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Stability of Kleinian groups

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We survey stability of Kleinian groups. Several results in this note are also contained in the forthcoming monograph [7].

Let Γ be a finitely generated non-elementary Kleinian group, which is identified with a discrete subgroup of $\mathrm{PSL}_2(\mathbf{C})$, with a fixed system of generators $\Gamma=\langle\gamma_1,\ldots,\gamma_N\rangle$. We consider the set of $\mathrm{PSL}_2(\mathbf{C})$-representations

$$\mathrm{Hom}(\Gamma)=\{\rho \mid \rho:\Gamma\to\mathrm{PSL}_2(\mathbf{C}) \text{ is a homomorphism}\}.$$

This is regarded as an analytic subset of $\mathrm{PSL}_2(\mathbf{C})^N$ by the correspondence

$$\rho\mapsto(\rho(\gamma_1),\ldots,\rho(\gamma_N))\in\mathrm{PSL}_2(\mathbf{C})^N.$$

We say that Γ is *structurally stable* if there is a neighborhood U of the identity representation id in $\mathrm{Hom}(\Gamma)$ such that any $\rho\in U$ is a faithful representation.

However, a weaker condition than structural stability is more interesting in deformation theories of Kleinian groups, where we treat an analytic set of all representations sending any parabolic element to a parabolic one or the identity. Letting

$$\mathrm{PHom}(\Gamma)=\{\rho\in\mathrm{Hom}(\Gamma)\mid \mathrm{tr}^2\rho(\gamma)=4 \text{ for any parabolic } \gamma\in\Gamma\},$$

we call Γ *weakly structurally stable* if the condition of structural stability is satisfied after replacing $\mathrm{Hom}(\Gamma)$ with $\mathrm{PHom}(\Gamma)$.

Here, the property that any $\rho\in U\subset\mathrm{PHom}(\Gamma)$ is faithful is actually equivalent to that ρ is a quasiconformal deformation. Indeed, we can apply

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the λ-lemma to a holomorphic family of isomorphisms defined over a complex disk holomorphically embedded in U that passes a given point of U. Thus the weakly structural stability is nothing but the following quasiconformal stability:

Let $T(\Omega(\Gamma)/\Gamma)$ be the Teichmüller space of the union of orbifolds $\Omega(\Gamma)/\Gamma$. For every $[\mu] \in T(\Omega(\Gamma)/\Gamma)$, we denote by f_μ a quasiconformal automorphism of $\hat{\mathbf{C}}$ that gives the deformation $[\mu]$ of the complex structure of $\Omega(\Gamma)/\Gamma$ and that satisfies a suitable normalization. Then a holomorphic map

$$\tilde{\Psi} : T(\Omega(\Gamma)/\Gamma) \times \mathrm{PSL}_2(\mathbf{C}) \to \mathrm{PHom}(\Gamma)$$

is defined by the conjugation of $A \circ f_\mu$, for any pair $([\mu], A) \in T(\Omega(\Gamma)/\Gamma) \times \mathrm{PSL}_2(\mathbf{C})$. It is known that $\tilde{\Psi}$ is well-defined. Moreover, the Sullivan rigidity theorem implies that the image of $\tilde{\Psi}$ coincides with the set of the whole representations induced by quasiconformal automorphisms of $\hat{\mathbf{C}}$ (cf. [7, Chapter 5]). This set is called the quasiconformal deformation space and denoted by $\mathrm{QHom}(\Gamma)$ ($\subset \mathrm{Hom}(\Gamma)$). We say that Γ is *quasiconformally stable* if there is a neighborhood U of the identity representation id such that

$$\mathrm{PHom}(\Gamma) \cap U \subset \mathrm{QHom}(\Gamma).$$

Gardiner and Kra [3] investigated the derivative $d\tilde{\Psi}|_{id}$ of $\tilde{\Psi}$ at id, representing the Zariski tangent space of the analytic set $\mathrm{PHom}(\Gamma)$ as Eichler cohomology. This is called *the Bers map*. They showed that $d\tilde{\Psi}|_{id}$ is injective and decomposed the tangent space into subspaces caused by $\mathrm{PSL}_2(\mathbf{C})$-conjugations, quasiconformal deformations and deformations of projective structure. In particular, they proved that surjectivity of $d\tilde{\Psi}|_{id}$ implies quasiconformal stability.

For a torsion-free Kleinian group, Marden [5] proved that if Γ is geometrically finite then it is quasiconformally stable, by investigation of fundamental polyhedra. Later Sullivan [10] proved the converse, namely, that quasiconformal stability implies geometric finiteness, by 3-dimensional topological arguments to compare the dimension of the Teichmüller space $T(\Omega(\Gamma)/\Gamma)$ with the dimension of $\mathrm{PHom}(\Gamma)$.

In [6], we extend this equivalence to Kleinian groups with torsion. One direction is easy if we pass Γ to a torsion-free subgroup of finite index by the Selberg lemma, but the other not. We consider a core in the 3-dimensional hyperbolic orbifold $\mathbf{H}^3/\Gamma$, where a core means a compact suborbifold such that the inclusion induces an isomorphism between the orbifold fundamental groups. Moreover, we require that the core is relative to the boundary at infinity $\Omega(\Gamma)/\Gamma$. If we can construct such a core, then we can obtain a relation between $\dim T(\Omega(\Gamma)/\Gamma)$ and $\dim \mathrm{PHom}(\Gamma)$ at id because both are topological quantities determined only by the topology of the relative core.

We can prove the existence of an orbifold relative core in the case that Γ is indecomposable as a free product in a certain sense. For the general case, we decompose Γ into indecomposable ones and use induction arguments to compare the dimensions.

Theorem 1 *The following conditions are equivalent for any finitely generated non-elementary Kleinian group* Γ*:*

1. Γ *is geometrically finite;*

2. Γ *is quasiconformally stable;*

3. *the Bers map* $d\tilde{\Psi}|_{id}$ *is an isomorphism.*

In particular:

Corollary 2 *If* Γ *is geometrically finite, then* $\mathrm{QHom}(\Gamma)$ *is a complex regular submanifold of* $\mathrm{PSL}_2(\mathbf{C})^N$.

Next we will prove that Corollary 2 is actually satisfied for any finitely generated Kleinian group. Since $\tilde{\Psi}$ is a holomorphic immersion onto $\mathrm{QHom}(\Gamma)$, the only problem is compatibility of the Teichmüller topology of $T(\Omega(\Gamma)/\Gamma) \times \mathrm{PSL}_2(\mathbf{C})$ and the topology of $\mathrm{QHom}(\Gamma)$, which is the algebraic topology for $\mathrm{PSL}_2(\mathbf{C})$-representations.

Problem If ρ_n converge to id in $\mathrm{QHom}(\Gamma)$, do there always exist $t_n \in \tilde{\Psi}^{-1}(\rho_n)$ such that t_n converge to the base point $0 \in T(\Omega(\Gamma)/\Gamma)$ as $n \to \infty$?

This problem is originated in Bers [1, p.578], where it was announced that the proof would appear elsewhere, however it has not appeared as far as the author knows. See also [2]. Later Krushkal published a series of papers (cf. [4]) concerning this problem. A finitely generated Kleinian group is called *conditionally stable* or *quasi-stable* if it satisfies the property in the problem above.

We will show that a result on geometric convergence of Kleinian groups yields the affirmative answer to this problem.

Theorem 3 *Any finitely generated Kleinian group* Γ *is conditionally stable. Hence* QHom(Γ) *is a complex regular submanifold of* $\mathrm{PSL}_2(\mathbf{C})^N$.

We first remark the following two facts.

Lemma 4 Γ *is conditionally stable if any component subgroup of* Γ *is conditionally stable.*

Proof. This follows easily from the definition of conditional stability. ■

Lemma 5 *Let* Γ *be a finitely generated Kleinian group and* Γ' *a subgroup of* Γ *of finite index. If* Γ' *is conditionally stable, then so is* Γ.

Proof. Suppose that Γ is not conditionally stable. Then there is a sequence $\rho_n \in$ QHom(Γ) converging to *id* such that the maximal dilatation of the extremal quasiconformal automorphism f_n inducing ρ_n does not tend to 1 as $n \to \infty$. Here the extremal quasiconformal map is the one with the smallest maximal dilatation among quasiconformal maps with the required property.

We restrict ρ_n to the subgroup Γ' and have $\rho'_n \in$ QHom(Γ'). Then ρ'_n converges to *id* and f_n induces ρ'_n. Since Γ' is of finite index in Γ, f_n is also the extremal quasiconformal automorphism that induces ρ'_n (cf. Ohtake [9]). But this contradicts the assumption that Γ' is conditionally stable. Thus we see that Γ is also conditionally stable. ■

By these facts, it suffice to consider torsion-free function groups Γ for proving Theorem 3. It is known that such Γ is constructed from elementary groups, quasifuchsian groups and totally degenerate groups without APT by a finite number of applications of the Maskit combination theorem. Moreover we can see that conditional stability is preserved under the Maskit combination theorem:

Lemma 6 *Assume that a torsion-free function group* Γ *is constructed from* Γ_1 *and* Γ_2 *(as the amalgamated free product or the HNN-extension) by the Maskit combination theorem. If both* Γ_1 *and* Γ_2 *are conditionally stable, then so is* Γ.

Proof. See [7, Section 7.3]. ■

Therefore Theorem 3 will complete if it is solved for totally degenerate groups without torsion nor APT. The crucial fact for this step is the following result due to Thurston (cf. Ohshika [9] and [7, Section 7.2]).

Proposition 7 *Let Γ_0 be a finitely generated torsion-free Fuchsian group and $\theta_n : \Gamma_0 \to \Gamma_n$ a sequence of type-preserving isomorphisms onto Kleinian groups, which converges algebraically to a type-preserving isomorphism $\theta : \Gamma_0 \to \Gamma$. If Γ is a totally degenerate group, then Γ_n also converge geometrically to Γ.*

Applying this proposition, we can assert:

Lemma 8 *Under the same circumstances as in Proposition 7, if Γ_n and Γ are totally degenerate groups, then the marked complex structures t_n of $\Omega(\Gamma_n)/\Gamma_n$ converge to t of $\Omega(\Gamma)/\Gamma$. In particular, any torsion-free, totally degenerate group without APT is conditionally stable.*

Proof. Let C_n be the convex core of the hyperbolic manifold $\mathbf{H}^3/\Gamma_n$ and ∂C_n the relative boundary of C_n, which is regarded as a pleated surface with a marked hyperbolic structure s_n. By Proposition 7, we can see that s_n converge to the marked hyperbolic structure s of the boundary surface of the convex core of $\mathbf{H}^3/\Gamma$. By Sullivan's theorem (cf. [7, Section 7.1]), s_n and t_n are in a bounded Teichmüller distance independent of n. Hence $\{t_n\}$ is a bounded sequence in the Teichmüller space and there is a subsequence $\{t_{n'}\}$ which converges to some t'. Then $\Gamma_{n'}$ converge algebraically to a b-group Γ' such that the marked complex structure of D'/Γ' is t', where D' is the invariant component of $\Omega(\Gamma')$. However, Γ' should coincide with Γ, and hence $t' = t$. ■

Thus we obtain Theorem 3.

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