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GENERATORS AND RELATIONS FOR THE MAPPING CLASS GROUP OF THE HANDLEBODY OF GENUS 2

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Introduction

A *handlebody* of genus g, H_g, is an orientable 3-manifold, which is constructed from a 3-ball with attaching g 1-handles. Let $Diff^+(H_g)$ be the group of orientation preserving diffeomorphisms on H_g, $\mathcal{H}_g$ be a group which consists of isotopy classes of $Diff^+(H_g)$. The groups $\mathcal{H}_g$ are interesting objects because of their relationships with Heegaard splittings of 3-manifolds and outer automorphism group of free groups. In this paper, we give a presentation of $\mathcal{H}_2$. Before we state this presentation, we set notations used there. We indicate an element of $Diff^+(H_g)$ by figure like left hand side of Figure 1, in this figure, the left hand side figure denotes an element given in the right hand side figure.

The symbol $\rightleftarrows$ means *commute with.* If L, M, N are any elements of $\mathcal{H}_g$, a relation $L \rightleftarrows M, N$ means that $LM = ML$, $LN = NL$. In this paper, we consider that the group $\mathcal{H}_g$ acting on H_g from the right.

Theorem 1. *Let a, b, c, d, t, e, f be the elements of $\pi_0(Diff^+(H_2))$ by the elements of $Diff^+(H_2)$ indicated in Figure 2. The group $\pi_0(Diff^+(H_2))$ admits a presentation with generators a, b, c, d, t, e, f and defining relations,*

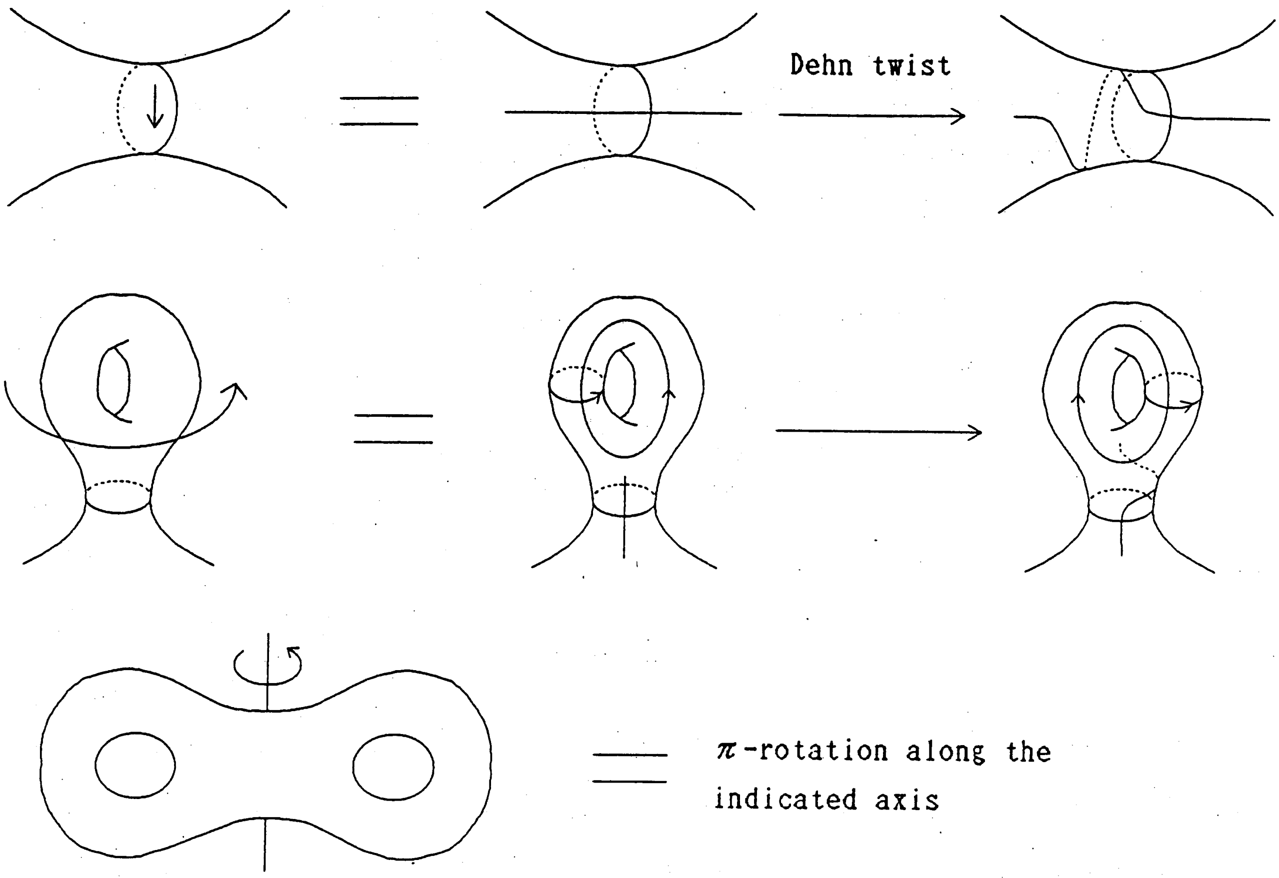

FIGURE 1

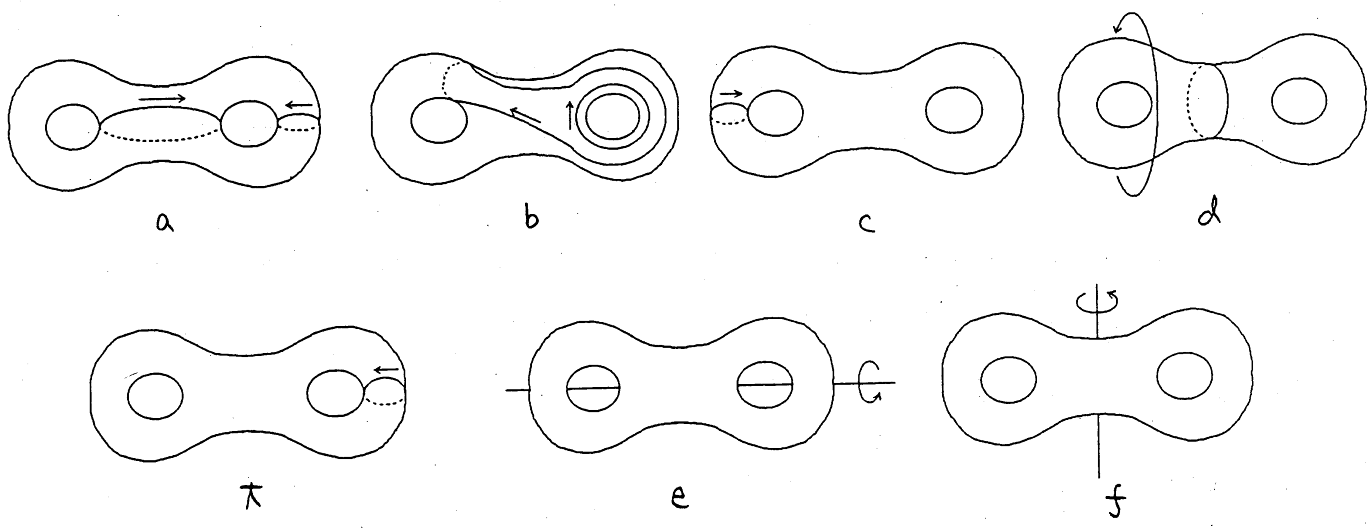

FIGURE 2

$$e^2 = f^2 = 1,$$
$$fbfb^{-1}fb = ca^{-1}de, aba^{-1}b^{-1} = d^2c^2,$$
$$dad^{-1} = d^2c^2a^{-1}, db^{-1}d^{-1} = d^2c^2b,$$
$$t^{-1}bt = ba, ftf = c^{-1}, e = fd^{-1}fd,$$
$$c \rightleftarrows a, b, d,$$
$$e \rightleftarrows a, b, c, d, t, f,$$
$$t \rightleftarrows a, c, d,$$
$$f \rightleftarrows a^{-1}t,\ d^2.$$

Contents are as follows: in section 1, we set notations, and review method, by Brown, to get a presentation of a group which act simplicially on a simply connected CW-complex, and introduce a simply connected CW-complex where $\mathcal{H}_2$ acts. In section 2, we obtain a presentation of subgroups of H_2 which is used to prove Theorem 1. In section 3, we prove Theorem 1. In section 4, we check this presentation by showing that there is a surjection from $\mathcal{H}_2$ to $SL(2, \mathbb{Z})$ and injection which is the inverse of this surjection.

1. Preliminaries

In this section, we set notations and review tools used in this paper.

1. Notations. Let X be any oriented manifold, and $K_1, \ldots, K_n, K_{n+1}$ be subsets of X. We introduce a notation as follows,

$$Diff^+(X, K_1, \ldots, K_n, rel(K_{n+1})) = \left\{ \begin{array}{c} \varphi : \text{an orientation preserving} \\ \text{self-diffeomorphism on } X, \\ \text{such that } \varphi(K_i) = K_i \\ \text{for } 1 \leq i \leq n, \varphi|_{K_{n+1}} = id \end{array} \right\}.$$

For abbreviation, we denote $Diff^+(X, K_1, \ldots, K_n, rel(\phi))$ by $Diff^+(X, K_1, \ldots, K_n)$, denote $Diff^+(X, \phi, rel(K_1))$ by $Diff^+(X, rel(K_1))$, and denote $Diff^+(X, \phi, rel(\phi))$

by $Diff^+(X)$. The set $Diff^+(X, K_1, \dots, K_n, rel(K_{n+1}))$ has a natural group structure. In this paper, we consider that the group $Diff^+(X, K_1, \dots, K_n, rel(K_{n+1}))$ acts on X from the right, that is, for elements φ_1 and φ_2 of $Diff^+(X, K_1, \dots, K_n, rel(K_{n+1}))$, $\varphi_1\varphi_2$ means that apply φ_1 first, and apply φ_2. The group $\pi_0(Diff^+(X, K_1, \dots, K_n, rel(K_{n+1})))$ consists of the isotopy classes of $Diff^+(X, K_1, \dots, K_n, rel(K_{n+1}))$ and its group law is induced from that of $Diff^+(X, K_1, \dots, K_n, rel(K_{n+1}))$. Especially, we denote $\pi_0(Diff^+(H_g))$ by $\mathcal{H}_g$. In this paper, we give a presentation of $\mathcal{H}_2$. We denote by $< a_1, \dots, a_n| >$ a free group generated by $a_1, \dots, a_n$.

2. Brown's method [Br]. Let G be a group, and X be a simply connected CW-complex on which G acts as cellular homeomorphisms. We call this X as simply connected G-CW-*complex*. In this paper, we regard the action of G as a right action. There is a method introduced by Brown [Br] to get a presentation of G from a simply connected G-CW-complex. In this subsection, we review this method.

At first, we need to introduce some notations and terminologies. For any CW-complex C, a 0-cell of C is called a *vertex*. A 1-cell of C with an orientation is called an *edge*. Any edge e has an initial vertex $o(e)$ and a final vertex $t(e)$. The notation $\bar{e}$ denote the edge corresponding to the same 1-cell as e and whose orientation is opposite to e. $V(C)$ denote the set of vertices of C, $\Sigma(C)$ denote the set of 1-cells of C, and $E(C)$ denote the set of edges of C.

From here to the end of this subsection, X denote a G-CW-complex. A 1-cell σ of X is called *inverted* if there is an element g of G such that $\sigma g = \sigma$ and reverse the orientation of σ. The notation $\tilde{\Sigma}^-$ denote the set of inverted 1-cells of X, and $\tilde{\Sigma}^+$ denote the set of non-inverted 1-cells of X. A simply conected CW-complex which consists of 0-cells and 1-cells is called a *tree*. A subtree T of X is called *a tree of reprsentatives* if $V(T)$ is the set of representatives of $V(X)/G$, and each element of $\Sigma(T)$ is an element of $\tilde{\Sigma}^+$. A tree of representatives of X is a 'fundamental domain' of the action of G on X. We can give an orientation for each element of $\tilde{\Sigma}^+$ such that these are preserved by the action of G. A subset P of $E(X)$ given in the above manner, is called *orientation* of X. Let E^+ be the set of representations of P/G such that for each $e \in E^+$ $o(e)$ is an element of E^+ and for each 1-cell of T given proper orientation is an element of E^+. Let Σ^+ be the set of 1-cells of X which correspond to the elements of E^+. Let E^- be the set of representatives of $\tilde{\Sigma}^+/G$ with an orientation such that for each $e \in E^-$, $o(e) \in V(T)$. Let Σ^- be the set of 1-cells of X which correspond to the elements of E^-. For each $e \in E^+$, $t(e)$ may not be an element of $V(T)$, but there is one and only one element of $V(T)$ in the orbit of $t(e)$ by the action of G. We denote this vertex by $w(e)$. By the definition of $w(e)$, there is at least one and not unique element g_e of G such that $w(e)g_e = t(e)$. When $e \in E(T)$, we choose $g_e = 1$. For any $g \in G_{t(e)}$, $g_e g g_e^{-1} \in G_{w(e)}$. Hence, we can define isomorphism c_e from $G_{t(e)}$ to $G_{w(e)}$ by $c_e(g) = g_e g g_e^{-1}$. Any element of G_e preserve $t(e)$, so we can naturally consider G_e as a subgroup of $G_{t(e)}$. We can define in the natural way the injection from G_e to $G_{w(e)}$, we denote this injection also by c_e. For the sake of giving the presentation, we need to present $g_e g g_e^{-1}$ as an element of $G_{t(e)}$ for each generator of G_e.

Each edge ϵ of X, such that $o(\epsilon) \in V(T)$, fall into the following three cases: (1) ϵ corresponds to a 1-cell in $\tilde{\Sigma}^+$ and there is an element $e \in E^+$, and an element $g \in G$ such that $eg = \epsilon$, (2) ϵ corresponds to a 1-cell in $\tilde{\Sigma}^+$ and there is an element $e \in E^+$, and an element $g \in G$ such that $\bar{e}g = \epsilon$, (3) ϵ corresponds to a 1-cell in $\tilde{\Sigma}^-$. In these cases, ϵ is as indicated in the following figures.

(1) $v \xrightarrow[eh=\epsilon]{} w(e)g_e h$ $\quad (h \in G_v,\ e \in E^+)$

(2) $w \xrightarrow[\bar{e}g_e^{-1}h=\epsilon]{} o(e)g_e^{-1}h$ $\quad (h \in G_w,\ e \in E^+)$

(3) $v \xrightarrow[eh=\epsilon]{} v t h$ $\quad (h \in G_v,\ t \in E^-,\ t \in G_\sigma$ reverse the orientation of $\sigma)$

We can consider ϵ as a bridge between T and Tg for some $g \in G$. The above figure indicates the way how to give this g for each ϵ. In (1), $g = g_e h$, in (2), $g = g_e^{-1}h$, in (3), $g = th$. Let α be a colsed path in X such that whose base point v_0 is in

$V(T)$. We choose an element g_α of G as follows. This path is a sequence of edges $e'_1 e'_2 \cdots e'_n$. The first one e'_1 is an edge such that $o(e'_1) = v_0 \in V(T)$, we can obtain elements $v_1 \in V(T)$, $g_1 \in G$ such that $v_1 g_1 = t(e'_1)$ in the above manner. The initial vertex of $e_2 = e'_2 g_1^{-1}$ is $v_1 \in V(T)$. Hence, in the same way, we can obtain elements $v_2 \in V(T)$, $g_2 \in G$ such that $v_2 g_2 = t(e_2)$. This means $t(e'_2) = v_2 g_2 g_1$. We continue this construction successively for other $e'_3, e'_4, \ldots$ and e'_n, then we can get a sequence $g_1, g_2, \ldots g_n$ of elements of G such that $v_n g_n \cdots g_2 g_1 = t(e'_n)$. In our situation, α is a loop, so $v_n = v_0$. Hence, $g_\alpha = g_n \cdots g_2 g_1$ is a element of G_{v_0}. Let $\hat{G} = (\underset{v \in V(T)}{*} G_v) * (\underset{\sigma \in \Sigma^-}{*} G_\sigma) * (\underset{e \in E^+}{*} < \hat{g}_e >)$. In the above construction, g_i is a product of g_e, $t \in G_\sigma$ and $h \in G_v$. The element $\hat{g}_i$ of $\hat{G}$ is given from g_i with replacing g_e with $\hat{g}_e$. Let F be the set of representatives of 2-cells of X modulo G such that, for each $\tau \in F$, $\partial\tau$ go through $V(T)$. In the above way, we construct $\hat{g}_{\partial\tau}$. By the following theorem, we can obtain a presentation of G.

Theorem [Br]. *In the above situation, G is presented as $\hat{G}$ with the following relations:*

(1) for $e \in E(T)$, $\hat{g}_e = 1$,

(2) for each $e \in E^+$ and $g \in G_e$, $\hat{g}_e i_e(g) \hat{g}_e^{-1} = c_e(g)$, where $i_e : G_e \hookrightarrow G_{o(e)}$ is the inclusion and $c_e : G_e \to G_{w(e)}$ is the injection given above,

(3) for each $e \in E^-$ and $g \in G_e$, $i_e(g) = j_e(g)$, where $i_e : G_e \hookrightarrow G_{o(e)}$, $j_e : G_e \hookrightarrow G_\sigma$ are inclusions,

(4) for each $\tau \in F$, $\hat{g}_{\partial\tau} = g_{\partial\tau}$. □

3. The disk complex [Jo]. In this subsection, we introduce simply connected CW-complex, where the group $\mathcal{H}_2$ acts simplicially.

The *disk complex* $\Delta(H_2)$ of H_2 is the simplicial complex whose m-simplices are isotopy classes of $(m+1)$-tuples $(D_0, D_1, \ldots, D_m)$ of essential and pairwise non-isotopic disjoint disks. In H_2, there is no more than three disks which define a simplex. Hence, $\Delta(H_2)$ is a 2-dimensional simplicial complex. By some cut and paste argument, we can see $\Delta(H_2)$ is simply-connected [Jo; Prop. 2.2]. This simplicial complex is not a

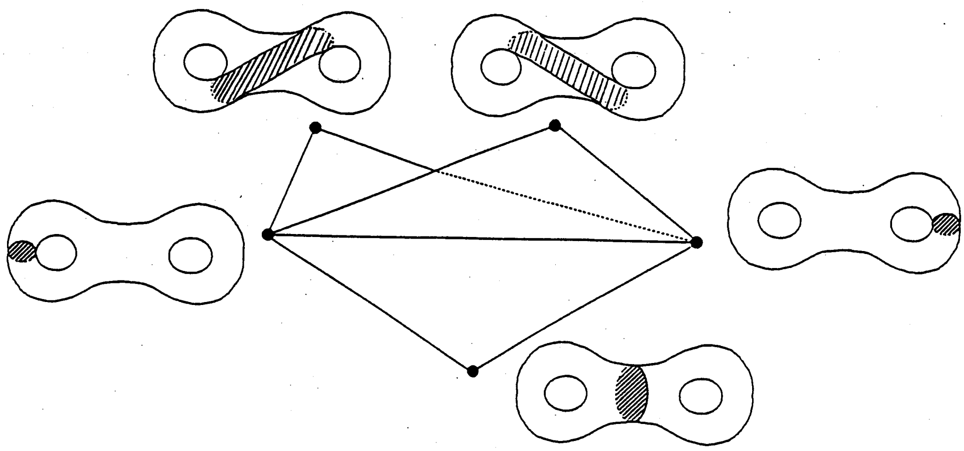

FIGURE 3

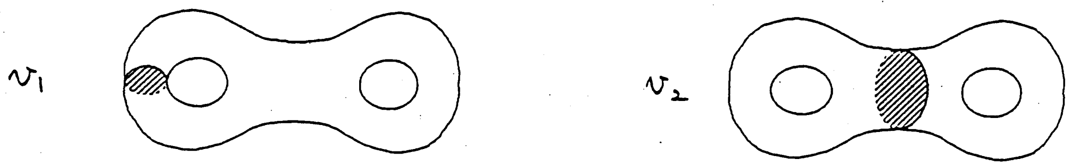

FIGURE 4

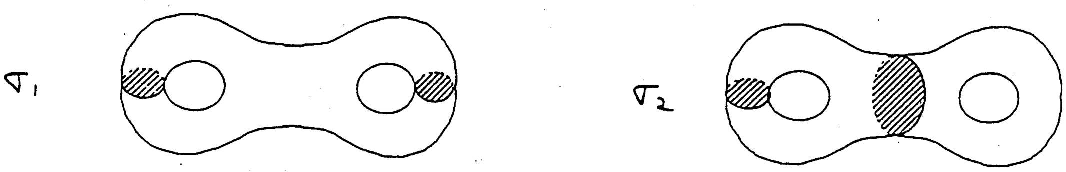

FIGURE 5

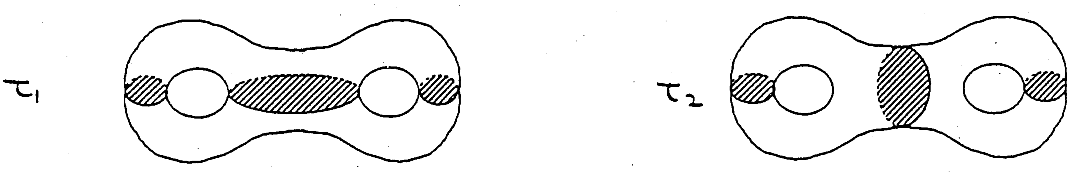

FIGURE 6

manifold, because, for each edges, there are more than three faces emanating from this (see Figure 3). By the elementary argument, we can see:

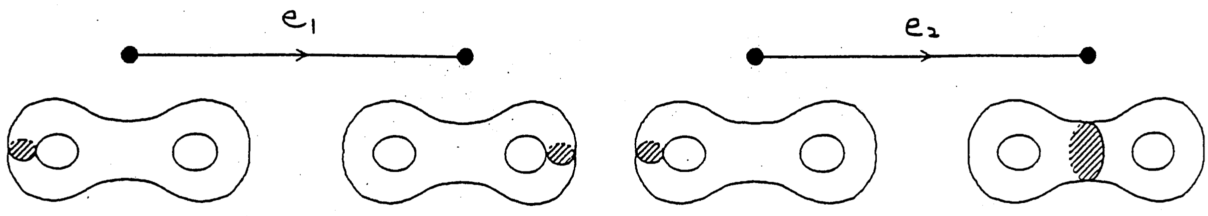

FIGURE 7

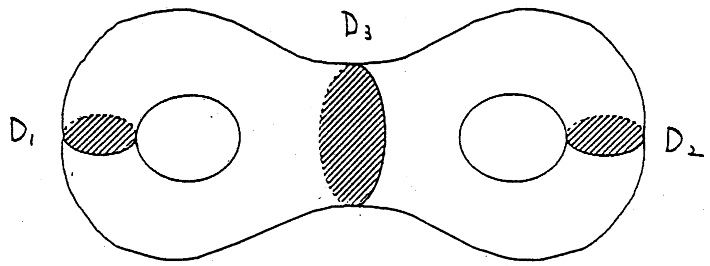

FIGURE 8

(1) the set of representatives of $V(H_2)/\mathcal{H}_2$ consists of two elements v_1 and v_2 (see Figure 4),

(2) the set of representatives of 1-cells of $\Delta(H_2)$ modulo $\mathcal{H}_2$ consists of two elements σ_1, σ_2, one of them σ_1 is inverted and the other σ_2 is non-inverted (see Figure 5),

(3) the set of representatives of 2-cells of $\Delta(H_2)$ mod $\mathcal{H}_2$ consists of 2-elements τ_1, τ_2 (see Figure 6).

Here, we make a choice. Let a tree of representatives T be the subcomplex of $\Delta(H_2)$ which consists of σ_2, v_1 and v_2. Let e_1 be an edge which is σ_1 with orientation given in Figure 7, e_2 be an edge which is σ_2 with orientation from v_1 to v_2. We set $E^+ = \{e_2\}$, $\Sigma^+ = \{\sigma_2\}$, $E^- = \{e_1\}$ and $\Sigma^- = \{\sigma_1\}$. Since $t(e_2) \in V(T)$, we choose $g_{e_1} = 1$ and $w(e_2) = t(e_2)$. In the next section, we give presentations for G_{v_1}, G_{v_2} and G_{σ_1}, sets of generators for G_{e_1} and G_{e_2}.

2. SUBGROUPS OF $\mathcal{H}_2$

In this section, we will give presentations for the groups which we use to give a presentation for $\mathcal{H}_2$. Let D_1, D_2, D_3 be disks properly embedded in H_2 indicated in Figure

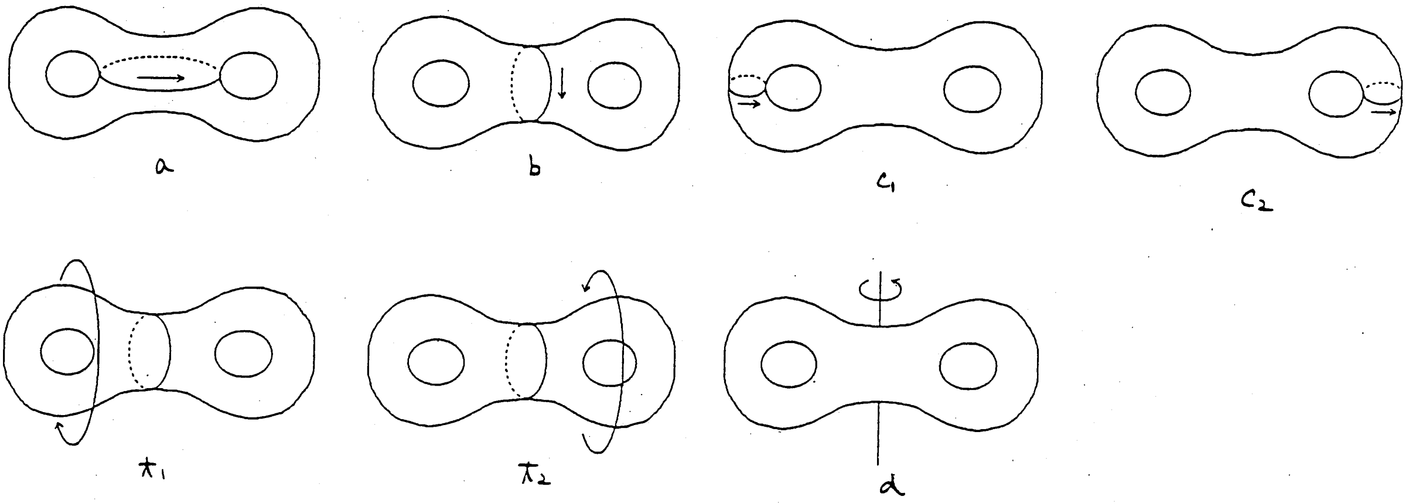

FIGURE 9

8. With these notations, $G_{v_1} = \pi_0(Diff^+(H_2, D_1))$, $G_{v_2} = \pi_0(Diff^+(H_2, D_3))$, $G_{\sigma_1} = \pi_0(Diff^+(H_2, D_1 \cup D_2))$, $G_{e_1} = \pi_0(Diff^+(H_2, D_1, D_2))$ and $G_{e_2} = \pi_0(Diff^+(H_2, D_1, D_3))$.

Proposition 1. *Let $a, b, c_1, c_2, d, t_1, t_2$ be the elements of $\pi_0(Diff^+(H_2, D_1 \cup D_2))$ given in Figure 9 This group admits a presentation with generators $a, b, c_1, c_2, d, t_1, t_2$, and defining relations,*

$$t_1^2 = t_2^2 = b, d^2 = 1, dt_1d = t_2, dc_1d = c_2$$

$$t_1^{-1}at_1 = t_2^{-1}at_2 = b^{-1}a^{-1}c_1^{-2}c_2^{-2}, \quad t_1 \rightleftarrows t_2,$$

$$a \rightleftarrows c_1, c_2, d, \quad b \rightleftarrows c_1, c_2, d, \quad c_1 \rightleftarrows c_2$$

$$t_1, t_2 \rightleftarrows b, c_1, c_2.$$

□

Proposition 2. *Let a, b, c, d, t, e be the elements of $\pi_0(Diff^+(H_2, D_1))$ given in Figure*

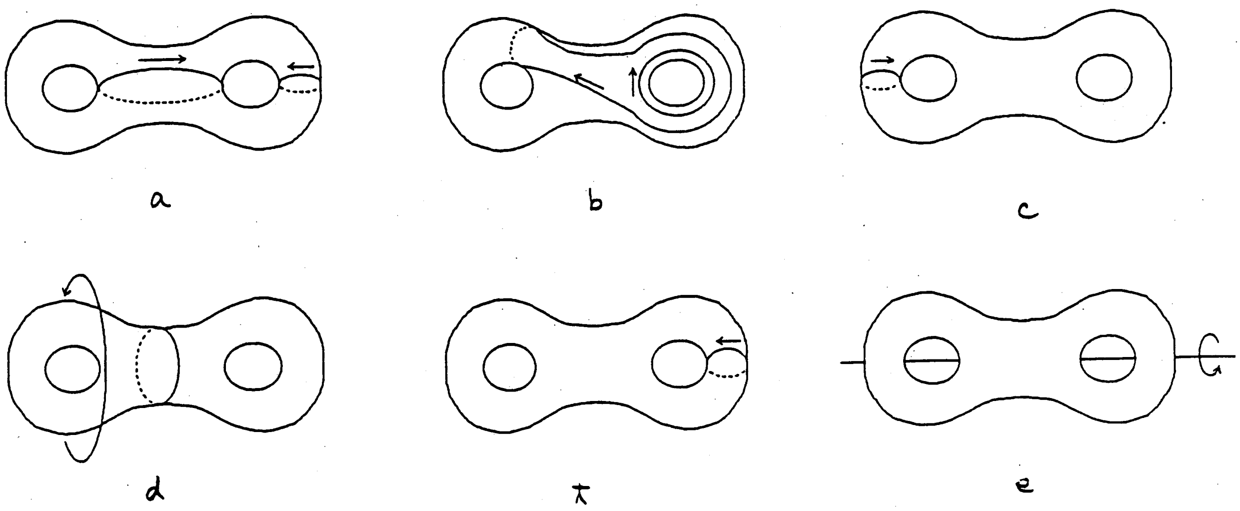

FIGURE 10

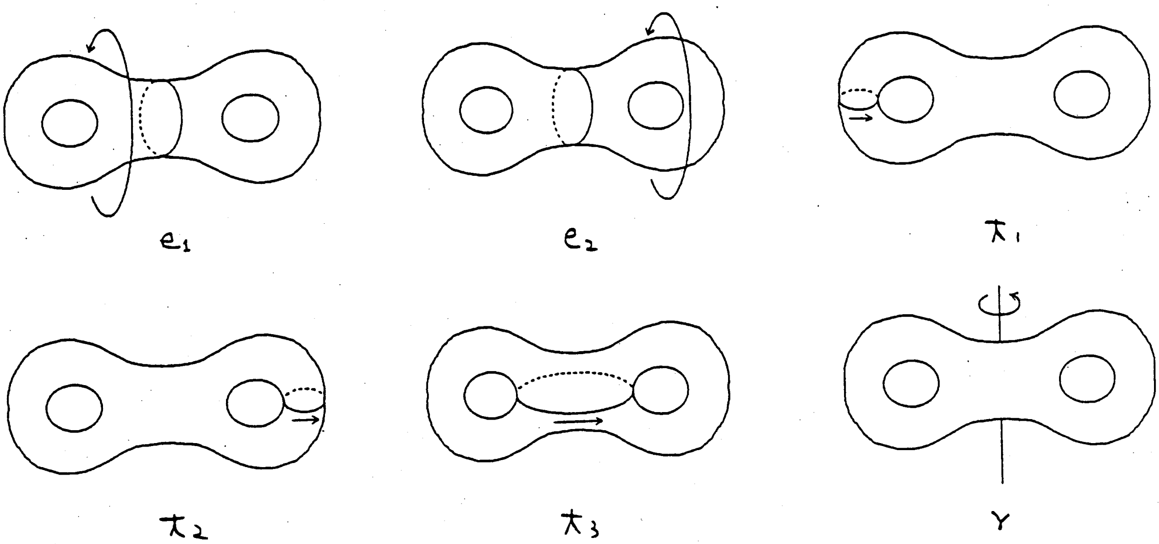

FIGURE 11

10. This group admits a presentation with generators a, b, c, d, t, e and defining relations,

$$aba^{-1}b^{-1} = d^2c^2,\ dad^{-1} = aba^{-1}b^{-1}a^{-1}, dbd^{-1} = ab^{-1}a^{-1},$$

$$e^2 = 1,\ t^{-1}bt = ba,$$

$$e \rightleftarrows a, b, c, d, t,\ t \rightleftarrows a, c, d,$$

$$c \rightleftarrows a, b, d.$$

□

Proposition 3. *Let e_1, e_2, t_1, t_2, r be the elements of $\pi_0(Diff^+(H_2, D_3))$ given in Figure 11. This group admits a presentation with generators above 6 elements and defining relations,*

$$e_1^2 = e_2^{-2}, r^2 = 1, rt_1r = t_2,$$

$$re_1r = e_2^{-1}, \quad e_1 \rightleftarrows t_1, e_2, t_2,$$

$$e_2 \rightleftarrows t_1, t_2, \quad t_1 \rightleftarrows t_2$$

□

Proposition 4. *The group $\pi_0(Diff^+(H_2, D_1, D_2))$ is generated by e_1, e_2, t_1, t_2, t_3 given in Figure 11.* □

Proposition 5. *The group $\pi_0(Diff^+(H_2, D_1, D_3))$ is generated by e_1, e_2, t_1, t_2 given in Figure 11.* □

3. A PRESENTATION FOR $\mathcal{H}_2$

Let $\hat{G} = G_{v_1} * G_{v_2} * G_{\sigma_1} * < \hat{g}_{e_2} >$. We put suffix α, β, γ, δ, ϵ for each element of G_{v_1}, G_{v_2}, G_{σ_1}, G_{e_1}; G_{e_2} respectively. For example, $a \in G_{v_1}$ is denoted by a_α, $t_1 \in G_{\sigma_1}$ is denoted by $t_{1,\gamma}$ and so on. Following the theorem by Brown, we give a presentation for $\mathcal{H}_2$. The edge e_2 is in the tree T, hence, (1) means $\hat{g}_{e_2} = 1$. Since we choose $g_{e_2} = 1$, $c_{e_2} : G_{e_2} \to G_{w(e_2)}$ is a natural inclusion. Since $o(e_2) = v_1$, $w(e_2) = t(e_2) = v_2$, (2) means, for each $g \in G_{e_2}$, $i_{e_2}(g) = j_{e_2}(g)$, where $i_{e_2} : G_{e_2} \hookrightarrow G_{v_1}$, $j_{e_2} : G_{e_2} \hookrightarrow G_{v_2}$ are inclusions. We get the following relations:

$$e_{1,\beta} = e_{1,\delta} = d_\alpha, \; e_{2,\beta} = e_{2,\delta} = e_\alpha d_\alpha^{-1},$$

$$t_{1,\beta} = t_{1,\delta} = c_\alpha^{-1}, \; t_{2,\beta} = t_{2,\delta} = t_\alpha$$

(3) means, for each $g \in G_{e_1}$, $i_{e_1}(g) = j_{e_1}(g)$, where $i_{e_1} : G_{e_1} \hookrightarrow G_{v_1}, j_{e_1} : G_{e_1} \hookrightarrow G_{\sigma_1}$.

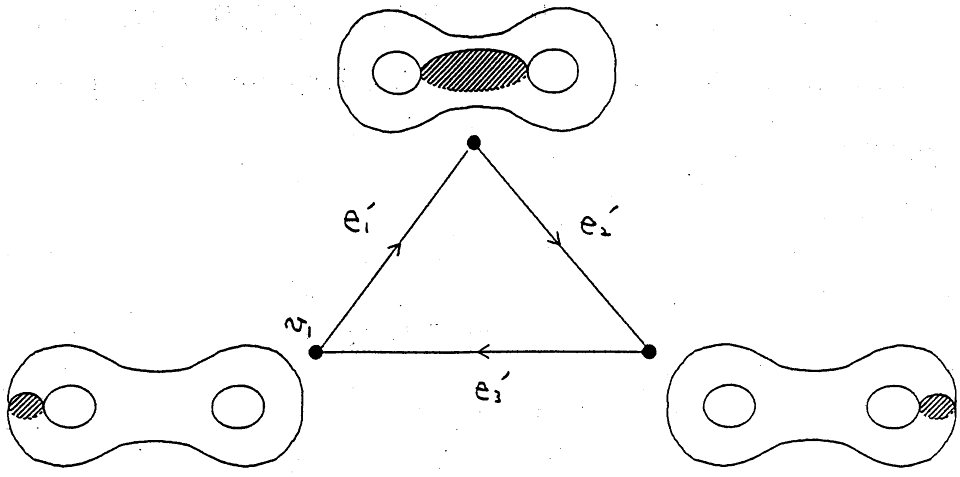

FIGURE 12

We get the following relations:

$$d_\alpha = e_{1,\epsilon} = t_{1,\beta}^{-1},\ e_\alpha d_\alpha^{-1} = e_{2,\epsilon} = t_{2,\beta},$$

$$c_\alpha^{-1} = t_{1,\epsilon} = c_{2,\beta},\ t_\alpha = t_{2,\epsilon} = c_{1,\beta},$$

$$a_\alpha^{-1} t_\alpha = t_{3,\epsilon} = a_\beta^{-1}.$$

To get the relations induced by (4), we need to give a presentation of $g_{\partial\tau_1}$ and $g_{\partial\tau_2}$. Let e_1', e_2', e_3' be the edges of $\partial\tau_1$ indicated in Figure 12. We choose a sequence g_1, g_2, g_3 of the elements of $\mathcal{H}_2$ corresponding to these edges. The element d_γ of G_{σ_1} satisfies $v_1 d_\gamma = t(e_1)$, and $b_\alpha \in G_{v_1}$ satisfies $e b_\alpha = e_1'$, therefore, we choose $g_1 = d_\gamma b_\alpha$. The element b_α^{-1} of G_{v_1} satisfies $e b_\alpha^{-1} = e_2' g_1^{-1}$, therefore, we choose $g_2 = d_\gamma b_\alpha^{-1}$. The element b_α of G_{v_1} satisfies $e b_\alpha = e_2'(g_2 g_1)^{-1}$, therefore, we choose $g_3 = d_\gamma b_\alpha$. The element $g_3 g_2 g_1$ is in G_v, namely we can check $g_3 g_2 g_1 = c_\alpha a_\alpha^{-1} d_\alpha e_\alpha$. Hence we get the following relation:

$$d_\gamma b_\alpha d_\gamma b_\alpha^{-1} d_\gamma b_\alpha = c_\alpha a_\alpha^{-1} d_\alpha e_\alpha.$$

Let e_1", e_2", e_3" be the edges of $\partial\tau_2$ indicated in Figure 13. By the same manner, we get the following relation:

$$d_\gamma = r_\beta^{-1}.$$

We get a presentation of $\mathcal{H}_2$, by Proposition 1 to 5, and relations given above. We apply

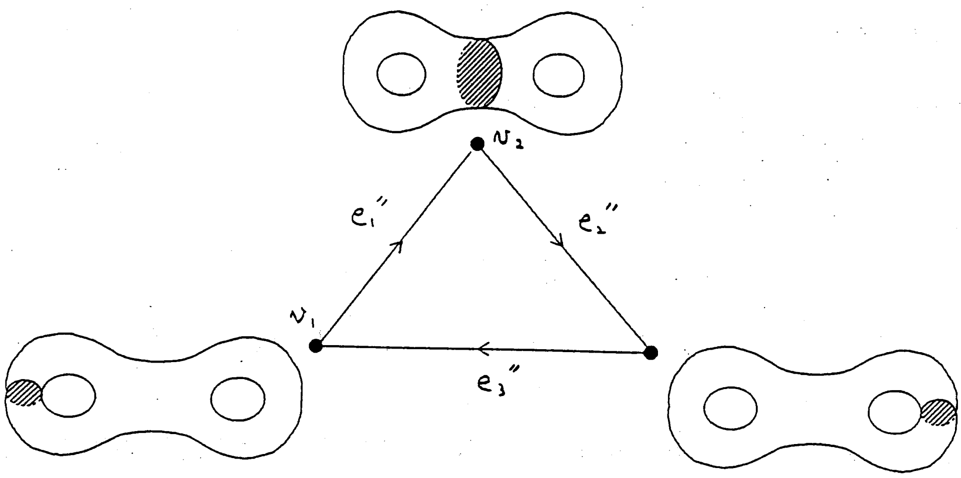

FIGURE 13

Tietze transformations [MKS; §1.5] to this presentation, then we obtain the presentation given in Theorem 1.

4. THE SURJECTION FROM $\mathcal{H}_2$ TO $GL(2,\mathbb{Z})$

There is a natural surjection from $\mathcal{H}_2$ to the outer automorphism group of the free group of rank 2, $Out(F_2)$, which is defined by the action of the elements of $\mathcal{H}_2$ on the fundamental group of the handle body of genus 2. For the sake of check the presentation given in Theorem 1, we show this result with using this presentation. The group $Out(F_2)$ is naturally identified with $GL(2,\mathbb{Z})$ (see [MKS; p.169]). The group $GL(2,\mathbb{Z})$ is generated by R_1, R_2, R_3:

$$R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the following is the set of relations between them which defines $GL(2,\mathbb{Z})$:

$$R_1^2 = R_2^2 = R_3^2 = E,$$
$$(R_1R_2)^3 = (R_1R_3)^2 = Z, \quad Z^2 = E,$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

(see [CM; Chapter 7]).

A homomorphism ψ from $\mathcal{H}_2$ to $GL(2,\mathbb{Z})$ defines by:

$$a, c, t \longmapsto E,$$

$$b \longmapsto R_1R_3R_2R_1, \quad d \longmapsto R_3,$$

$$e \longmapsto R_1R_3R_1R_3(=Z), \quad f \longmapsto R_1R_3R_1R_3R_1,$$

is a natural surjection. A homomorphism ϕ from $GL(2,\mathbb{Z})$ to $\mathcal{H}_2$ defined by considering natural identification of (once punctured torus) $\times[0,1]$ with H_2 :

$$R_1 \longmapsto fe,$$

$$R_2 \longmapsto d^{-1}ac^{-1}fbf,$$

$$R_3 \longmapsto d^{-1}ac^{-1},$$

is a injection and satisfies $\psi \circ \phi = id_{GL(2,\mathbb{Z})}$. The above two facts are verified by using the representation of $\mathcal{H}_2$ given as a Theorem 1.

Problem. A injection ϕ from $GL(2,\mathbb{Z})$ to $\mathcal{H}_2$, which satisfies $\psi \circ \phi = id_{GL(2,\mathbb{Z})}$ is not unique. Is it unique up to conjugation ?

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