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CYCLICALLY PRESENTED GROUPS AND TAKAHASHI MANIFOLDS

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Abstract

We consider groups with cyclic presentations which arise as fundamental groups of a family of closed 3-dimensional manifolds. These manifolds were firstly described by Takahashi using Dehn surgeries on links. We demonstrate that the cyclic automorphisms of groups induce cyclic coverings of the 3-sphere branched over 2-bridge knots. Moreover, polynomials associated with cyclic presentations are equal to Alexander polynomials of corresponding knots.
Keywords: fundamental group, 3-manifold, cyclic covering.

1. Cyclically Presented Groups

The cyclically presented groups comprise a rich source of groups which are interesting from a topological point of views. The connection between cyclically presented groups and cyclic branched coverings of knots and links was studied, in particular, in [2], [4], [5] and [6].

Let $F_n = \langle x_1, \ldots, x_n \quad | \quad \rangle$ be the free group of rank n and $\eta : F_n \to F_n$ be an automorphism of order n such that $\eta(x_i) = x_{i+1}$ for $i = 1, \ldots, n$, where all indices by mod n. We recall [8, §9] that for a reduced word $w \in F_n$ the *cyclically presented group* $G_n(w)$ is given by

$$G_n(w) = \langle x_1, \ldots, x_n \quad | \quad w, \eta(w), \ldots, \eta^{n-1}(w) \rangle.$$

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A group G is said to *have a cyclic presentation* if $G \cong G_n(w)$ for some n and w. Clearly, the automorphism η of F_n induces an automorphism of $G_n(w)$. Such automorphism defines an action of the group $\mathbb{Z}_n = \langle \eta \mid \eta^n = 1 \rangle$ on $G_n(w)$. Let us consider a split extension $H_n = G_n(w) \lambda \mathbb{Z}_n$. The group H_n is said to be a *natural extension of a cyclically presented group.* It was remarked in [4] that the group $H_n = H_n(v)$ always has a 2-generator, 2-relator presentation of the form

$$H_n(v) \cong \langle \eta, x \mid \eta^n = v(\eta, x) = 1 \rangle,$$

where $v = v(\eta, x) = w(x, \eta^{-1} x \eta, \ldots, \eta^{-(n-1)} x \eta^{n-1})$.

Following [8] we define the *polynomial* $f_w(t)$ *associated with* the cyclically presented group $G \cong G_n(w)$ as

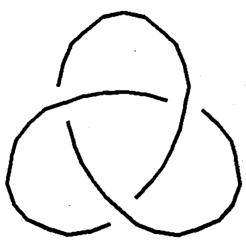
$$f_w(t) = \sum \alpha_i t^i,$$

where α_i is the exponent sum of x_i in w, $1 \leq i \leq n$.

Example 1.1. Let us consider the *Sieradski groups* defined by the presentation

$$S(n) = \langle x_1, \ldots, x_n \mid x_i x_{i+2} = x_{i+1}, \quad i = 1, \ldots, n \rangle,$$

where all indices are taken mod n. This presentation is cyclic and $w(x_i, x_{i+1}, x_{i+2}) = x_i \, x_{i+2} \, x_{i+1}^{-1}$.



Therefore,

$$f_w(t) = t^i + t^{i+2} - t^{i+1} = t^i (t^2 - t + 1).$$

We recall that $\Delta(t) = t^2 - t + 1$ is the Alexander polynomial of the *trefoil knot* [1].

Consider the 2-generator group

$$H_n(v) = \langle \eta, x \mid \eta^n = v(\eta, x) = 1 \rangle,$$

where $v(\eta, x) = w(x, \eta^{-1}x\eta, \eta^{-2}x\eta^2) = x\eta^{-2}x\eta^2(\eta^{-1}x\eta)^{-1} = x\eta^{-2}x\eta x^{-1}\eta$.

Let λ be such that $x = \eta\lambda$. Then we get

$$H_n(v) \cong \langle \lambda, \eta \mid \eta^n = \lambda^n = 1, \quad \eta\lambda\eta^{-1}\lambda\eta\lambda^{-1} = 1 \rangle,$$

and we recall that the group

$$\langle \lambda, \eta \mid \lambda\eta^{-1}\lambda = \eta^{-1}\lambda\eta^{-1} \rangle$$

is isomorphic to the fundamental group of the trefoil knot [1], where generators λ and η are meridians.

Example 1.2. Let us consider the *Fibonacci groups* defined by the presentation

$$F(2, 2n) = \langle x_1, \ldots, x_{2n} \mid x_i x_{i+1} = x_{i+2}, \quad i = 1, \ldots, 2n \rangle,$$

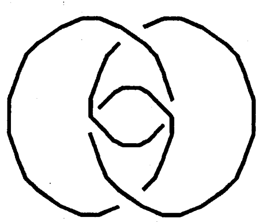
where all indices are taken mod $2n$. This presentation is cyclic and for this case $w(x_i, x_{i+1}, x_{i+2}) = x_i\, x_{i+1}\, x_{i+2}^{-1}$. Therefore

$$f_w(t) = t^i + t^{i+1} - t^{i+2} = t^i\,(-t^2 + t + 1).$$

Unfortunately, the polynomial $\Delta(t) = -t^2 + t + 1$ cannot be the Alexander polynomial of a knot [1]. So, we would like to consider another cyclic presentation of the group $F(2, 2n)$. Suppose $y_i = x_{2i}$ for $i = 1, \ldots, n$. Then $x_{2i+1} = x_{2i}^{-1}\, x_{2i+2} = y_i^{-1}\, y_{i+1}$. Hence

$$F(2, 2n) \cong \langle y_1, \ldots, y_n \mid (y_i^{-1} y_{i+1})\, y_{i+1} = (y_{i+1}^{-1} y_{i+2}) \quad i = 1, \ldots, n \rangle.$$

Thus we got the cyclic presentation with $w(y_i, y_{i+1}, y_{i+2}) = y_i^{-1} y_{i+1}^2\, y_{i+2}^{-1}\, y_{i+1}$.



Therefore

$$f_w(t) = -t^i + 2t^{i+1} - t^{i+2} + t^{i+1} = -t^i\,(t^2 - 3t + 1).$$

We recall that $\Delta(t) = t^2 - 3t + 1$ is the Alexander polynomial of the *figure-eight knot* [1].

Consider the 2-generator group

$$H_n(v) = \langle \eta\, y \quad | \quad \eta^n = v(\eta, y) = 1 \rangle,$$

where $v(\eta, y) = w(y, \eta^{-1}y\eta, \eta^{-2}y\eta^2) = y^{-1}\eta^{-1}y^2\eta^{-1}y^{-1}\eta y\eta$.

Let λ be such that $y = \eta\lambda$. Then we get

$$H_n(v) \cong \langle \lambda, \eta \quad | \quad \eta^n = \lambda^n = 1, \quad \eta^{-1}[\lambda, \eta] = [\lambda, \eta]\, \lambda \rangle,$$

where $[\lambda, \eta] = \lambda^{-1}\eta^{-1}\lambda\eta$. We recall that the group

$$\langle \lambda, \eta \quad | \quad \eta^{-1}[\lambda, \eta] = [\lambda, \eta]\, \lambda \rangle$$

is the fundamental group of the figure-eight knot [1], where generators λ and η are meridians.

2. Takahashi Manifolds

In this section we describe a series of closed orientable 3-manifolds whose fundamental groups were studied by M. Takahashi [13].

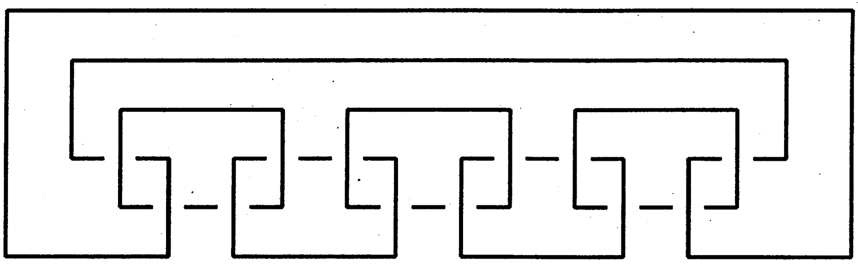
For any integer $n \geq 2$ we consider a link L_{2n} with $2n$ components, each of which is unknotted and is linked with exactly two adjacent components, similar to the figure below, where the link L_6 is pictured.

It was shown by W. Thurston [14, Section 6.8.7] that for $n \geq 3$ the link L_{2n} is hyperbolic and the hyperbolic volume of the complement $\mathcal{S}^3 \setminus L_{2n}$ is given by the formula

$$vol(\mathcal{S}^3 \setminus L_{2n}) = 8\, n \left[\Lambda \left(\frac{\pi}{4} + \frac{\pi}{2n} \right) + \Lambda \left(\frac{\pi}{4} - \frac{\pi}{2n} \right) \right],$$

where $\Lambda(x)$ is the *Lobachevsky function* [14]:

$$\Lambda(x) = - \int_0^x \ln |2 \sin \theta| \, d\theta.$$



The link L_6.

Let us cyclically enumerate components of L_{2n}, and consider closed manifolds $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ obtained by Dehn surgeries on components of L_{2n}, where a surgery coefficient p_i/q_i, $i = 1, \ldots, 2n$, corresponds to the i-th component of L_{2n}. The manifolds $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ are refer as *Takahashi manifolds.* The presentations of the fundamental groups of manifolds $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ where studied in [13] where the following nice result was obtained.

Theorem 2.1.[13] *The fundamental group of a manifold $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ has the following presentation*

$$\langle a_1, \ldots, a_{2n} \quad | \quad a_{2k-1}^{q_{2k-1}} \, a_{2k}^{-p_{2k}} = a_{2k+1}^{q_{2k+1}}, \, a_{2k}^{q_{2k}} \, a_{2k+1}^{p_{2k+1}} = a_{2k+2}^{q_{2k+2}}, \, k = 1, \ldots, n \rangle$$

where all indices are taken mod $2n$.

Let us consider the following particular cases.

Example 2.1. Assume that $p_i = q_i = 1$ for all $i = 1, \ldots, 2n$. Then the fundamental group of the manifold $M_{2n}(1, 1, \ldots, 1, 1)$ is given by

$$\begin{aligned}
&\pi_1(M_{2n}(1, 1, \ldots, 1, 1)) \\
&\quad = \langle a_1, \ldots, a_{2n} \quad | \quad a_{2k-1} \, a_{2k}^{-1} = a_{2k+1}, \, a_{2k} \, a_{2k+1} = a_{2k+2}, \, k = 1, \ldots, n \rangle \\
&\quad = \langle a_1, \ldots, a_{2n} \quad | \quad a_{2k-1} = a_{2k+1} \, a_{2k}, \, a_{2k+1} = a_{2k}^{-1} \, a_{2k+2}, \, k = 1, \ldots, n \rangle \\
&\quad = \langle a_1, \ldots, a_{2n} \quad | \quad a_{2k+1} = a_{2k+3} \, a_{2k+2}, \, a_{2k+1} = a_{2k}^{-1} \, a_{2k+2}, \, k = 1, \ldots, n \rangle.
\end{aligned}$$

Therefore $a_{2k+3} = a_{2k}^{-1}$ for all $k = 1, \dots, n$. Hence using the notation $x_i = a_{2i}$ for $i = 1, \dots, n$, we get

$$\begin{aligned}
&\pi_1(M_{2n}(1,1,\dots,1,1)) \\
&\quad = \langle x_1, \dots, x_n \mid x_{k-1}^{-1} = x_k^{-1} x_{k+1},\ k = 1, \dots, n\rangle \\
&\quad = \langle x_1, \dots, x_n \mid x_k = x_{k-1} x_{k+1},\ k = 1, \dots, n\rangle,
\end{aligned}$$

that is isomorphic to the Sieradski group $S(n)$. It was shown in [2] that groups $S(n)$ are isomorphic to fundamental groups of the n-fold cyclic coverings of the 3-sphere $\mathcal{S}^3$ branched over the trefoil knot. Indeed, the Takahashi manifold $M_{2n}(1,1,\dots,1,1)$ is homeomorphic to the n-fold cyclic covering of $\mathcal{S}^3$ branched over the trefoil knot (see discussion in [12] for small n).

Example 2.2. Assume that $q_i = 1$ for each i and $p_i = (-1)^{i+1}$, where $i = 1, \dots, 2n$. Then the fundamental group of the manifold $M_{2n}(1,-1,\dots,1,-1)$ is given by

$$\begin{aligned}
&\pi_1(M_{2n}(1,-1,\dots,1,-1)) \\
&\quad = \langle a_1, \dots, a_{2n} \mid a_{2k-1} a_{2k} = a_{2k+1},\ a_{2k} a_{2k+1} = a_{2k+2},\ k = 1, \dots, n\rangle \\
&\quad = \langle a_1, \dots, a_{2n} \mid a_i a_{i+1} = a_{i+2},\ i = 1, \dots, 2n\rangle,
\end{aligned}$$

that is isomorphic to the Fibonacci group $F(2,2n)$. It follows from [5] and [6] that the group $F(2,2n)$ is isomorphic to the fundamental group of the n-fold cyclic covering of the 3-sphere $\mathcal{S}^3$ branched over the figure–eight knot. Indeed, it was shown in [3] that the Takahashi manifold $M_{2n}(1,-1,\dots,1,-1)$ is homeomorphic to the n-fold cyclic covering of $\mathcal{S}^3$ branched over the figure-eight knot.

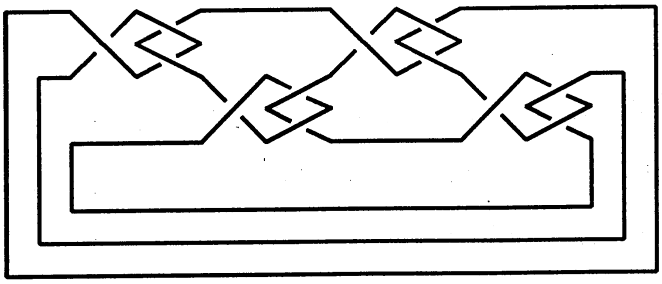
3. Two-fold branched coverings

Let us define a family of knots and links which are closely connected with the Takahashi manifolds. We recall [1] that any link can be obtained as a closed braid.

For coprime integers p and q we denote by $\sigma_i^{p/q}$ a rational p/q-tangle [1] whose incoming arcs are i-th and $(i+1)$-th strings of the braid. For $n \geq 1$ and pairs of coprime integers p_i and q_i, $i = 1, \ldots, 2n$, we denote by $K_{2n}(p_1/q_1, \ldots p_{2n}/q_{2n})$ a *closed rational 3-strings braid*

$$\sigma_1^{p_1/q_1} \sigma_2^{p_2/q_2} \cdots \sigma_1^{p_{2n-1}/q_{2n-1}} \sigma_2^{p_{2n}/q_{2n}}.$$

As an example, the diagram of the link $K_4(3/2, -3/2, 3/2, -3/2)$ is pictured below.



The link $K_4(3/2, -3/2, 3/2, -3/2)$.

There is the following connection between the Takahashi manifolds and the above links.

Theorem 3.1.[9] *Any Takahashi manifold $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ can be obtained as the two-fold branched covering of the link $K_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$.*

The proof of the theorem is based on the Montesinos algorithm [11], which admits to describe a two-fold covering presentation for a manifold obtained by Dehn surgeries on a strongly invertible link.

Because for the case $q_i = 1$ for all $i = 1, \ldots, 2n$, a rational 3-strings braid becomes an ordinary 3-strings braid, we get

Corollary 3.1. *For any closed 3-strings braid its two-fold branched covering is a Takahashi manifold.*

The following particular case of Theorem 3.1 was discussed in [12] for small n and in [2] for the general case.

Corollary 3.2. *Takahashi manifolds* $M_{2n}(1,1,\ldots,1,1)$ *are two-fold coverings of 3-strings torus knots* $T_{n,3} = (\sigma_1\sigma_2)^n$, *where in particular,* $T_{2,3}$ *is the trefoil knot* 3_1, *and* $T_{3,3} = 8_{19}$.

As a particular case of Theorem 3.1, in virtue of [3], we get the following result remarked in [10].

Corollary 3.3. *Takahashi manifolds* $M_{2n}(1,-1,\ldots,1,-1)$ *are two-fold coverings of Turks head links* $Th_n = (\sigma_1\sigma_2^{-1})^n$, *where in particular,* Th_2 *is the figure-eight knot* 4_1, Th_3 *are Borromean rings* 6_2^3, *and* Th_4 *is the knot* 8_{18}.

Because the link $K_2(p_1/q_1, p_2/q_2)$ is a connected sum of 2-bridge (p_1/q_1)–link and (p_2/q_2)–link, the manifolds $M_2(p_1/q_1, p_2/q_2)$ can be easy described.

Corollary 3.4. *Takahashi manifolds* $M_2(p_1/q_1, p_2/q_2)$ *are connected sums of lens spaces* L_{p_1,q_1} *and* $L_{p_2.q_2}$.

4. Cyclic branched coverings

In this section we consider Takahashi manifolds with cyclic symmetries. We will be say that a manifold $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ is *n-periodic* if surgery parameters are such that $p_{2i-1}/q_{2i-1} = a/b$ and $p_{2i}/q_{2i} = c/d$ for $i = 1, \ldots, n$, where a/b and c/d are some rational. In this case we will be use a notation

$$M_n(a/b, c/d) = M_{2n}(a/b, c/d, \ldots, a/b, c/d).$$

According to this notations the Fibonacci manifolds can be written in the form $M_n(1,-1)$ where $\pi_1(M_n(1,-1)) = F(2,2n)$.

Analogously, we consider n-periodic closed rational 3-strings braid $K_n(a/b, c/d)$, that is the closure of $(\sigma_1^{a/b}\sigma_2^{c/d})^n$. In particular cases we get following 2-bridge knots.

Lemma 4.1. *For any integers k and l the link* $K_n(1/k, -1/l)$ *is the 2-bridge* $(2k+\frac{1}{2l})$-*knot.*

Lemma 4.2. *For any integers k and l the link $K_n(1/k, 1/l)$ is the 2-bridge $(2k - \frac{1}{2l})$-knot.*

Similar to [7], we use the following notations for orbifolds whose singular set is a two-bridge knot or link. We denote by $(p/q)(n)$ an orbifold whose underlying space is the 3-sphere $\mathcal{S}^3$ and whose singular set is the 2-bridge p/q–knot with index n. By $(p/q)(m,n)$ we denote an orbifold whose underlying space is the 3-sphere $\mathcal{S}^3$ and whose singular set is the 2-bridge p/q–link with indices m and n corresponding to its components. By $K_n(a/b, c/d)(2)$ we denote an orbifold whose underlying space is the 3-sphere $\mathcal{S}^3$ and whose singular set is the n-periodic closed rational 3-strings braid $K_n(a/b, c/d)$ with index 2 corresponding to each component of $K_n(a/b, c/d)$.

Theorem 4.1. *Let $M_n(1/b, -1/d)$, $n \geq 2$, $b > 0$, $d > 0$, be a n-periodic Takahashi manifold. Then the following covering diagram holds:*

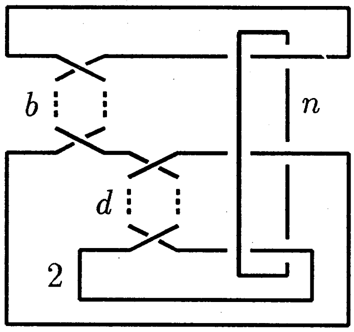
$$\begin{tikzcd}[column sep=small] & M_n(\frac{1}{b}, -\frac{1}{d}) \arrow[dl, "2"'] \arrow[dr, "n"] & \\ K_n(\frac{1}{b}, -\frac{1}{d})(2) \arrow[dr, "n"'] & & (2b + \frac{1}{2d})(n) \arrow[dl, "2"] \\ & (p/q)(2,n) & \end{tikzcd}$$

where $p = 8bd + 2$ and $0 < q < 4bd + 1$ such odd that $2dq = \pm 1 \pmod{4bd + 1}$.

Proof. The 2-fold covering

$$M_n(1/b, -1/d) \xrightarrow{2} K_n(1/b, -1/d)$$

holds by Theorem 3.1. Obviously the orbifold $K_n(1/b, -1/d)(2)$ has the symmetry ρ of order n. Consider the quotient orbifold $\mathcal{O}^{b,d}(2,n) = K_n(1/b, -1/d)(2)/\rho$. Its singular set $\mathcal{L}^{b,d}$ is the 2-component link in the figure below with indices 2 and n corresponding to components. We remark that components of $\mathcal{L}^{b,d}$ are unknotted and equivalent.



The link $\mathcal{L}^{b,d}$

Using the method from [10], we can define an epimorphism $\theta : \pi_1(\mathcal{O}^{b,d}(2,n)) \to \mathbb{Z}_2 \oplus \mathbb{Z}_n$ such that $\theta^{-1}(\mathbb{Z}_n) = \pi_1((2b + 1/2d)(n))$, $\theta^{-1}(\mathbb{Z}_2) = \pi_1(K_n(1/b, -1/d)(2))$, and $Ker(\theta) = \pi_1(M_n(1/b, -1/d))$. The covering diagram holds from the diagram of subgroups and Lemma 4.1. □

<u>Theorem 4.2.</u> *Let $M_n(1/b, 1/d)$, $n \geq 2$, $b > 1$, $d > 0$, be a n-periodic Takahashi manifold. Then the following covering diagram holds:*

$$\begin{tikzcd}
 & M_n(\frac{1}{b}, \frac{1}{d}) \arrow[dl, "2"'] \arrow[dr, "n"] & \\
K_n(\frac{1}{b}, \frac{1}{d})(2) \arrow[dr, "n"'] & & (2b - \frac{1}{2d})(n) \arrow[dl, "2"] \\
 & (p/q)(2,n) &
\end{tikzcd}$$

where $p = 8bd - 2$ and $0 < q < 4bd - 1$ such odd that $2dq = \pm 1 \pmod{4bd - 1}$.

Proof. Analogously to the proof of Theorem 4.1. □

5. Group presentations

<u>Theorem 5.1.</u> *Denote by $\mathcal{M}_n(p/q)$ the n-fold cyclic branched covering of the 2-bridge (p/q)-knot. (i) If $p/q = 2k + \frac{1}{2l}$, then*

$$\pi_1(\mathcal{M}_n(p/q)) = \langle x_1, \ldots, x_n \mid (x_i^{-l} x_{i+1}^{l})^k x_{i+1} = (x_{i+1}^{-l} x_{i+2}^{l})^k, \quad i = 1, \ldots, n \rangle.$$

(ii) If $p/q = 2k - \frac{1}{2l}$, then

$$\pi_1(\mathcal{M}_n(p/q)) = \langle y_1, \ldots, y_n \mid (y_i^{-l} y_{i+1}^{l})^k y_{i+1}^{-1} = (y_{i+1}^{-l} y_{i+2}^{l})^k, \quad i = 1, \ldots, n\rangle.$$

Proof. (i) By Theorem 4.1 the manifold $\mathcal{M}_n(2k + \frac{1}{2l})$ is the Takahashi manifold $M_n(1/k, -1/l)$ whose fundamental group can be found by Theorem 2.1:

$$\pi_1(\mathcal{M}_n(2k + 1/2l)) = \pi_1(M_n(1/k, -1/l)) =$$

$$\langle a_1, \ldots, a_{2n} \mid a_{2i+1}^k a_{2i+2} = a_{2i+3}^k, a_{2i}^l a_{2i+1} = a_{2i+2}^l, \quad i = 1, \ldots, n\rangle.$$

The formula from the statement of the theorem will be obtained if we suppose $x_i = a_{2i}$, $i = 1, \ldots, n$.

(ii) Analogously, by Theorem 4.2 the manifold $\mathcal{M}_n(2k - \frac{1}{2l})$ is the Takahashi manifold $M_n(1/k, 1/l)$ whose fundamental group can be found by Theorem 2.1:

$$\pi_1(\mathcal{M}_n(2k - 1/2l)) = \pi_1(M_n(1/k, 1/l)) =$$

$$\langle a_1, \ldots, a_{2n} \mid a_{2i+1}^k a_{2i+2}^{-1} = a_{2i+3}^k, a_{2i}^l a_{2i+1} = a_{2i+2}^l, \quad i = 1, \ldots, n\rangle.$$

The formula from the statement of the theorem will be obtained if we suppose $y_i = a_{2i}$, $i = 1, \ldots, n$. □

Corollary 5.1. *The polynomial associated with the cyclic presentation of $\pi_1(\mathcal{M}_n)(p/q)$ from Theorem 5.1 is equivalent to the Alexander polynomial of the two-bridge p/q-knot: if $p/q = 2k + \frac{1}{2l}$, then*

$$\Delta(t) = klt^2 - (2kl + 1)t + kl,$$

and if $p/q = 2k - \frac{1}{2l}$, then

$$\Delta(t) = klt^2 - (2kl - 1)t + kl.$$

In conclusion we remark that the present paper was inspired in part by the nice paper of M. Dunwoody [4], where he constructed a family of 3-manifolds whose fundamental groups are cyclically presented, and asked are these manifolds cyclic branched coverings of knots or links.

It is easy to check that all cyclically presented groups from [4] with $w = w(x_i, x_{i+1}, x_{i+2})$ are of the type (i) or of the type (ii) from Theorem 5.1 for some k and l.

Corollary 5.2. *Each cyclically presented group from [4] with $w = w(x_i, x_{i+1}, x_{i+2})$ is isomorphic to the fundamental group of the cyclic branched covering of the 2-bridge $(2k + \frac{1}{2l})$-knot or $(2k - \frac{1}{2l})$-knot for some k and l.*

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