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WHITEHEAD LINK から得られる双曲的多様体について
ON THE HYPERBOLIC MANIFOLDS OBTAINED FROM THE WHITEHEAD LINK

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1. Introduction

Let W be the Whitehead link complement and $W(p,q)$ the manifold obtained from W by p/q-Dehn filling on one end. It is well known that W possesses a complete hyperbolic structure of finite volume, thus due to the work of Thurston, hyperbolic Dehn surgery theorem, $W(p,q)$ also possesses a complete hyperbolic structure of finite volume for sufficiently large (p,q). (See [**T**, **NZ**].) In fact, for any pair of coprime integers (p,q) which lies outside the parallelogram with vertices $(\pm 4,\mp 1)$ and $(0,\pm 1)$, $W(p,q)$ possesses a complete hyperbolic structure of finite volume. Let $\mathcal{W}$ be the family of hyperbolic manifolds $W(p,q)$. It is also known that $\mathcal{W}$ contains two famous families of hyperbolic manifolds, which are:

1. $\{W(1,q)\}$ is the family of the twist knot complements, and
2. $\{W(p,1)\}$ is the family of the tunnel number one once-punctured torus bundles.

(See Figure 1. The first assertion is easily observed and the second is due to the works of [**HMW**, **Jh**, **K**].)

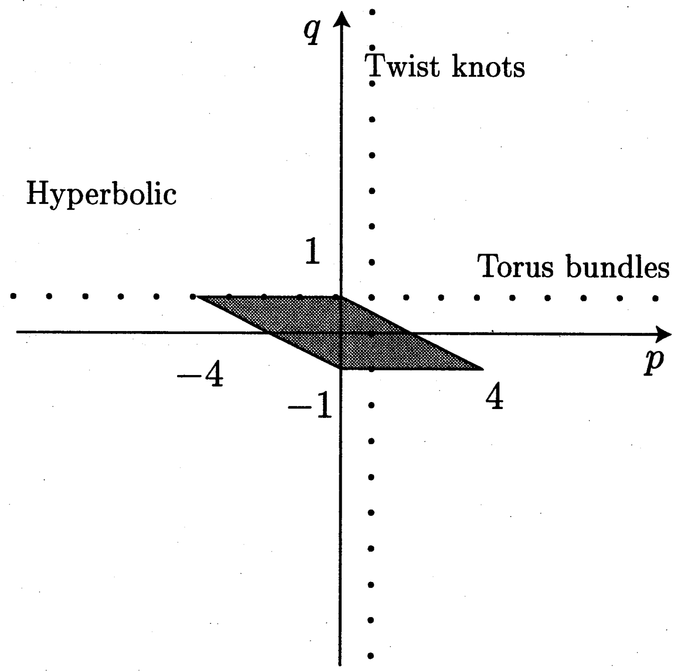

Figure 1

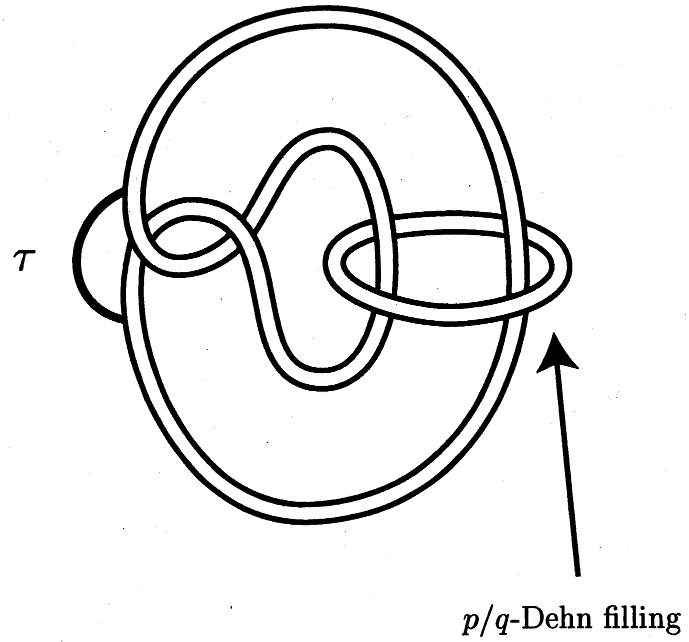

Figure 2

In this article, we study two topics concerning $\mathcal{W}$. The first topic is the review on the work of [**ANS**], where the shortest vertical geodesics of the manifolds in $\mathcal{W}$ are determined. In the second topic, we study the canonical decompositions of the manifolds in $\mathcal{W}$ to obtain the result which asserts that the canonical decompositions of those manifolds are ideal tetrahedral.

1.1. **On the shortest vertical geodesics.** Let M be an orientable hyperbolic 3-manifold of finite volume with a cusp. Geometrically, the cusp lifts to a disjoint set of horoballs in hyperbolic 3-space $\mathbb{H}^3$. A *vertical geodesic* is a geodesic which is perpendicular to the cusp at each of its ends. Once the size of the cusp has been fixed, the *length* of a vertical geodesic with respect to the cusp is defined to be the length of that part of the geodesic that lies between the two points on the geodesic where it intersects the cusp boundary perpendicularly. The shortest vertical geodesics can be determined independently from the choice of the size of the cusp. We can characterize them as follows: By expanding the cusp until it touches itself, we obtain the *maximal cusp.* (Thus it lifts to a set of horoballs in $\mathbb{H}^3$ with disjoint interiors but such that some of the horoballs are tangent to one another.) A vertical geodesic is the shortest if and only if it intersects the maximal cusp orthogonally at a point of self-tangency of the maximal cusp.

Let τ be the arc in W depicted in Figure 2, and $\tau(p,q)$ the image of τ by the inclusion $W \hookrightarrow W(p,q)$.

The following is the main theorem for the first topic.

Theorem 1.1. *For any hyperbolic manifold $W(p,q)$, $\tau(p,q)$ is isotopic to a shortest vertical geodesic. Moreover, if (p,q) is not equal to $\pm(1,1)$ nor $\pm(-5,1)$ then $\tau(p,q)$ is the unique shortest vertical geodesic. If (p,q) is equal to $\pm(1,1)$ or $\pm(-5,1)$, $W(p,q)$ has precisely one other shortest vertical geodesic besides $\tau(p,q)$.*

We can easily see that $\tau(p,q)$ is an *unknotting tunnel* for $W(p,q)$, i.e., the complement of $\tau(p,q)$ in $W(p,q)$ is an open handlebody. In particular, we have the followings.

Corollary 1.2. *The upper tunnel of a hyperbolic twist knot is isotopic to a shortest vertical edge.*

By using the classification theorem of the unknotting tunnels for punctured torus bundles over S^1 due to Johannson [**Jh**] (cf. Kobayashi [**K**]), we obtain the following corollary.

Corollary 1.3. *A properly embedded arc in a tunnel number one punctured torus bundle over S^1 with hyperbolic monodromy is an unknotting tunnel if and only if it is isotopic to a shortest vertical geodesic.*

1.2. **On the canonical decompositions.** In [**T**], the figure eight knot complement is decomposed into two hyperbolic ideal tetrahedra. Such decompositions give a nice "visualization" of hyperbolic manifolds with cusps and the following conjecture is known.

Conjecture 1.4. Every cusped hyperbolic 3-manifold can be decomposed into hyperbolic ideal tetrahedra.

The decomposition of the figure eight knot complement is also an example of the canonical cell decomposition due to Epstein-Penner and Weeks [**EP**, **W**], which is determined for all cusped hyperbolic 3-manifolds, even though it is not generally an ideal tetrahedral one (namely, the canonical decomposition generally consists of convex ideal polyhedra).

The main theorem for the second topic is the following.

Theorem 1.5. *For any hyperbolic manifold $W(p,q)$, the canonical decomposition of $W(p,q)$ is ideal tetrahedral.*

2. Constructions of $W(p,q)$

Since our proof for both of the main theorems require deep observations on the manifolds, we give concrete constructions of $W(p,q)$ following [**NR**]. Due to the symmetry of the Whitehead link, there are two ways of constructions which are mutually similar to each other.

For any point in the upper half of the complex plane, denoted by $\mathbb{C}_+$, let $\mathcal{O}_x$ and $\mathcal{O}'_x$ be ideal octahedra in $\mathbb{H}^3$ with the following vertices:

$$\mathcal{O}_x:\ \infty,\ 0,\ 1,\ x,\ -1,\ -x$$
$$\mathcal{O}'_x:\ \infty,\ 0,\ 1,\ x,\ x^2,\ -x.$$

Both $\mathcal{O}_x$ and $\mathcal{O}'_x$ have the same combinatorial gluing patterns. (See Figures 3a and 3b.)

Let A_x, B_x, C_x, D_x be the orientation preserving isometries in $\mathbb{H}^3$ which maps $\mathcal{A}'$ in Figure 3a to $\mathcal{A}$ and so on, precisely, they map the triples to the other triples as follows.

$$A_x:(0,1,x)\to(\infty,-x,1) \qquad B_x:(0,x,-1)\to(0,1,-x)$$
$$C_x:(\infty,x,-1)\to(0,-1,-x) \qquad D_x:(\infty,x,1)\to(\infty,-1,-x)$$

Similarly, let A'_x, B'_x, C'_x, D'_x be the orientation preserving isometries in $\mathbb{H}^3$ which maps $\mathcal{A}'$ in Figure 3b to $\mathcal{A}$ and so on, precisely, they map the triples to the other triples as follows.

$$A'_x:(0,1,x)\to(\infty,-x,1) \qquad B'_x:(0,x,x^2)\to(0,1,-x)$$
$$C'_x:(\infty,x,x^2)\to(0,x^2,-x) \qquad D'_x:(\infty,x,1)\to(\infty,x^2,-x)$$

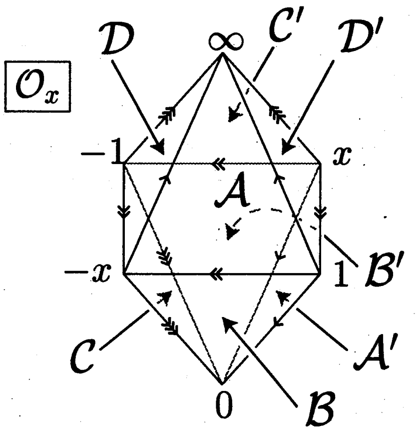

Figure 3a

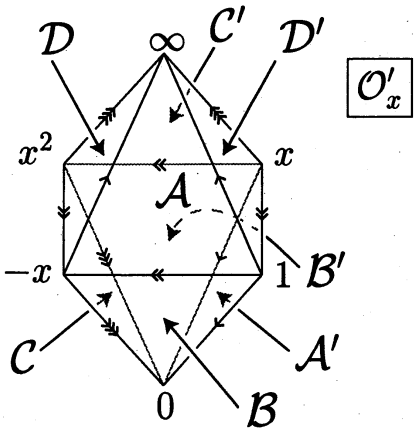

Figure 3b

Let W_x (W'_x resp.) be the manifold obtained from $\mathcal{O}_x$ ($\mathcal{O}'_x$ resp.) by gluing the four pairs of faces using A_x, B_x, C_x, D_x (A'_x, B'_x, C'_x, D'_x resp.). It is observed in [**T**] that both W_x and W'_x are (generally) incomplete hyperbolic manifolds which are homeomorphic to the Whitehead link complement. Precisely,

1. both the end of W_x formed by the vertices ∞ and 0 and the end of W'_x formed by the vertices 1, x, x^2, $-x$ are complete for any $x \in \mathbb{C}_+$, and
2. both the end of W_x formed by the vertices 1, x, -1, $-x$ and the end of W'_x formed by the vertices ∞ and 0 are (generally) incomplete.

Concerning the shortest vertical geodesics, we can see that the preimage of τ is

1. the geodesic connecting ∞ with 0 in $\mathcal{O}_x$, and
2. the four edges $[1, x]$, $[x, x^2]$, $[x^2, -x]$, $[-x, 1]$ in $\mathcal{O}'_x$.

The real Dehn surgery parameters (p_x, q_x) of the incomplete end can be calculated associated to each $x \in \mathbb{C}_+$ as follows.

$$p_x = \frac{-8\pi \log|x|}{\log\left|\frac{x(x+1)}{x-1}\right|(4\arg x - 2\pi) - 4\log|x| \arg\frac{x(x+1)}{x-1}}$$

$$q_x = \frac{2\pi \log\left|\frac{x(x+1)}{x-1}\right|}{\log\left|\frac{x(x+1)}{x-1}\right|(4\arg x - 2\pi) - 4\log|x| \arg\frac{x(x+1)}{x-1}}$$

Proposition 2.1. *Concerning (p, q), the following properties hold.*

1. *There is a well defined continuous map*
$$(p, q) : \mathbb{C}_+ \ni x \mapsto (p_x, q_x) \in \mathbb{R}^2 \cup \{\infty\}$$
which has pole exactly at $x = i$.
2. $(p_{-1/x}, q_{-1/x}) = (-p_x, -q_x)$
3. *When (p_x, q_x) is a pair of coprime integers, the metric completion of W_x and W'_x are complete hyperbolic manifold both of which are homeomorphic to $W(p_x, q_x)$.*
4. *$W_i \cong W'_i$ is itself a complete hyperbolic manifold, therefore realize the complete hyperbolic structure of the Whitehead link complement.*

Let Γ_x (Γ'_x resp.) be the subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ generated by A_x, B_x, C_x, D_x (A'_x, B'_x, C'_x, D'_x resp.). When the Dehn surgery parameter (p_x, q_x) is a pair of coprime integers,

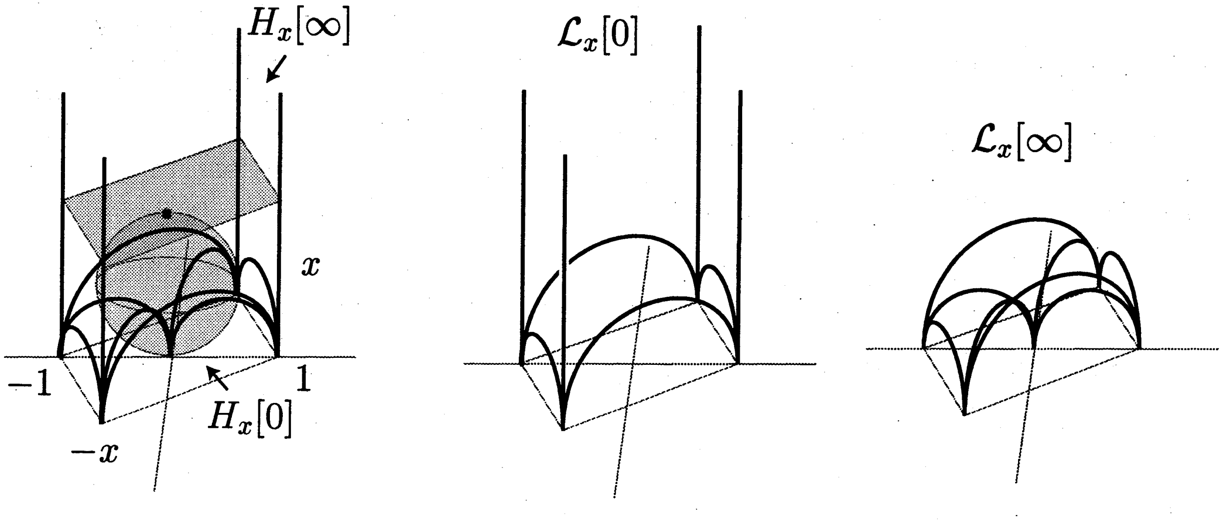

Figure 4

both Γ_x and Γ'_x are discrete and torsion free and there are coverings

$$\Psi_x : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma_x \cong W(p_x, q_x)$$

$$\Psi'_x : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma'_x \cong W(p_x, q_x)$$

moreover, Γ_x (Γ'_x resp.) acts $\mathbb{H}^3 - {\Psi_x}^{-1}(W(p_x, q_x) - W_x)$ ($\mathbb{H}^3 - {\Psi'_x}^{-1}(W(p_x, q_x) - W'_x)$ resp.) and $\mathcal{O}_x$ ($\mathcal{O}'_x$ resp.) is the fundamental domain for the action. Since both $W(p_x, q_x) - W_x$ and $W(p_x, q_x) - W'_x$ are closed geodesics, and thus both ${\Psi_x}^{-1}(W(p_x, q_x) - W_x)$ and ${\Psi'_x}^{-1}(W(p_x, q_x) - W'_x)$ are the union of countable geodesics, we may say that $\mathcal{O}_x$ ($\mathcal{O}'_x$ resp.) is an "almost fundamental domain" for the action of Γ_x (Γ'_x resp.) on $\mathbb{H}^3$.

3. Sketch of the proof of Theorems

3.1. **On Theorem 1.1 (following [ANS]).** We will use $\mathcal{O}_x$ as a fundamental domain. Let τ_x be the geodesic connecting ∞ with 0, which is naturally embedded in W_x, we also denote it by the same symbol τ_x. Let $H_x[0]$ and $H_x[\infty]$ the horoball components of the inverse image under Ψ_x of the maximal cusp. Then τ_x is shortest, if and only if $H_x[0]$ and $H_x[\infty]$ touches at a point in τ_x. On the other hand, since both $H_x[0]$ and $H_x[\infty]$ projects to the same maximal cusp in $W(p_x, q_x)$, any element of Γ_x sending 0 to ∞ must bring $H_x[0]$ to $H_x[\infty]$. In particular, $A_x(H_x[0]) = H_x[\infty]$. Hence, $H_x[0]$ and $H_x[\infty] = A_x(H_x[0])$ touches at a point in τ_x if and only if $h_E(H_x[0]) = h_E(\partial H_x[\infty]) = \sqrt{|x(x+1)/(x-1)|}$. (Here, h_E is the Euclidean height of a set in the upper half space.)

Keeping the above observation in mind, put $t_x = \sqrt{|x(x+1)/(x-1)|}$ and let T_x be the point in τ_x with Euclidean height t_x. Define $H_x[0]$ (resp. $H_x[\infty]$) to be the horoball centered at 0 (resp. ∞) with $h_E(H_x[0]) = t_x$ (resp. $h_E(\partial H_x[\infty]) = t_x$) anew. Then these two horoballs touches at T_x, and we have $A_x(H_x[0]) = H_x[\infty]$. Let $\mathcal{L}_x[\infty]$ (resp. $\mathcal{L}_x[0]$) be the union of faces of $\mathcal{O}_x$ which do not have ∞ (resp. 0) as a vertex.

The following is the key proposition for the proof of Theorem 1.1.

Proposition 3.1. *Suppose $H_x[\infty] \cap \mathcal{L}_x[\infty] = \emptyset$ and $H_x[0] \cap \mathcal{L}_x[0] = \emptyset$. Then both $H_x[\infty]$ and $H_x[0]$ project to the maximal cusp of $W(p_x, q_x)$, and T_x projects to the unique point of self-tangency of the maximal cusp.*

Idea of the proof. By the conditions in the statement, we can see that $H_x[\infty]$ is included in $\Gamma_x((H_x[\infty] \cup H_x[0]) \cap \mathcal{O}_x)$, thus $(H_x[\infty] \cup H_x[0]) \cap \mathcal{O}_x$ is a fundamental domain for the inverse image under Ψ_x of the cusplike region $\Psi_x((H_x[\infty] \cup H_x[0]) \cap \mathcal{O}_x)$. Since $\mathcal{O}_x$ is an "almost fundamental domain", if there is a pair of horoballs which has nontrivial intersection, the intersection can be mapped into $\mathcal{O}_x$, and since $(H_x[\infty] \cap H_x[0]) \cap \mathcal{O}_x$ consists of just one point T_x, the horoball pair must be equivalent to $H_x[\infty] \cup H_x[0]$. □

By Proposition 2.1-2, we will assume $\Re(x) \geq 0$. The conditions in Proposition 3.1 can be interpreted to an algebraic inequality using the following lemma.

Lemma 3.2. 1. $H_x[\infty] \cap \mathcal{L}_x[\infty] = H_x[0] \cap \mathcal{L}_x[0] = \emptyset \Leftrightarrow t_x > h_E(\mathcal{L}[\infty] \cup A_x\mathcal{L}[0])$.

2. $h_E(\mathcal{L}[\infty] \cup A_x\mathcal{L}[0]) = \dfrac{|x+1|^2}{2|x-1|}$.

Sketch of proof. 1. This is rather trivial since

$$H_x[0] \cap \mathcal{L}_x[0] = A_x{}^{-1}(A_xH_x[0] \cap A_x\mathcal{L}_x[0]) = A_x{}^{-1}(H_x[\infty] \cap A_x\mathcal{L}_x[0]), \text{ and}$$
$$t_x = h_E(\partial H[\infty]).$$

2. We can see that the Euclidean height of $\mathcal{L}[\infty] \cup A_x\mathcal{L}[0]$ is achieved at the top of an edge, so we only need to decide the longest edge among the projections of the faces to $\mathbb{C}$.

□

We will determine the region in the plane of the real Dehn surgery parameters in which the conditions in Proposition 3.1 hold.

1. Figure 5 is the region in $\mathbb{C}_+$ where $t_x > h_E(\mathcal{L}[\infty] \cup A_x\mathcal{L}[0])$ holds. (The region is extended by the symmetry $x \mapsto -1/x$.)

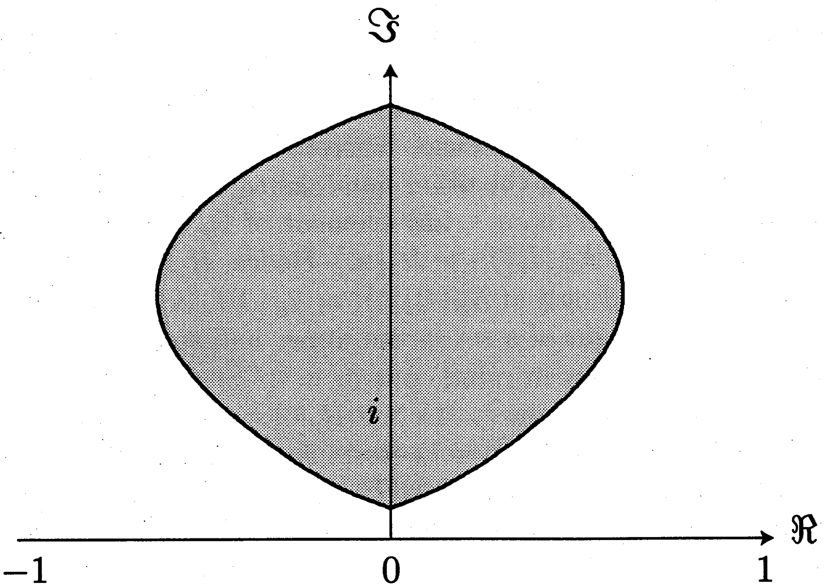

Figure 5

2. Figure 6 is the image of the region in Figure 5 by the real Dehn surgery parameter map, thus for any pair of coprime integers (p, q) which is contained in the region, $\tau(p, q)$ is the unique shortest vertical geodesic of $W(p, q)$.

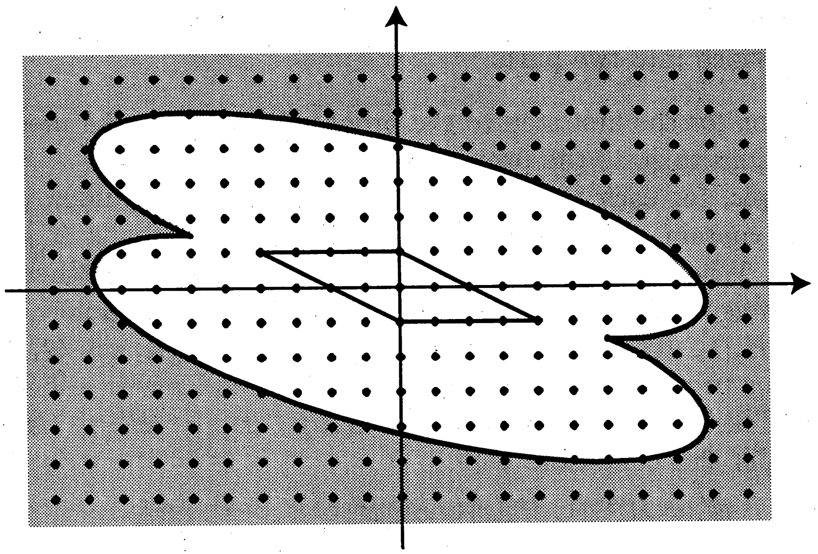

Figure 6

By the above observations, if there are exceptions for Theorem 1.1, they must be contained in Table 1. Since the number of the entries in the table is just 64, we can

$q = 5$	$p = -7, -6$
$q = 4$	$p = -7, -5, -3, -1, 1$
$q = 3$	$p = -7, -5, -4, -2, -1, 1, 2, 4$
$q = 2$	$p = -5, -3, -1, 1, 3, 5$
$q = 1$	$p = -8, -7, -6, -5, 1, 2, 3, 4, 5, 6, 7$
$q = -1$	$p = -7, -6, -5, -4, -3, -2, -1, 5, 6, 7, 8$
$q = -2$	$p = -5, -3, -1, 1, 3, 5$
$q = -3$	$p = -4, -2, -1, 1, 2, 4, 5, 7$
$q = -4$	$p = -1, 1, 3, 5, 7$
$q = -5$	$p = 6, 7$

Table 1

prove Theorem 1.1 in any way. One of the ways will be using computer program SnapPea (maybe anyone can do this, if he has much time and patience), so we will omit the rest of the proof, however, we remark that: In [**ANS**], another sufficient condition for τ_x be shortest which is stronger than Proposition 3.1 is presented, in fact, the exceptions left for us become $W(\pm 1, \pm 1)$ and $W(\pm 5, \mp 1)$.

3.2. **On Theorem 1.5.** Our starting point is the following proposition, which is easily seen from the definition (thus the proof is omitted).

Proposition 3.3. *Let M be a cusped hyperbolic 3-manifold of finite volume. Take horoball neighborhoods for all cusps so that the volumes of them coincide and lift them to the universal cover $\mathbb{H}^3$ and denote the set of horoballs by $\mathcal{H}$. The canonical decomposition of M is ideal tetrahedral if and only if the number of the nearest horoballs in $\mathcal{H}$ is at most 4 for any point in $\mathbb{H}^3$.*

Due to Proposition 3.3, we only need to count the nearest horoballs in $\mathbb{H}^3$. The proof of Theorem 1.5 is divided into two parts.

The key observation for the first step to the proof is Proposition 3.4 mentioned below. We need more notations to state the proposition.

As in the first topic, for each vertex z of $\mathcal{O}'_x$, let $\mathcal{L}'_x[z]$ be the union of the faces of $\mathcal{O}'_x$ which does not contain z as a vertex. We will define horoballs $H'_x[z]$ centered at $z \in \Gamma'_x(1)$ as follows. When the size of $H'_x[1]$ is fixed, the sizes of the other horoballs can be determined unambiguously so that they respect the Γ'_x-action, namely, for $\gamma \in \Gamma'_x$ we define $H'_x[\gamma(1)] = \gamma H'_x[1]$. There is always a geodesic quadrangle, say Q_x, with vertices 1, x, x^2, $-x$, and thus two geodesics connecting 1 with x^2 and x with $-x$ always have an intersection, say p. Take the sizes of horoballs so that $p \in H'_x[1] \cap H'_x[x] \cap H'_x[x^2] \cap H'_x[-x]$ and are minimal under this condition. Put

$$H'_x = H'_x[1] \cup H'_x[x] \cup H'_x[x^2] \cup H'_x[-x], \text{ and}$$
$$\mathcal{H}'_x = \{\gamma H'_x[z] \mid z \in \Gamma'_x(1)\},$$

then the following proposition holds, whose proof is again omitted.

Proposition 3.4. *The number of the nearest horoballs in $\mathcal{H}'_x$ is at most four for any point in $H'_x \cap \mathcal{O}'_x$ when the following conditions are satisfied.*

1. $H'_x[1] \cap \mathcal{L}'_x[1] = \emptyset$
2. $H'_x[x] \cap \mathcal{L}'_x[x] = \emptyset$
3. $H'_x[x^2] \cap \mathcal{L}'_x[x^2] = \emptyset$
4. $H'_x[-x] \cap \mathcal{L}'_x[-x] = \emptyset$

It is also easy to observed that:

Lemma 3.5. *When $W(p_x, q_x)$ is a hyperbolic manifold, H'_x contains entire Q_x, thus $\mathcal{O}'_x - H'_x$ has two connected component which are regular neighborhoods of 0 and ∞ in $\mathcal{O}'_x$ respectively.*

In the following, as the second step, we make an analysis on the points near the completed end. Fix $x \in \mathbb{C}_+$ satisfying the condition in Proposition 3.4 and such that $W(p_x, q_x)$ is a hyperbolic manifold. Then two isometries $\mu_2 = D'_x$ and $\lambda_2 = \sqrt{A'_x C'_x}$ become hyperbolic elements which commute each other. (Here $\sqrt{A'_x C'_x}$ is the square root of $A'_x C'_x \in \Gamma'_x$ in $PSL(2, \mathbb{C})$.)

Let Γ'_0 be the abelian group generated by μ_2 and λ_2 and consider the developed image of $\mathcal{O}'_x$ by Γ'_0, namely $\Gamma'_0\mathcal{O}'_x$. Since $\Gamma'_0\mathcal{O}'_x$ wraps around the incomplete geodesic, the axis of μ_2 and λ_2, and all the segment of the horoballs which appear in $\mathcal{O}'_x$ as the developed image of $H'_x[1]$ by Γ'_x are $H'_x[1]$, $H'_x[x]$, $H'_x[x^2]$, $H'_x[-x]$, we may assume that no horoballs can appear above the horoballs which are the developed images of $H'_x[1]$ by Γ'_0. Here, even though $\lambda_2 \notin \Gamma'_x$, a direct calculation shows that $\Gamma'_0 H'_x[1] \subset \Gamma'_x H'_x[1]$. Thus the nearest horoballs in $\mathcal{H}'_x$ to a point in the neighborhood of the incomplete geodesic, which is the gray region in Figure 7, are contained in $\{\gamma H'_x[1] \mid \gamma \in \Gamma'_0\}$. (By lemma 3.5, those horoballs cut out a neighborhood of ∞ so the term 'above' has a meaning.)

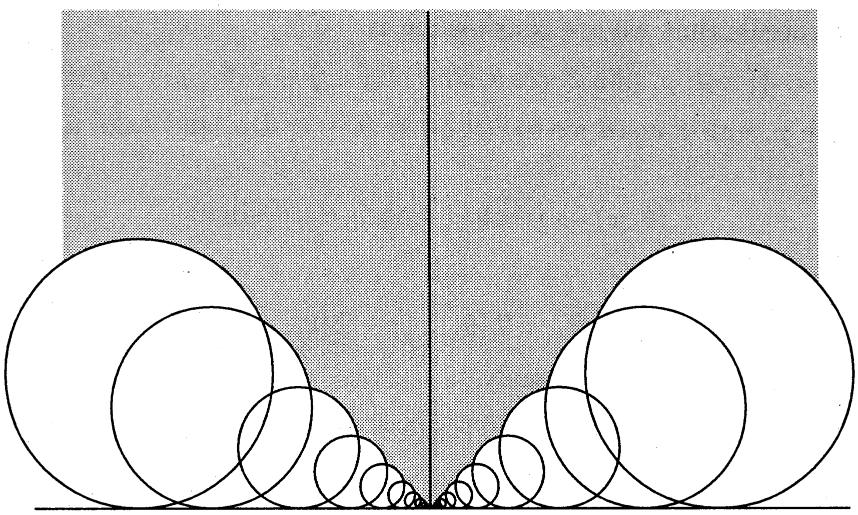

Figure 7

Now we change our view point to the Minkowski 4-space $\mathbb{M}^4$ which is a 4-dimensional vector space with (3, 1)-bilinear form $\langle \cdot, \cdot \rangle$, where the original definition of the canonical decompositions is made in [**EP**]. In this model,

1. $\mathbb{H}^3 = \{x = (x_0, x_1, x_2, x_3) \mid \langle x, x \rangle = -1,\ x_0 > 0\}$
2. The set of horoballs is identified with the positive light cone
$$L_+ = \{v = (v_0, v_1, v_2, v_3) \mid \langle v, v \rangle = 0,\ v_0 > 0\}$$
by the correspondence
$$v \in L_+ \leftrightarrow \{x \in \mathbb{H}^3 \mid \langle v, x \rangle \geq -1\}.$$
3. Each point at infinity is identified with a half line contained in L_+ and the center of the horoball corresponding to $v \in L_+$ is the half line determined by v.
4. $$\begin{aligned}\mathrm{Isom}^+(\mathbb{H}^3) &= SO^+(1,3)\\ &= \{A \in SL(4,\mathbb{R}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle,\ \forall x \in \mathbb{M}^4,\ A\mathbb{H}^3 = \mathbb{H}^3\}\end{aligned}$$
5. For a hyperbolic manifold with a cusp, take a horoball neighborhood of the cusp and lift to $\mathbb{H}^3$, then make the convex hull in $\mathbb{M}^4$ of the points in L_+ corresponding to those horoballs. The canonical decomposition is the boundary pattern of the hull quotiented by the π_1-action. (This is called the convex hull construction.)

We will use the following notation.

$$(\theta, \varphi) = \begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \in SO^+(1,3) = \mathrm{Isom}^+(\mathbb{H}^3)$$

Make a coordinate change so that

$$\mu_2 = (\theta_1, \varphi_1), \quad \lambda_2 = (\theta_2, \varphi_2) \quad \text{for some } (\theta_1, \varphi_1), (\theta_2, \varphi_2) \in \mathbb{R}^2,$$

and $H'_x[1]$ is mapped to the horoball H_0 corresponding to $(t, 0, t, 0) \in L_+$ for some $t > 0$. (Since the convex hull construction does not depend on the fixed sizes of horoballs, we may assume $t = 1$.)

The set of horoballs that we are considering is

$$\Gamma'_0 H_0 = \{(\cosh\varphi, \sinh\varphi, \cos\theta, \sin\theta) \mid (\theta, \varphi) \in \mathbb{Z}(\theta_1, \varphi_1) + \mathbb{Z}(\theta_2, \varphi_2)\}.$$

Put $S = \{\{(\cosh\varphi, \sinh\varphi, \cos\theta, \sin\theta) \mid (\theta, \varphi) \in \mathbb{R}^2\} \subset L_+$, and take a covering $\mathbb{R}^2 \to S$ defined by

$$(\theta, \varphi) \mapsto (\cosh\varphi, \sinh\varphi, \cos\theta, \sin\theta),$$

then lift $\Gamma'_0 H_0$ to $\mathbb{R}^2$

$$\widetilde{\Gamma'_0 H_0} = \mathbb{Z}(\theta_1, \varphi_1) + \mathbb{Z}(\theta_2, \varphi_2).$$

For an ellipsoidal hyperplane V, namely,

$$V = \{x = (x_0, x_1, x_2, x_3) \mid a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = b\}$$

for some $a = (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$ with $\langle a, a\rangle < 0$ and $b \in \mathbb{R}$, we can see that each point $(\cosh\varphi, \sinh\varphi, \cos\theta, \sin\theta) \in V \cap S$ satisfies

$$a_0 \cosh\varphi + a_1 \sinh\varphi + x_2 \cos\theta + a_3 \sin\theta = b. \tag{3.1}$$

Since $\langle a, a\rangle < 0$, (3.1) is equivalent to one of the followings.

$$\cosh(\varphi + \varphi_0) = \beta_1, \quad \text{if } a_2^2 + a_3^2 = 0 \tag{3.2}$$

$$\alpha\cosh(\varphi + \varphi_0) + \cos(\theta + \theta_0) = \beta_2, \quad \text{if } a_2^2 + a_3^2 \neq 0 \tag{3.3}$$

$$\alpha > 1 \quad \beta_1, \beta_2 \in \mathbb{R}$$

In the case (3.2) holds, put

$$E' = \{(\theta, \varphi) \mid \cosh(\varphi + \varphi_0) \leq \beta_1\},$$

then $V \cap S$ lifts to $\partial E'$. Since Γ'_x is torsion free and $\langle \mu_2, {\lambda_2}^2\rangle \subset \Gamma'_x$, the only elliptic elements which are contained in $\langle \mu_2, \lambda_2\rangle$ are, if exists, of order 2. Thus

$$\min\{|\varphi - \varphi'| \mid (\theta, \varphi), (\theta, \varphi') \in \widetilde{\Gamma'_0 H_0}\} = \pi \text{ or } 2\pi,$$

hence $\partial E'$ contains at most 4 points of $\widetilde{\Gamma'_0 H_0}$ modulo the equivalence $(\theta, \varphi) \sim (\theta + 2\pi, \varphi)$.

In the case (3.3) holds, put

$$D' = \{(\theta, \varphi) \mid \alpha\cosh(\varphi + \varphi_0) + \cos(\theta + \theta_0) \leq \beta_2\},$$

then $V \cap S$ lifts to $\partial D'$ and therefore the horoballs which lie on V have the form $(\cosh\varphi, \sinh\varphi, \cos\theta, \sin\theta)$ for $(\theta, \varphi) \in (\mathbb{Z}(\theta_1, \varphi_1) + \mathbb{Z}(\theta_2, \varphi_2)) \cap \partial D'$, and moreover, we can see that if V supports a face of the convex hull,

$$(\mathbb{Z}(\theta_1, \varphi_1) + \mathbb{Z}(\theta_2, \varphi_2)) \cap \mathrm{int}(D') = \emptyset.$$

We will end the second step of the proof of Theorem 1.5 by the following proposition, which is rather technical but has an elementary proof, which is omitted.

Proposition 3.6. *Let D be the region in $\mathbb{R}^2$ defined by*

$$D = \{(x, y) \mid \alpha\cosh y + \cos x \leq a\} \quad \textit{for } \alpha > 1$$

and L a lattice in $\mathbb{R}^2$ satisfying

$$(x, y) \in L \Rightarrow (x + 2m\pi, y) \in L \quad \textit{for } \forall m \in \mathbb{Z}.$$

Then $\#(L \cap \partial D)/2\pi \geq 5$ implies $L \cap \mathrm{int}(D) \neq \emptyset$.
(Here, L is called a lattice in $\mathbb{R}^2$ if there is a set of linearly independent vectors $\{u, v\} \subset \mathbb{R}^2$ and $w \in \mathbb{R}^2$ such that $L = \mathbb{Z}u + \mathbb{Z}v + w$, and $\cdot/2\pi$ means something quotiented by $(x, y) \mapsto (x + 2\pi, y)$ symmetry.)

The following claim is a summation of what we have proved till now.

Claim 3.7. *If $x \in \mathbb{C}_+$ satisfies the conditions in Proposition 3.4 and produces a hyperbolic manifold $W(p_x, q_x)$, the canonical decomposition of $W(p_x, q_x)$ is ideal tetrahedral.*

We can make an estimation for the region of $x \in \mathbb{C}_+$ satisfying the conditions of Proposition 3.4 by a method similar to the one which is used in the first topic. Here we will note only the result of the estimation.

The conditions in Proposition 3.4 are satisfied if

$$\min\left\{\frac{|x-1|}{2}, \left|\frac{x}{x+1}\right|\right\} > \max\left\{\frac{1}{2}, \frac{|x|}{2}, \left|\frac{x-1}{2(x+1)}\right|, \left|\frac{x(x-1)}{2(x+1)}\right|\right\}. \tag{3.4}$$

By Proposition 2.1-2, we may assume $\Re(x) \geq 0$ hence $|x-1|/|x+1| \leq 1$, thus

$$\max\left\{\frac{1}{2}, \frac{|x|}{2}, \left|\frac{x-1}{2(x+1)}\right|, \left|\frac{x(x-1)}{2(x+1)}\right|\right\} = \max\left\{\frac{1}{2}, \frac{|x|}{2}\right\},$$

therefore the following region satisfies Inequality (3.4):

$$\{x \in \mathbb{C}_+ \mid |x-1| > 1,\ |x-1| > |x|,\ |x+1| < 2,\ |x+1| < 2|x|\},$$

which can be visualized as in Figure 8, and is mapped by the real Dehn surgery coefficient map to the region depicted in Figure 9. (The region in Figure 8 is extended to the region $\Re(x) < 0$ using the symmetry $x \mapsto -1/x$.)

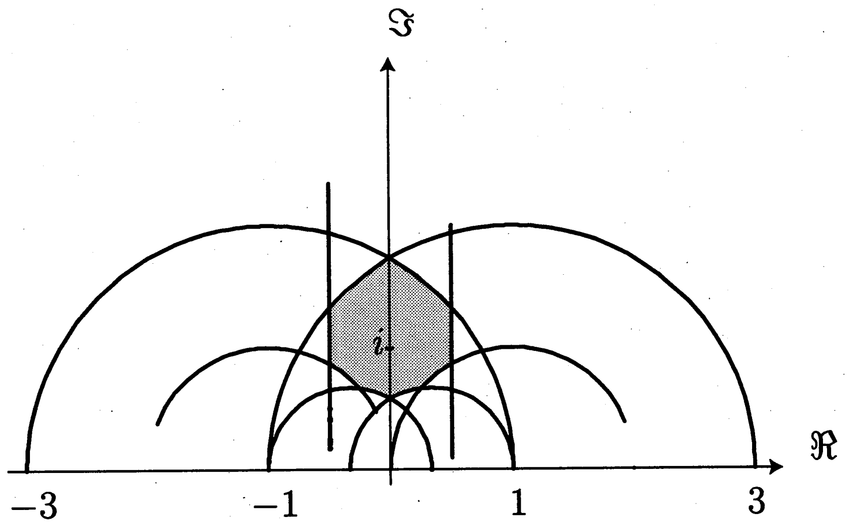

Figure 8

By Figure 9 together with Claim 3.7, there are at most 70 exceptions for Theorem 1.5, however by calculations using SnapPea shows there are no exceptions. This completes the proof of Theorem 1.5.

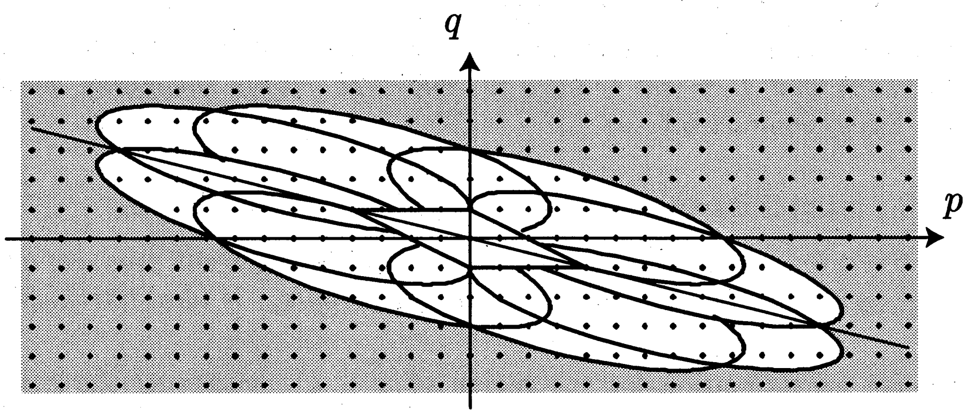

Figure 9

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