CYCLICALLY PRESENTED GROUPS AND TAKAHASHI MANIFOLDS

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Abstract

We consider groups with cyclic presentations which arise as fundamental groups of a family of closed 3-dimensional manifolds. These manifolds were firstly described by Takahashi using Dehn surgeries on links. We demonstrate that the cyclic automorphisms of groups induce cyclic coverings of the 3-sphere branched over 2-bridge knots. Moreover, polynomials associated with cyclic presentations are equal to Alexander polynomials of corresponding knots.

Keywords: fundamental group, 3-manifold, cyclic covering.

1. CYCLICALLY PRESENTED GROUPS

The cyclically presented groups comprise a rich source of groups which are interesting from a topological point of views. The connection between cyclically presented groups and cyclic branched coverings of knots and links was studied, in particular, in [2], [4], [5] and [6].

Let $F_n = \langle x_1, \ldots, x_n \mid \rangle$ be the free group of rank $n$ and $\eta : F_n \to F_n$ be an automorphism of order $n$ such that $\eta(x_i) = x_{i+1}$ for $i = 1, \ldots, n$, where all indices by mod $n$. We recall [8, §9] that for a reduced word $w \in F_n$ the cyclically presented group $G_n(w)$ is given by

$$G_n(w) = \langle x_1, \ldots, x_n \mid w, \eta(w), \ldots, \eta^{n-1}(w) \rangle.$$
A group $G$ is said to have a cyclic presentation if $G \cong G_n(w)$ for some $n$ and $w$. Clearly, the automorphism $\eta$ of $F_n$ induces an automorphism of $G_n(w)$. Such automorphism defines an action of the group $\mathbb{Z}_n = \langle \eta \mid \eta^n = 1 \rangle$ on $G_n(w)$. Let us consider a split extension $H_n = G_n(w) \lambda \mathbb{Z}_n$. The group $H_n$ is said to be a natural extension of a cyclically presented group. It was remarked in [4] that the group $H_n = H_n(v)$ always has a 2-generator, 2-relator presentation of the form

$$H_n(v) \cong \langle \eta, x \mid \eta^n = v(\eta, x) = 1 \rangle,$$

where $v = v(\eta, x) = w(x, \eta^{-1}x\eta, \ldots, \eta^{-1+(n-1)}x\eta^{n-1})$.

Following [8] we define the polynomial $f_w(t)$ associated with the cyclically presented group $G \cong G_n(w)$ as

$$f_w(t) = \sum \alpha_i t^i,$$

where $\alpha_i$ is the exponent sum of $x_i$ in $w$, $1 \leq i \leq n$.

Example 1.1. Let us consider the Sieradski groups defined by the presentation

$$S(n) = \langle x_1, \ldots, x_n \mid x_ix_{i+2} = x_{i+1}, i = 1, \ldots, n \rangle,$$

where all indices are taken mod $n$. This presentation is cyclic and $w(x_i, x_{i+1}, x_{i+2}) = x_i x_{i+2} x_{i+1}^{-1}$.

Therefore,

$$f_w(t) = t^i + t^{i+2} - t^{i+1} = t^i(t^2 - t + 1).$$

We recall that $\Delta(t) = t^2 - t + 1$ is the Alexander polynomial of the trefoil knot [1].

Consider the 2-generator group

$$H_n(v) = \langle \eta, x \mid \eta^n = v(\eta, x) = 1 \rangle,$$
where $v(\eta, x) = w(x, \eta^{-1}x\eta, \eta^{-2}x\eta^2) = x\eta^{-2}x\eta^2(\eta^{-1}x\eta)^{-1} = x\eta^{-2}x\eta x^{-1}\eta$.

Let $\lambda$ be such that $x = \eta\lambda$. Then we get

$$H_n(v) \cong \langle \lambda, \eta \mid \eta^n = \lambda^n = 1, \ \eta\lambda^{-1}\lambda\eta^{-1} = 1 \rangle,$$

and we recall that the group

$$\langle \lambda, \eta \mid \lambda\eta^{-1}\lambda = \eta^{-1}\lambda\eta^{-1} \rangle$$

is isomorphic to the fundamental group of the trefoil knot [1], where generators $\lambda$ and $\eta$ are meridians.

**Example 1.2.** Let us consider the Fibonacci groups defined by the presentation

$$F(2, 2n) = \langle x_1, \ldots, x_{2n} \mid x_i x_{i+1} = x_{i+2}, \ i = 1, \ldots, 2n \rangle,$$

where all indices are taken mod $2n$. This presentation is cyclic and for this case $w(x_i, x_{i+1}, x_{i+2}) = x_i x_{i+1} x_{i+2}^{-1}$. Therefore

$$f_w(t) = t^i + t^{i+1} - t^{i+2} = t^i (-t^2 + t + 1).$$

Unfortunately, the polynomial $\Delta(t) = -t^2 + t + 1$ cannot be the Alexander polynomial of a knot [1]. So, we would like to consider another cyclic presentation of the group $F(2, 2n)$. Suppose $y_i = x_{2i}$ for $i = 1, \ldots, n$. Then $x_{2i+1} = x_{2i}^{-1} x_{2i+2} = y_{i}^{-1} y_{i+1}$. Hence

$$F(2, 2n) \cong \langle y_1, \ldots, y_n \mid (y_i^{-1} y_{i+1}) y_{i+1} = (y_{i+1}^{-1} y_{i+2}) i = 1, \ldots, n \rangle.$$

Thus we got the cyclic presentation with $w(y_i, y_{i+1}, y_{i+2}) = y_i^{-1} y_{i+1}^{-1} y_{i+2}^{-1}$. Therefore

$$f_w(t) = -t^i + 2t^{i+1} - t^{i+2} + t^{i+1} = -t^i (t^2 - 3t + 1).$$

We recall that $\Delta(t) = t^2 - 3t + 1$ is the Alexander polynomial of the figure-eight knot [1].
Consider the 2-generator group

\[ H_n(v) = \langle \eta y \mid \eta^n = v(\eta, y) = 1 \rangle, \]

where \( v(\eta, y) = w(y, \eta^{-1}y\eta, \eta^{-2}y\eta^2) = y^{-1}\eta^{-1}y^2\eta^{-1}y\eta y\eta. \)

Let \( \lambda \) be such that \( y = \eta\lambda \). Then we get

\[ H_n(v) \cong \langle \lambda, \eta \mid \eta^n = \lambda^n = 1, \eta^{-1}[\lambda, \eta] = [\lambda, \eta] \lambda \rangle, \]

where \([\lambda, \eta] = \lambda^{-1}\eta^{-1}\lambda\eta\). We recall that the group

\[ \langle \lambda, \eta \mid \eta^{-1}[\lambda, \eta] = [\lambda, \eta] \lambda \rangle \]

is the fundamental group of the figure-eight knot [1], where generators \( \lambda \) and \( \eta \) are meridians.

2. Takahashi Manifolds

In this section we describe a series of closed orientable 3-manifolds whose fundamental groups were studied by M. Takahashi [13].

For any integer \( n \geq 2 \) we consider a link \( L_{2n} \) with \( 2n \) components, each of which is unknotted and is linked with exactly two adjacent components, similar to the figure below, where the link \( L_6 \) is pictured.

It was shown by W. Thurston [14, Section 6.8.7] that for \( n \geq 3 \) the link \( L_{2n} \) is hyperbolic and the hyperbolic volume of the complement \( S^3 \setminus L_{2n} \) is given by the formula

\[ \text{vol}(S^3 \setminus L_{2n}) = 8n \left[ \Lambda \left( \frac{\pi}{4} + \frac{\pi}{2n} \right) + \Lambda \left( \frac{\pi}{4} - \frac{\pi}{2n} \right) \right], \]

where \( \Lambda(x) \) is the Lobachevsky function [14]:

\[ \Lambda(x) = -\int_0^x \ln |2\sin \theta| d\theta. \]
Let us cyclically enumerate components of $L_{2n}$, and consider closed manifolds $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ obtained by Dehn surgeries on components of $L_{2n}$, where a surgery coefficient $p_i/q_i$, $i = 1, \ldots, 2n$, corresponds to the $i$-th component of $L_{2n}$. The manifolds $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ are refer as Takahashi manifolds. The presentations of the fundamental groups of manifolds $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ where studied in [13] where the following nice result was obtained.

**Theorem 2.1.** [13] The fundamental group of a manifold $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ has the following presentation

$$\langle a_1, \ldots, a_{2n} \mid a_{2k-1}^{q_{2k-1}} a_{2k}^{-p_{2k}} a_{2k+1}^{q_{2k+1}}, a_{2k}^{q_{2k}} a_{2k+1}^{p_{2k+1}} = a_{2k+2}^{q_{2k+2}}, k = 1, \ldots, n \rangle$$

where all indices are taken mod $2n$.

Let us consider the following particular cases.

**Example 2.1.** Assume that $p_i = q_i = 1$ for all $i = 1, \ldots, 2n$. Then the fundamental group of the manifold $M_{2n}(1,1, \ldots, 1,1)$ is given by

$$\pi_1(M_{2n}(1,1, \ldots, 1,1)) = \langle a_1, \ldots, a_{2n} \mid a_{2k-1} a_{2k}^{-1} a_{2k+1}, a_{2k} a_{2k+1} = a_{2k+2}, k = 1, \ldots, n \rangle$$

$$= \langle a_1, \ldots, a_{2n} \mid a_{2k-1} = a_{2k+1} a_{2k}, a_{2k+1} = a_{2k}^{-1} a_{2k+2}, k = 1, \ldots, n \rangle$$

$$= \langle a_1, \ldots, a_{2n} \mid a_{2k+1} = a_{2k+3} a_{2k+2}, a_{2k+1} = a_{2k}^{-1} a_{2k+2}, k = 1, \ldots, n \rangle.$$
Therefore $a_{2k+3} = a_{2k}^{-1}$ for all $k = 1, \ldots, n$. Hence using the notation $x_i = a_{2i}$ for $i = 1, \ldots, n$, we get

$$
\pi_1(M_{2n}(1,1,\ldots,1,1)) = \langle x_1, \ldots, x_n \mid x_{k-1}^{-1} = x_k^{-1} x_{k+1}, k = 1, \ldots, n \rangle \cong \langle x_1, \ldots, x_n \mid x_k = x_{k-1} x_{k+1}, k = 1, \ldots, n \rangle,
$$

that is isomorphic to the Sieradski group $S(n)$. It was shown in [2] that groups $S(n)$ are isomorphic to fundamental groups of the $n$-fold cyclic coverings of the 3-sphere $S^3$ branched over the trefoil knot. Indeed, the Takahashi manifold $M_{2n}(1,1,\ldots,1,1)$ is homeomorphic to the $n$-fold cyclic covering of $S^3$ branched over the trefoil knot (see discussion in [12] for small $n$).

**Example 2.2.** Assume that $q_i = 1$ for each $i$ and $p_i = (-1)^{i+1}$, where $i = 1, \ldots, 2n$. Then the fundamental group of the manifold $M_{2n}(1, -1, \ldots, 1, -1)$ is given by

$$
\pi_1(M_{2n}(1, -1, \ldots, 1, -1)) = \langle a_1, \ldots, a_{2n} \mid a_{2k-1} a_{2k} = a_{2k+1}, a_{2k} a_{2k+1} = a_{2k+2}, k = 1, \ldots, n \rangle \cong \langle a_1, \ldots, a_{2n} \mid a_i a_{i+1} = a_{i+2}, i = 1, \ldots, 2n \rangle,
$$

that is isomorphic to the Fibonacci group $F(2,2n)$. It follows from [5] and [6] that the group $F(2,2n)$ is isomorphic to the fundamental group of the $n$-fold cyclic covering of the 3-sphere $S^3$ branched over the figure-eight knot. Indeed, it was shown in [3] that the Takahashi manifold $M_{2n}(1, -1, \ldots, 1, -1)$ is homeomorphic to the $n$-fold cyclic covering of $S^3$ branched over the figure-eight knot.

### 3. Two-fold branched coverings

Let us define a family of knots and links which are closely connected with the Takahashi manifolds. We recall [1] that any link can be obtained as a closed braid.
For coprime integers $p$ and $q$ we denote by $\sigma_i^{p/q}$ a rational $p/q$-tangle [1] whose incoming arcs are $i$-th and $(i + 1)$-th strings of the braid. For $n \geq 1$ and pairs of coprime integers $p_i$ and $q_i$, $i = 1, \ldots, 2n,$ we denote by $K_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ a closed rational 3-strings braid

$$\sigma_1^{p_1/q_1} \sigma_2^{p_2/q_2} \cdots \sigma_1^{p_{2n-1}/q_{2n-1}} \sigma_2^{p_{2n}/q_{2n}}.$$ 

As an example, the diagram of the link $K_4(3/2, -3/2, 3/2, -3/2)$ is pictured below.

![Link Diagram](image)

The link $K_4(3/2, -3/2, 3/2, -3/2)$.

There is the following connection between the Takahashi manifolds and the above links.

**Theorem 3.1.[9]** Any Takahashi manifold $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ can be obtained as the two-fold branched covering of the link $K_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$.

The proof of the theorem is based on the Montesinos algorithm [11], which admits to describe a two-fold covering presentation for a manifold obtained by Dehn surgeries on a strongly invertible link.

Because for the case $q_i = 1$ for all $i = 1, \ldots, 2n$, a rational 3-strings braid becomes an ordinary 3-strings braid, we get

**Corollary 3.1.** For any closed 3-strings braid its two-fold branched covering is a Takahashi manifold.

The following particular case of Theorem 3.1 was discussed in [12] for small $n$ and in [2] for the general case.
Corollary 3.2. Takahashi manifolds $M_{2n}(1, 1, \ldots, 1, 1)$ are two-fold coverings of 3-strings torus knots $T_{n,3} = (\sigma_1 \sigma_2)^n$, where in particular, $T_{2,3}$ is the trefoil knot $3_1$, and $T_{3,3} = 8_{19}$.

As a particular case of Theorem 3.1, in virtue of [3], we get the following result remarked in [10].

**Corollary 3.3.** Takahashi manifolds $M_{2n}(1, -1, \ldots, 1, -1)$ are two-fold coverings of Turks head links $Th_{n} = (\sigma_1 \sigma_2^{-1})^n$, where in particular, $Th_{2}$ is the figure-eight knot $4_1$, $Th_{3}$ are Borromean rings $6_{2}^{3}$, and $Th_{4}$ is the knot $8_{18}$.

Because the link $K_2(p_1/q_1, p_2/q_2)$ is a connected sum of 2-bridge $(p_1/q_1)$-link and $(p_2/q_2)$-link, the manifolds $M_2(p_1/q_1, p_2/q_2)$ can be easy described.

**Corollary 3.4.** Takahashi manifolds $M_2(p_1/q_1, p_2/q_2)$ are connected sums of lens spaces $L_{p_1,q_1}$ and $L_{p_2,q_2}$.

### 4. Cyclic branched coverings

In this section we consider Takahashi manifolds with cyclic symmetries. We will be say that a manifold $M_{2n}(p_1/q_1, \ldots, p_{2n}/q_{2n})$ is $n$-periodic if surgery parameters are such that $p_{2i-1}/q_{2i-1} = a/b$ and $p_{2i}/q_{2i} = c/d$ for $i = 1, \ldots, n$, where $a/b$ and $c/d$ are some rational. In this case we will be use a notation

$$M_n(a/b, c/d) = M_{2n}(a/b, c/d, \ldots, a/b, c/d).$$

According to this notations the Fibonacci manifolds can be written in the form $M_n(1, -1)$ where $\pi_1(M_n(1, -1)) = F(2, 2n)$.

Analogously, we consider $n$-periodic closed rational 3-strings braid $K_n(a/b, c/d)$, that is the closure of $(\sigma_1^{a/b} \sigma_2^{c/d})^n$. In particular cases we get following 2-bridge knots.

**Lemma 4.1.** For any integers $k$ and $l$ the link $K_n(1/k, -1/l)$ is the 2-bridge $(2k + \frac{1}{2l})$-knot.
Lemma 4.2. For any integers $k$ and $l$ the link $K_n(1/k, 1/l)$ is the 2-bridge $(2k - \frac{1}{2l})$-knot.

Similar to [7], we use the following notations for orbifolds whose singular set is a two-bridge knot or link. We denote by $(p/q)(n)$ an orbifold whose underlying space is the 3-sphere $S^3$ and whose singular set is the 2-bridge $p/q$-knot with index $n$. By $(p/q)(m, n)$ we denote an orbifold whose underlying space is the 3-sphere $S^3$ and whose singular set is the 2-bridge $p/q$-link with indices $m$ and $n$ corresponding to its components. By $K_n(a/b, c/d)(2)$ we denote an orbifold whose underlying space is the 3-sphere $S^3$ and whose singular set is the $n$-periodic closed rational 3-strings braid $K_n(a/b, c/d)$ with index 2 corresponding to each component of $K_n(a/b, c/d)$.

Theorem 4.1. Let $M_n(1/b, -1/d)$, $n \geq 2$, $b > 0$, $d > 0$, be a $n$-periodic Takahashi manifold. Then the following covering diagram holds:

$$
\begin{array}{c}
M_n(\frac{1}{b}, -\frac{1}{d}) \\
\downarrow 2 \downarrow n \\
K_n(\frac{1}{b}, -\frac{1}{d})(2) \quad (2b + \frac{1}{2d})(n) \\
\downarrow n \downarrow 2 \\
(p/q)(2, n)
\end{array}
$$

where $p = 8bd + 2$ and $0 < q < 4bd + 1$ such odd that $2dq = \pm 1 \pmod{4bd + 1}$.

Proof. The 2-fold covering

$$
M_n(1/b, -1/d) \xrightarrow{2} K_n(1/b, -1/d)
$$

holds by Theorem 3.1. Obviously the orbifold $K_n(1/b, -1/d)(2)$ has the symmetry $\rho$ of order $n$. Consider the quotient orbifold $\mathcal{O}^{b,d}(2, n) = K_n(1/b, -1/d)(2)/\rho$. Its singular set $\mathcal{L}^{b,d}$ is the 2-component link in the figure below with indices 2 and $n$ corresponding to components. We remark that components of $\mathcal{L}^{b,d}$ are unknotted and equivalent.
Using the method from [10], we can define an epimorphism \( \theta : \pi_1(\mathcal{O}^{b,d}(2,n)) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_n \) such that \( \theta^{-1}(\mathbb{Z}_n) = \pi_1((2b + 1/2d)(n)) \), \( \theta^{-1}(\mathbb{Z}_2) = \pi_1(K_n(1/b, -1/d)(2)) \), and \( \text{Ker}(\theta) = \pi_1(M_n(1/b, -1/d)) \). The covering diagram holds from the diagram of subgroups and Lemma 4.1. \( \square \)

**Theorem 4.2.** Let \( M_n(1/b, 1/d), n \geq 2, b > 1, d > 0 \), be a \( n \)-periodic Takahashi manifold. Then the following covering diagram holds:

![Diagram](image)

\( \pi_1(M_n(p/q)) \) is the \( n \)-fold cyclic branched covering of the 2-bridge \( (p/q) \)-knot. (i) If \( p/q = 2k + \frac{1}{2l} \), then

\[
\pi_1(M_n(p/q)) = \langle x_1, \ldots, x_n | (x_i^{-l}x_{i+1}^l)^k x_{i+1} = (x_{i+1}^{-l}x_{i+2}^l)^k, \ i = 1, \ldots, n \rangle.
\]
(ii) If $p/q = 2k - \frac{1}{2l}$, then

$$\pi_1(\mathcal{M}_n(p/q)) = \langle y_1, \ldots, y_n \mid (y_i^{-l}y_{i+1})^k y_i^{-1} = (y_{i+1}^{-l}y_{i+2})^k, \ i = 1, \ldots, n \rangle.$$ 

Proof. (i) By Theorem 4.1 the manifold $\mathcal{M}_n(2k + \frac{1}{2l})$ is the Takahashi manifold $M_n(1/k, -1/l)$ whose fundamental group can be found by Theorem 2.1:

$$\pi_1(\mathcal{M}_n(2k + 1/2l)) = \pi_1(M_n(1/k, -1/l)) =$$

$$\langle a_1, \ldots, a_{2n} \mid a_{2i+1}^k a_{2i+2} = a_{2i+3}^k, a_{2i} a_{2i+1} = a_{2i+2}^{l}, \ i = 1, \ldots, n \rangle.$$ 

The formula from the statement of the theorem will be obtained if we suppose $x_i = a_{2i}, i = 1, \ldots, n$.

(ii) Analogously, by Theorem 4.2 the manifold $\mathcal{M}_n(2k - \frac{1}{2l})$ is the Takahashi manifold $M_n(1/k, 1/l)$ whose fundamental group can be found by Theorem 2.1:

$$\pi_1(\mathcal{M}_n(2k - 1/2l)) = \pi_1(M_n(1/k, 1/l)) =$$

$$\langle a_1, \ldots, a_{2n} \mid a_{2i+1}^k a_{2i+2}^{-1} = a_{2i+3}^k, a_{2i} a_{2i+1} = a_{2i+2}^{l}, \ i = 1, \ldots, n \rangle.$$ 

The formula from the statement of the theorem will be obtained if we suppose $y_i = a_{2i}, i = 1, \ldots, n$. □

Corollary 5.1. The polynomial associated with the cyclic presentation of $\pi_1(\mathcal{M}_n)(p/q)$ from Theorem 5.1 is equivalent to the Alexander polynomial of the two-bridge $p/q$-knot: if $p/q = 2k + \frac{1}{2l}$, then

$$\Delta(t) = klt^2 - (2kl + 1)t + kl,$$

and if $p/q = 2k - \frac{1}{2l}$, then

$$\Delta(t) = klt^2 - (2kl - 1)t + kl.$$

In conclusion we remark that the present paper was inspired in part by the nice paper of M. Dunwoody [4], where he constructed a family of 3-manifolds whose
fundamental groups are cyclically presented, and asked are these manifolds cyclic branched coverings of knots or links.

It is easy to check that all cyclically presented groups from [4] with \( w = w(x_i, x_{i+1}, x_{i+2}) \) are of the type (i) or of the type (ii) from Theorem 5.1 for some \( k \) and \( l \).

Corollary 5.2. Each cyclically presented group from [4] with \( w = w(x_i, x_{i+1}, x_{i+2}) \) is isomorphic to the fundamental group of the cyclic branched covering of the 2-bridge \((2k + \frac{1}{2l})\)-knot or \((2k - \frac{1}{2l})\)-knot for some \( k \) and \( l \).

References


