq f \cdot G$.

Proof:

(i) Using the properties of the operation $*$.
(ii) We can show that $0^r$ is the unit element 1.
(iii) By operating $*$ to both sides of the equation $0 = 0 \cdot 0^*$, we get that $0^* = 0$. Similarly we can show that $1^* = 1$.
(iv) Not so hard.

The above Lemma 1.2 (iv) assures us the possibility of introducing a partially order on $P(G)$.

Definition 1.3 (Order on $P(G)$) Let $\mathcal{G} = (G, \cdot, *)$ be a Rickart * semigroup. Define a partial order $\leq$ on $P(G)$ as follows: for $e, f \in P(G)$, $e \leq f$ iff $e \cdot f = e$. 

\[ \square \]
It is obvious that 1 is the maximum and that 0 is the minimum with respect to this order. Hence \( P(G) \) can be regarded as a bounded partial ordered set.

In the proof of Lemma 1.2, we have defined the unary operation \( \cdot \) from \( G \) to \( P(G) \). Here we will see some of the basic properties of the operation \( \cdot \) in detail, which will be used in the later discussion.

**Lemma 1.4 (Properties of the operation \( \cdot \))** Let \( G = \langle G, \cdot, * \rangle \) be a Rickart * semigroup. For any \( x, y \in G \) and for any \( e, f \in P(G) \), the following statements can be verified.

(i) \( 0^r = 1 \), and \( 1^r = 0 \).
(ii) \( x \cdot x^r = 0 \), and \( x^r \cdot x^r = 0 \). (vi) \( x = x \cdot x^r \), and \( e \leq e^r \).
(iii) If \( x \cdot e = 0 \), then \( e \leq x^r \). (vii) \( x^r = x^{rr} \).
(iv) \( x^r \leq (y \cdot x)^r \). (viii) If \( e \cdot x = x \cdot e \), then \( e^r \cdot x = x \cdot e^r \).

**Proof**: Here we prove only (vi) and (viii). Rest is not so hard.

(vi) By (ii), \( x^r \in \{x^r\}^{(i)} = x^{rr} \cdot G \). Then there exists some \( s \in G \), such that \( x^r = x^{rr} \cdot s \). By operating * to this equation, we have that \( x = x^{**} = s^* \cdot x^{*r} = s^* \cdot x^r \). Further operating \( x^{rr} \) from the right to the equation \( x = s^* \cdot x^{*r} \), we can derive that \( x \cdot x^{rr} = (s^* \cdot x^{rr}) \cdot x^{rr} = s^* \cdot x^{rr} = x \). In particular, when \( x \) is equal to a projection \( e \), we have that \( e \cdot e^{rr} = e \), that is, \( e \leq e^{rr} \).

(viii) Suppose that \( e \cdot x = x \cdot e \). Then we have \( e \cdot x \cdot e^r = x \cdot e \cdot e^r = 0 \), since \( e \cdot e^r = 0 \). So \( x \cdot e^r \in \{e\}^{(i)} = e^{rr} \cdot G \), and there exists some \( s \in G \) satisfying that \( x \cdot e^r = e^r \cdot s \). By multiplying \( e^r \) from the left to both sides of this equation, we have that \( e^r \cdot x \cdot e^r = e^r \cdot e^r \cdot s = e^r \cdot s = x \cdot e^r \)  \( \cdots (1) \)

On the other hand, by operating * to the supposition \( e \cdot x = x \cdot e \), so we have that \( x^* \cdot e = e \cdot x^* \). Then \( e \cdot x^* \cdot e^r = x^* \cdot e \cdot e^r = 0 \), which means that \( x^* \cdot e^r \in \{e\}^{(i)} = e^{rr} \). So there exists some \( t \in G \) such that \( x^* \cdot e^r = e^r \cdot t \). By multiplying \( e^r \) from the left to both sides of this equation, we have that \( e^r \cdot x^* \cdot e^r = e^r \cdot e^r \cdot t = e^r \cdot t = x^* \cdot e^r \).

Further operating * again, we get that \( e^r \cdot x \cdot e^r = e^r \cdot x \)  \( \cdots (2) \).

From (1) and (2), we can conclude that \( x \cdot e^r = e^r \cdot x \).

\( \square \)

Now we will consider a particular class of projections, called *closed projections*.

**Definition 1.5 (Closed projection)** A projection \( f \in P(G) \) is called *closed* iff there exists an element \( x \in G \) such that \( f \) is the right annihilating projection for \( x \). This means that a closed projection \( f \) can be written as \( f = x^r \) for some element \( x \in G \). We denote the set of all closed projections in \( G \) by \( P_c(G) \).

In other words, the set \( P_c(G) \) is the range of the function \( \cdot \) from \( G \) to \( P(G) \). We give here a necessary and sufficient condition on a projection to be closed.

**Proposition 1.6** For any \( e \in P(G) \), \( e \in P_c(G) \) if and only if \( e^{rr} = e \). \( \square \)
We will show that in $P_c(G)$ we can always find the supremum and the infimum of any two elements of it and hence this partially ordered set forms a lattice. Moreover we can show that $P_c(G)$ is an orthomodular lattice.

**Lemma 1.7 (Existence of meet in $P_c(G)$)**

(i) For any closed projections $e$ and $f$ such that $e \cdot f = f \cdot e$, $e \cdot f \in P_c(G)$ holds, and there exists the infimum $(e \cap f)$ of $e, f$, which satisfies the equation $e \cap f = e \cdot f$.

(ii) In general, for any closed projections $e$ and $f$, there exists the infimum $(e \cap f)$ of $e, f$ and the equation $e \cap f = e \cdot (f^r \cdot e)^r = (f^r \cdot e)^r \cdot e = e \cap (f^r \cdot e)^r$ holds.

**Proof**:

(i) Suppose that $e \cdot f = f \cdot e$. We show that $e \cdot f \in P_c(G)$. Since $e, f \in P(G)$ and $e \cdot f = f \cdot e$, we can derive:

$$(e \cdot f)^* = f^* \cdot e^* = f \cdot e = e \cdot f, \quad \text{and} \quad (e \cdot f) \cdot (e \cdot f) = e \cdot e \cdot f \cdot f = e \cdot f.$$  

Thus, $e, f \in P(G)$. To prove that $e \cdot f \in P_c(G)$, by Proposition 1.5, it is enough to show that $(e \cdot f)^r = e \cdot f$. Then we have only to show that $(e \cdot f)^r \leq e \cdot f$ as the converse inequality holds always by Lemma 1.4 (vi). Considering the Lemma 1.4 (iv), we have that $e^r \leq (e \cdot f)^r$. Then by the Lemma 1.4 (v), we can derive that $(e \cdot f)^r \leq e^r = e$, which means $e \cdot (e \cdot f)^r = (e \cdot f)^r$. Similarly we can derive that $f \cdot (e \cdot f)^r = (e \cdot f)^r$. Therefore $e \cdot f \cdot (e \cdot f)^r = e \cdot (e \cdot f)^r = (e \cdot f)^r$. Thus $(e \cdot f)^r \leq e \cdot f$.

It is easy to see that $e \cdot f$ is the infimum of $e$ and $f$.

(ii) We put $u := f^r \cdot e$. By Lemma 1.4 (iv), we have that $e^r \leq (f^r \cdot e)^r = u^r$. This means that $e^r \cdot u^r = e^r = u^r \cdot e^r$. By applying Lemma 1.4 (viii), we have that $e \cdot u^r = u^r \cdot e$. Then by (i) of the present lemma, we can conclude that $e \cdot u^r \in P_c(G)$, and that $e \cap u^r = e \cdot u^r$. So it remains to show that $e \cap f = e \cdot u^r$.

(a) Clearly, $e \cdot (e \cdot u^r) = e \cdot u^r$. So we have $e \cdot u^r \leq e$. On the other hand, $f^r \cdot e \cdot u^r = f^r \cdot e \cdot (f^r \cdot e)^r = 0$. So from Lemma 1.4 (iii), we derive that $e \cdot u^r \leq f^r = f$. Thus $e \cdot u^r$ is a lower bound of $\{e, f\}$.

(b) Take any $g \in P_c(G)$ such that $g \cdot e = g$ and $g \cdot f = g$. Then because $f \cdot f^r = 0$, we have that $g \cdot f \cdot f^r \cdot e = 0$. By our assumption on $g$, $g \cdot f \cdot f^r \cdot e = 0$, which means that $g \cdot u = 0$. By Lemma 1.4 (iii), we can derive that $u \leq g^r$. So by Lemma 1.4 (v), $g = g^{rr} \leq u^r$. This is equivalent to $g \cdot u^r = g$. Again using the assumption on $g$, $g \cdot e \cdot u^r = g$. So we have derived that $g \leq e \cdot u^r$.

Thus we have shown that $e \cap f = e \cdot u^r$. 

$\square$
Therefore we have the following Proposition.

**Proposition 1.8** For any $e, f \in P_c(G)$, the following equation holds:
\[ e \cdot G \cap f \cdot G = (e \cap f) \cdot G. \]

Next we will see that $P_c(G)$ is an orthomodular lattice.

**Theorem 1.9** $P_c(G)$ forms an orthomodular lattice, where the orthocomplement is the operation $\cdot$.

**Proof**: We can easily check the conditions in Definition 0.1.

Next, in Section 2, we will introduce a semantics for orthomodular logic by using Rickart * semigroups, and prove the soundness.

### 2 Semigroup semantics and soundness theorem

**Definition 2.1** (Orthomodular model) $\mathcal{M} = \langle G, u \rangle$ is a orthomodular model (OM model for short) iff $G = \langle G, \cdot, * \rangle$ is a Rickart * semigroup and $u$ is a function assigning to each propositional variable $p_i$ an element $u(p_i)$ of $P_c(G)$.

The notion of truth in OM models is defined inductively as follows: the symbol $'(\mathcal{M}, x) \models \alpha'$ is read as "a formula $\alpha$ is true at $x$ in $\mathcal{M}"$.

1. $\langle \mathcal{M}, x \rangle \models p_i$ iff $x \in u(p_i) \cdot G$.
2. $\langle \mathcal{M}, x \rangle \models \alpha \land \beta$ iff $\langle \mathcal{M}, x \rangle \models \alpha$ and $\langle \mathcal{M}, x \rangle \models \beta$.
3. $\langle \mathcal{M}, x \rangle \models \neg \alpha$ iff $\forall y \in G, [ (\mathcal{M}, y) \models \alpha \text{ only if } y^* \cdot x = 0 ]$.

For each formula $\alpha$, define $\|\alpha\|^{\mathcal{M}} := \{ x \in G \mid (\mathcal{M}, x) \models \alpha \}$. Then we can restate the above conditions in the following way:

1. $\|p_i\|^{\mathcal{M}} = u(p_i) \cdot G$.
2. $\|\alpha \land \beta\|^{\mathcal{M}} = \|\alpha\|^{\mathcal{M}} \cap \|\beta\|^{\mathcal{M}}$.
3. $\|\neg \alpha\|^{\mathcal{M}} = \{ x \in G \mid \forall y \in \|\alpha\|^{\mathcal{M}} \ (y^* \cdot x = 0) \}$.

**Definition 2.2** Let $\alpha$ and $\beta$ be formulas.

1. $\alpha$ implies $\beta$ at $x$ in an OM model $\mathcal{M}$ $(\mathcal{M}, x) : \alpha \models \beta$) iff either $(\mathcal{M}, x) \models \alpha$ does not hold or $(\mathcal{M}, x) \models \beta$ holds.
2. $\alpha$ implies $\beta$ in an OM model $\mathcal{M}$ $(\mathcal{M} : \alpha \models \beta)$ iff for all $x$ in the model $\mathcal{M}$, $(\mathcal{M}, x) : \alpha \models \beta$ holds.

It is easy to see that $\mathcal{M} : \alpha \models \beta$ is equivalent to $\|\alpha\|^{\mathcal{M}} \subseteq \|\beta\|^{\mathcal{M}}$. 


Lemma 2.3 Let $\mathcal{M} = (\mathcal{G}, u)$ be an orthomodular model and $e$ such an orthomodular valuation from $\Phi$ to $P_2(G)$ that $e(p_i) = u(p_i)$ holds for all propositional variables. Then for any formula $\alpha$, $\|\alpha\|^{\mathcal{M}} = e(\alpha) \cdot G$ holds.

Proof: Induction on the construction of the formula $\alpha$. $\square$

Now we can prove the soundness theorem.

Theorem 2.4 (Soundness theorem) For given formulas $\alpha$ and $\beta$, let $(S)$ and $(T)$ be the statements as follows:

(S): for any orthomodular lattice $A$ and any orthomodular valuation $v : \Phi \rightarrow A$,
$v(\alpha) \leq v(\beta)$.

(T): for any orthomodular model $\mathcal{M}$, $\mathcal{M} : \alpha \models \beta$.

Then $(S)$ implies $(T)$. $\square$

3 Monotone, residuated maps on an ordered set

Next, we will prove the Completeness Theorem. To show the direction $((S) \iff (T))$, we need to know how to build up an orthomodular model from a given orthomodular lattice. To do this, we need some preparations.

Definition 3.1 (Residuated, monotone maps on an ordered set) Let $\langle A, \leq \rangle$ be an ordered set.

(i) A map $\varphi$ from $A$ to $A$ is called monotone iff it satisfies the following condition: for any $x, y \in A$, if $x \leq y$, then $\varphi(x) \leq \varphi(y)$.

We denote the set of all monotone maps from $A$ to $A$ by $\mathcal{G}(A)$.

(ii) A map $\varphi \in \mathcal{G}(A)$ is called residuated iff there exists a map $\varphi^\dag \in \mathcal{G}(A)$ such that for any $x \in A$, $\varphi^\dag(\varphi(x)) \geq x$ and $\varphi(\varphi^\dag(x)) \leq x$.

We call this map $\varphi^\dag$ a residual map for $\varphi$, and denote the set of all residuated, monotone maps on $A$ by $G(A)$.

Lemma 3.2 (Properties of residual maps) Let $\langle A, \leq \rangle$ be an ordered set. Then the following holds.

(i) For any $\varphi \in G(A)$, the residual map for $\varphi$ is uniquely determined.

(ii) For any $\varphi, \psi \in G(A)$, $(\varphi \cdot \psi)^\dag = \psi^\dag \cdot \varphi^\dag$ holds, where $\cdot$ means the composition operator for maps. Therefore $G(A)$ is closed under this operation $\cdot$.

Proof: Using the monotonicity and the inequalities which hold for $\varphi \in G(A)$ and its residual map $\varphi^\dag$. $\square$

It is guaranteed by (i) of Lemma 3.2 that we can write the residual map for $\varphi$ as $\varphi^\dag$. And (ii) of Lemma 3.2 means that $G(A)$ is a semigroup with respect to the operation $\cdot$. 

Lemma 3.3 Let \( \langle A, \leq, 0, 1 \rangle \) be an ordered set with the minimum element 0 and the maximum element 1 and let \( \theta \) be a map defined by the condition: for all \( x \in A \), \( \theta(x) = 0 \). Then \( \theta \) is the zero element in the semigroup \( G(A) \).

Lemma 3.4 Let \( \mathcal{A} = \langle A, \leq, \sqcap, \sqcup, \perp, 1, 0 \rangle \) be an ortholattice. Let \(*\) be defined by the following: for any \( \varphi \in G(A) \), \( \varphi^*(x) := (\varphi^!(x^\perp))^\perp \) for any \( x \in A \). Then \( \varphi^* \in G(A) \).

Moreover the following conditions hold for every \( \varphi, \psi \in G(A) \).

(a) \( \varphi^{**} = \varphi \).

(b) \((\varphi \cdot \psi)^* = \psi^* \cdot \varphi^* \).

Proof: We put \( \psi(x) := (\varphi(x^\perp))^\perp \) for any \( x \in A \) and show that \( \psi = \varphi^! \).

(i) First we will show that \( \psi \) is monotone. Suppose that \( x \leq y \) for \( x, y \in A \). Then by the properties of the operation \( \perp \), we have \( x^\perp \geq y^\perp \). Since \( \varphi \) is monotone, we have \( \varphi(x^\perp) \geq \varphi(y^\perp) \). Again by the properties of \( \perp \), we have \( (\varphi(x^\perp))^\perp \leq (\varphi(y^\perp))^\perp \), which means \( \psi(x) \leq \psi(y) \). Therefore \( \psi \) is monotone.

(ii) Next we will show that \( \psi \) is the residual map for \( \varphi \). By the properties of the operation \( \perp \) and the properties of \( \varphi^! \), we can derive: \( \psi \cdot \varphi^*(x) = \psi \cdot (\varphi^!(x^\perp))^\perp = [\varphi((\varphi^!(x^\perp))^\perp)]^\perp = [\varphi((\varphi^!(x^\perp))^\perp)]^\perp \geq x^\perp = x \). So we have \( \psi \cdot \varphi^*(x) \geq x \). Similarly we can derive: \( \varphi^* \cdot \psi(x) = \varphi^* \cdot (\varphi^!(x^\perp))^\perp = [\varphi^!(\varphi^!(x^\perp))^\perp] \leq (\varphi^!(\varphi^!(x^\perp))^\perp)^\perp \leq x^\perp = x \). So we have \( \varphi^* \cdot \psi(x) \leq x \).

Hence we can conclude that \( \psi = \varphi^! \) since the residual map of \( \varphi^* \) is unique. By (i) and (ii) in the above, we have that \( \varphi^* \in G(A) \). Thus \( \ast \) is a unary operator on \( G(A) \). Now we will check the conditions (a) and (b). By the properties of the operation \( \perp \), and the definition of \( \varphi^* \), we calculate as follows: for any \( \varphi, \psi \), and for any \( x \in A \),

(a): \( \varphi^{**}(x) = [(\varphi^!(x^\perp))^\perp]^\perp = \varphi(x) \).

(b): \( \psi^* \cdot \varphi^*(x) = \psi^* (\varphi^!(x^\perp))^\perp = [\psi^!(\varphi^!(x^\perp))^\perp]^\perp = [\psi^!(\varphi^!(x^\perp))^\perp]^\perp = [\psi^!(\varphi^!(x^\perp))^\perp]^\perp = \varphi^*(x) \).

Consequently this \( \ast \) satisfies conditions for the operator \( \ast \) in Rickart \( \ast \) semigroups.

From the above consideration, we can define the notions of projection, closed projection and right annihilator for an element in \( G(A) \). In order to get a Rickart \( \ast \) semigroup from \( G(A) \), we must show that for any element \( \varphi \in G(A) \), there exists some closed projection \( \mu \) such that \( \{ \varphi \}^{(\mu)} := \{ \psi \in G(A) \mid \varphi \cdot \psi = \theta \} = \mu \cdot G(A) \).

Lemma 3.5 Let \( \mathcal{A} = \langle A, \leq, \sqcap, \sqcup, \perp, 1, 0 \rangle \) be an orthomodular lattice. For each \( a \in A \), define a map \( \gamma_a \) by \( \gamma_a(x) := (x \sqcup a^\perp) \cap a \) for every \( x \in A \).

(i) \( \gamma_a \) is a projection in \( G(A) \) for any \( a \in A \).

(ii) For any \( \varphi \in G(A) \), if we put \( a := \varphi^!(0) \), then \( \{ \varphi \}^{(\mu)} = \gamma_a \cdot G(A) \) holds.
Proof: By our assumption, the following orthomodular law holds. For \( a, b, c \in A \),

1. \( a \leq b \) implies \( b = (b \cap a^\perp) \cup a \).
2. \( c \leq a \) implies \( c = (c \cup a^\perp) \cap a \).

It is easy to see that (2) follows from (1) and vice versa.

(i) First we will show that \( \gamma_a \in G(A) \). It is obvious that \( \gamma_a \) is monotone. We put \( \psi(x) := (x \cap a) \cup a^\perp \) for any \( x \in A \). Clearly \( \psi \) is also monotone. Moreover, as shown below, it is the residual map for \( \gamma_a \).

\[
\gamma_a \cdot \psi(x) = \left[(x \cap a) \cup a^\perp \right] \cap a
= \left[(x \cap a) \cup a \right] \cap a
= x \cap a \leq x.
\]

In the last equation in the above, we used (2) since \( x \cap a \leq a \).

\[
\psi \cdot \gamma_a(x) = \left[(x \cup a^\perp) \cap a \right] \cup a^\perp
= \left[(x \cup a^\perp) \cap a \right] \cup a^\perp
= x \cup a^\perp \geq x
\]

Also, we used (1) since \( x \cup a^\perp \geq a^\perp \).

Therefore \( \gamma_a \cdot \psi(x) = \psi(x) = (x \cap a) \cup a^\perp \). So \( \gamma_a \in G(A) \).

Next we will show that \( \gamma_a \) satisfies the conditions for projections.

\[
\gamma_a^* = (\gamma_a \cdot (x^\perp))^\perp = \left[(x^\perp \cap a) \cup a^\perp \right]^\perp
= (x^\perp \cap a) \cup a^\perp
= (x \cup a^\perp) \cap a = \gamma_a(x)
\]

\[
\gamma_a \cdot \gamma_a(x) = \left[(x \cup a^\perp) \cap a \right] \cup a^\perp
= (x \cup a^\perp) \cap a = \gamma_a(x)
\]

Since \( (x \cup a^\perp) \cap a \leq a \), we used (2) in the above calculation. Thus \( \gamma_a \) is a projection.

(ii) First we will prove that \( \gamma_a : G(A) \subseteq \{ \varphi \}^{(t)} \). Take any \( \psi \in \gamma_a \cdot G(A) \). Then there exists some element \( \lambda \in G(A) \) such that \( \psi = \gamma_a \cdot \lambda \). For any \( x \in A \), \( \gamma_a(x) = (x \cup a^\perp) \cap a \leq a = \varphi(x) \). So by the monotonicity of \( \varphi \), we have that \( \varphi \cdot \gamma_a(x) \leq \varphi \cdot \varphi(x) \). This means that \( \varphi \cdot \gamma_a = \theta \). Then \( \varphi \cdot \psi = \varphi \cdot \gamma_a \cdot \lambda = \theta \), that is \( \psi \in \{ \varphi \}^{(t)} \).

Thus we conclude that \( \gamma_a : G(A) \subseteq \{ \varphi \}^{(t)} \). Next we will show that \( \{ \varphi \}^{(t)} \subseteq \gamma_a \cdot G(A) \). Take any \( \psi \in \{ \varphi \}^{(t)} \).

Then \( \psi \) satisfies that \( \varphi \cdot \psi = \theta \), which means that for any \( x \in A \), we have that \( \varphi \cdot \psi(x) = 0 \). Taking 1 for \( x \), we have \( \varphi \cdot \psi(1) = 0 \), and hence \( a = \varphi(x) \cap a \leq a \). Therefore we have that for any \( x \in A \), \( \psi(x) \leq \psi(1) \leq a \).

By combining this result with the orthomodular law (2), we have that \( \gamma_a \cdot \psi(x) = (\psi(x) \cup a^\perp) \cap a = \psi(x) \). Consequently \( \psi = \gamma_a \cdot \psi \in \gamma_a \cdot G(A) \).

Thus we have proved \( \{ \varphi \}^{(t)} = \gamma_a \cdot G(A) \).

\[\square\]

Moreover, we can show the following lemma on the set of maps \( \gamma_a \).
Lemma 3.6 For any orthomodular lattice $\mathcal{A} = \langle A, \leq, \cap, \cup, \perp, 1, 0 \rangle$, the relation $P_c(G(A)) = \{\gamma_a \mid a \in A\}$ holds.

Proof: Take any $\lambda \in P_c(G(A))$. Then there exists some $\mu \in G(A)$ such that $\{\mu\}^{(i)} = \lambda \cdot G(A)$. Now putting $\vartheta := \mu\vartheta(0)$, we have $\{\mu\}^{(i)} = \gamma_\vartheta \cdot G(A)$ by Lemma 3.5 (ii). So the uniqueness of the right annihilating projection gives us that $\lambda = \gamma_\vartheta \in \{\gamma_a \mid a \in A\}$.
Conversely, consider $\gamma_a$ for $a \in A$. Since $\gamma_a$ is a projection, $\gamma_a = \gamma_a \cdot \gamma_a = \gamma_a^*$ holds. We have that $\gamma_a \cdot \gamma_a^* = \theta$. So by operating * to this equation, we get $\gamma_a^* \cdot \gamma_a = \theta$. Then of course, $\gamma_a^* \cdot \gamma_a \cdot \lambda = \theta$ for any $\lambda \in G(A)$ holds. Therefore we get $\{\gamma_a^*\}^{(i)} = \gamma_a \cdot G(A)$. Thus $\gamma_a \in P_c(G(A))$.
Consequently we have proved that $P_c(G(A)) = \{\gamma_a \mid a \in A\}$. $\square$

By all the lemmas 3.2, 3.3, 3.4 and 3.5, we can prove the following theorem.

Theorem 3.7 Let $\mathcal{A} = \langle A, \leq, \cap, \cup, \perp, 1, 0 \rangle$ be an orthomodular lattice. Then $G(\mathcal{A}) = \langle G(A), \cdot, * \rangle$ is a Rickart * semigroup, where $\cdot$ is a composition operator of maps and $*$ is a unary operator defined in Lemma 3.3.

4 Corresponding model and Completeness Theorem

Now we have prepared all the notions for constructing the corresponding model for orthomodular logic.

Definition 4.1 (Corresponding model) Let $\mathcal{A} = \langle A, \leq, \cap, \cup, \perp, 1, 0 \rangle$ be an orthomodular lattice, and $\nu: \Phi \rightarrow A$ an orthomodular valuation. The corresponding model to $\mathcal{A}$ and $\nu$ is the structure $\mathcal{M}_A = \langle G(A), \cdot, *, u_\mathcal{A} \rangle$, where

(i) $G(A)$ is the set of all residuated monotone maps on $A$,
(ii) $\cdot$ is the composition operator of maps on $A$,
(iii) $*$ is the unary operator on $G(A)$ defined in Lemma 3.4, that is, for any $\varphi \in G(A)$, $\varphi^*(x) := (\varphi(x^\perp))^\perp$ for all $x \in A$,
(iv) $u_\mathcal{A}$ is a function assigning to each propositional variable $p_i$ an element of the set $\{\gamma_a \mid a \in A\}$, such that, $u_\mathcal{A}(p_i) := \gamma_{u(p_i)}$.

Lemma 4.2 Let $\mathcal{A}$ be an orthomodular lattice and $\nu$ an orthomodular valuation. Then the corresponding model $\mathcal{M}_A = \langle G(A), \cdot, *, u_\mathcal{A} \rangle$ is an orthomodular model.

Proof: This is obvious from Lemma 3.6 and Lemma 3.7. $\square$

Since $\mathcal{M}_A$ is an orthomodular model, the notion of truth in $\mathcal{M}_A$ can be defined similarly in Definition 2.1 as follows Let $\alpha, \beta$ be formulas, $\varphi, \psi$ elements in $G(A)$. Then:


(i) \((\mathcal{M}_{A}, \varphi) \models p_i \iff p_i \in u(p_i) \cdot G(A)\).

(ii) \((\mathcal{M}_{A}, \varphi) \models \alpha \land \beta \iff (\mathcal{M}_{A}, \varphi) \models \alpha \text{ and } (\mathcal{M}_{A}, \varphi) \models \beta.\)

(iii) \((\mathcal{M}_{A}, \varphi) \models \neg \alpha \iff \forall \psi \in G(A), [(\mathcal{M}_{A}, \varphi) \models \alpha \text{ only if } \psi^* \cdot \varphi = 0].\)

By denoting \(\|\alpha\|^\mathcal{M}_{A} := \{\varphi \in G(A) \mid (\mathcal{M}_{A}, \varphi) \models \alpha\}\), we can restate the above conditions in the following way.

(i) \(\|p_i\|^\mathcal{M}_{A} = u(p_i) \cdot G(A)\).

(ii) \(\|\alpha \land \beta\|^\mathcal{M}_{A} = \|\alpha\|^\mathcal{M}_{A} \land \|\beta\|^\mathcal{M}_{A}\).

(iii) \(\|\neg \alpha\|^\mathcal{M}_{A} = \{\varphi \in G(A) \mid \forall \psi \in \|\alpha\|^\mathcal{M}_{A} (\psi^* \cdot \varphi = 0)\}.\)

Here we will make a comment about the order on \(P_c(G(A))\), where \(A\) is an orthomodular lattice. Because \(\gamma_a \in P_c(G)\) is a projection, the order on the set \(\{\gamma_a \mid a \in A\}\) is defined as in Definition 1.4, that is,

\[
\text{For } a, b \in A, \quad \gamma_a \leq \gamma_b \quad \text{iff} \quad \gamma_a \cdot \gamma_b = \gamma_a
\]

By Lemma 1.3, we have that \(\gamma_a \leq \gamma_b\) is equivalent to \(\gamma_a \cdot G(A) \subseteq \gamma_b \cdot G(A)\).

We can show the following lemma on this order relation.

**Lemma 4.3** Let \(A = \langle A, \leq, \cap, \cup, \perp, 1, 0 \rangle\) be an orthomodular lattice. Then the following two conditions are equivalent.

(i) \(a \leq b\) on \(A\).

(ii) \(\gamma_a \leq \gamma_b\) on \(P_c(G(A))\).

**Proof:** ( (i)⇒(ii) ): Suppose that \(a \leq b\). Then, for all \(x \in A\) the following holds:

\[
\gamma_b \cdot \gamma_a(x) = ((x \cup a^\perp) \cap a) \cup b^\perp \cap b
\]

\[
= (x \cup a^\perp) \cap a = \gamma_a(x)
\]

Since we have \((x \cup a^\perp) \cap a \leq a \leq b\), we used the orthomodular law (2) in the proof of Lemma 3.5. Thus we conclude that \(\gamma_a \leq \gamma_b\).

( (i)⇐(ii) ): Suppose that \(\gamma_a \leq \gamma_b\). This means that \(\gamma_a \cdot \gamma_b = \gamma_b \cdot \gamma_a = \gamma_a\). Since \(\gamma_a(1) \leq 1\), \(\gamma_a(1) = \gamma_b \cdot \gamma_a(1) = \gamma_b(\gamma_a(1)) \leq \gamma_b(1)\). Recall here that \(\gamma_a(x) := (x \cup a^\perp) \cap a\) for any \(x \in A\), then we have that \(a = \gamma_a(1) \leq \gamma_b(1) = b\).

\(\Box\)

As in Lemma 3.3, we can also extend the domain of valuation function \(u_A\) from the set of propositional variables to the set of all formulas \(\Phi\).
Lemma 4.4  Let $\mathcal{A} = (A, \leq, \cap, \cup, ^{\perp}, 1, 0)$ be an orthomodular lattice and $v$ an orthomodular valuation. Let $\mathcal{M}_A$ be the canonical orthomodular model corresponding to $A$. Then for any formula $\alpha$, $\|\alpha\|_{\mathcal{M}_A} = \gamma_{\nu} \cdot \Gamma(A)$.

Proof : Induction on the construction of the formula $\alpha$. \hfill $\square$

We have now reached the following Completeness Theorem.

Theorem 4.5 (Completeness theorem) For given formulas $\alpha$ and $\beta$, let (S) and (T) be the same statements in Theorem 2.4. That is,

(S): for any orthomodular lattice $A$ and any orthomodular valuation $v : \Phi \to A$, $v(\alpha) \leq v(\beta)$.

(T): for any orthomodular model $\mathcal{M}$, $\mathcal{M} : \alpha \models \beta$.

Then (T) implies (S). \hfill $\square$

5 Relation between two types of models

Theorem 5.1 Let $\mathcal{M} = \langle G, u \rangle = \langle G, \cdot, *, u \rangle$ be an orthomodular model. Then $\mathcal{Q} = \langle G', R, \zeta, V \rangle$ is a quantum model, where,

- $G' := G \setminus \{0\}$,
- $\zeta := \{e \cdot G' \mid e \in P_c(G')\}$,
- $R$ is a binary relation on $G'$ defined as the following: for $x, y \in G'$, $xRy \iff x^* \cdot y = 0$,
- $V$ is a function assigning to each $p_i$ an element $V(p_i)$ of $\zeta$.

Proof : Check the conditions for quantum model in Definition 0.4. \hfill $\square$

References

