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The chordal norm of Kleinian groups

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Let $M\ddot{o}b$ denote the group of all orientation preserving mobius transformations of the extended complex plane $\hat{C}=C\cup\{\infty\}$. We associate with each

$$f(z)=\frac{az+b}{cz+d}\in M\ddot{o}b, ad-bc=1,$$

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$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,C)$$

and set $tr(f)=tr(A)$, where $tr(A)$ denote the trace of A. Note that $tr(f)$ is defined up to sign. The matrix norm $m(f)$ is defined by

$$m(f)=\| A-A^{-1}\|=(2|a-d|^{2}+4|b|^{2}+4|c|^{2})^{\frac{1}{2}}.$$

The quantity $m(f)$ is independent of the choice of A representing f.

For each f and g in $M\ddot{o}b$ we let $[f,g]$ denote the multiplicative commutator $fgf^{-1}g^{-1}$. We call the three complex numbers

$$\beta(f)=tr^{2}(f)-4, \beta(g)=tr^{2}(g)-4, \gamma(f,g)=tr([f,g])-2,$$

the parameters of the two generator group $<f,g>$. These parameters are independent of the choice of representative matrices for f and g, and they determine $<f,g>$ up to conjugacy whenever $\gamma(f,g)\neq 0$. We derive a lower bound for the distance in the metric of

(1) $$d(f,g)=\sup\{q(f(z),g(z)); z\in\hat{C}\}$$

where $<f,g>$ is a Kleinian group generated by f,g in $M\ddot{o}b$ and q denotes the chordal distance in $\hat{C}$,

$$q^{2}(z,w)=\frac{4|z-w|^{2}}{(|z|^{2}+1)(|w|^{2}+1)}.$$

A mobius transformation h is said to be a chordal isometry if

$$q(h(z),h(w))=q(z,w)$$

for all $z,w\in\hat{C}$, then $m(f)$ is invariant with respect to conjugation by chordal isometries.

If $f\in M\ddot{o}b$ with $fix(f)=\{z_{1},z_{2}\}$,then

Lemma 1. Let f be loxodromic or elliptic with $fix(f) = \{z_1, z_2\}$ then

$$2|\beta(f)| = \frac{q(z_1,z_2)^2}{8 - q(z_1,z_2)^2} m(f)^2. \tag{2}$$

Proof. All quantities in (2) are invariant with respect to conjugation by chordal isometries. Therefore by means of such a conjugation we may arrange that $z_1 = -r$ and $z_2 = r$ where $0 < r \le 1$. Then f is represented by

$$A = \begin{pmatrix} a & br \\ br^{-1} & a \end{pmatrix}, b^2 = a^2 - 1.$$

Hence $\beta(f) = 4b^2$,$q(-r,r) = \frac{4r}{r^2+1}$,and $m(f)^2 = \| A - A^{-1} \|^2 = (r^2 + r^{-2})|2b|^2$. Thus we have

$$m(f)^2 = 2\frac{8 - q(-r,r)^2}{q(-r,r)^2} |\beta(f)|.$$

Lemma 2. Suppose that f is in $M\ddot{o}b \setminus \{id\}$ with $d(f,id) < 2$. If f has two fixed points and if 2θ is the argument of its multiplier, then

$$m(f)^2 \le 8\cos^2\theta \frac{d(f,id)^2}{4 - d(f,id)^2} \tag{3}$$

with equality if and only if f has antipodal fixed points.

Lemma 3. If f is elliptic of order n,then

$$d(f,id) \ge 2\sin(\frac{\pi}{n}) \tag{4}$$

with equality if and only if $n = 2$ or f is primitive with antipodal fixed points.
Proof. If f has order n and $fix(f) = \{-r, r\}$, then $m(f)^2 = 4(r^2 + r^{-2})\sin^2\theta$ where $\theta = k\pi/n$ and $1 \le k < n$. Thus if $d(f,id) < 2$, we have

$$d(f,id)^2 \ge \frac{16(r^2 + r^{-2})\sin^2\theta}{8\cos^2\theta + 4(r^2 + r^{-2})\sin^2\theta} \ge 4\sin^2\theta$$

from Lemma 2, and the right hand side of above inequality is an incresing in (r^2+r^{-2}). We derive $d(f,id)^2 \ge 4\sin^2\theta \ge 4\sin^2\pi/n$ with equality when $n > 2$ if and only if $r = k = 1$.

Lemma 4. If f and g are in $M\ddot{o}b$, then

$$\max\{d(f,id), d(g,id)\} \ge \left\{ \frac{4|\gamma(f,g)|^{\frac{1}{2}}}{2 + |\gamma(f,g)|^{\frac{1}{2}}} \right\}^{\frac{1}{2}} \tag{5}$$

Proof. If A and B are in $SL(2,C)$, then $\| AB - BA \|^2 \le \frac{1}{8} \| A - A^{-1} \|^2 \| B - B^{-1} \|^2$. Suppose that f and g are represented by A and B in $SL(2,C)$, then we have $|\gamma(f,g)| =$

$|tr([A,B])-2| = |det((AB)(BA)^{-1}-I)| = |det(AB-BA)| \leq \frac{1}{2} \parallel AB-BA \parallel^2 \leq \frac{1}{16} \parallel A-A^{-1} \parallel^2 \parallel B-B^{-1} \parallel^2 = \frac{1}{16} m(f)^2 m(g)^2$.It is easily seen that

$$m(f)^2 \leq \frac{8d(f,id)^2}{4-d(f,id)^2},$$

from Lemma 2 where equality holds if f is either the identity or hyperbolic with antipodal fixed points. Without loss of generality, set $m(g) \leq m(f)$, then we have

$$4|\gamma(f,g)|^{\frac{1}{2}} \leq m(f)m(g) \leq m(f)^2 \leq \frac{8d^2}{4-d^2}.$$

Therefore we have the result.

Lemma 5 Suppose that f and g are in $Möb$ and that $f(H) = g(H) = H$. If f and g are hyperbolic with $\gamma(f,g) < 0$, then there exists h in $Möb$ such that $h(H) = H$ and $2\beta(f_1) = m(f_1)^2, 2\beta(g_1) = m(g_1)^2$, where $f_1 = hfh^{-1}, g_1 = hgh^{-1}$ and each of f_1, g_1 have antipodal fixed points.

Theorem 6 Suppose that $< f, g >$ is a Kleinian subgroup of $Möb$ and that f and g have no common fixed point and are not both of order 2. Then

$$\max\{d(f,g), d(f^{-1},g^{-1})\} \geq k_1 \quad (6)$$

where k_1 is an absolute constant, $0.853 \leq k_1 \leq 0.911..$.

Proof. Let $\gamma = \gamma(f,g) = \gamma(fg^{-1},g^{-1})$ and $\beta = \beta(fg^{-1}) = \beta(g^{-1}f)$. If fg^{-1} is of order n and where $n = 2,3,4$, or 6, then $d(f,g) = d(fg^{-1},id) \geq 2\sin(\pi/n)$ from Lemma 3. If $\gamma(fg^{-1},g^{-1}) = \beta(fg^{-1})$, then [M2] Implies that fg^{-1} is elliptic of order 2,3,4,or 6 or g is elliptic of order 2. Therefore we assume that $\gamma \neq \beta$,and consider the subgroup $< fg^{-1}, g^{-1}f > = < fg^{-1}, g^{-1}(fg^{-1})g >$ of $< fg^{-1}, g^{-1} >$ with $\gamma(fg^{-1},g^{-1}f) = \gamma(fg^{-1},g^{-1})\{\gamma(fg^{-1},g^{-1}) - \beta(fg^{-1})\} = \gamma(\gamma-\beta) \neq 0$.Thus we have

$$|\gamma(fg^{-1},g^{-1}f)| \geq 2 - 2\cos(\pi/7) \quad (7)$$

Therefore we have

$$\begin{aligned} \max\{d(f,g), d(f^{-1},g^{-1}\} &= \max\{d(fg^{-1},id), d(g^{-1}f,id)\} \\ &\geq \left(\frac{4(2\cos(\frac{\pi}{7}) - 2\cos(\frac{\pi}{7}) + 1)}{2\cos(\frac{\pi}{7}) - 2\cos(\frac{\pi}{7}) + 3}\right)^{\frac{1}{2}} \geq 0.853. \end{aligned}$$

From now on, we show an upper bound for d, let $< \phi, \psi >$ denote the (2,3,7) triangle group acting on the upper half plane H with $\phi^2 = \psi^3 = (\phi\psi)^7 = id$ and set $f = \phi\psi$ and $g = \psi\phi$ Then $\gamma = tr([\phi,\psi]) - 2 = 2\cos(\frac{2\pi}{7}) - 1$, $\beta = tr^2([\phi,\psi]) - 4 = 2(\cos(\frac{2\pi}{7}) + \cos(\frac{\pi}{7}) - 1) > 0$ where $fg^{-1} = [\phi,\psi]$.Hence fg^{-1} and $g^{-1}f$ are hyperbolic with $\gamma(fg^{-1},g^{-1}f) = \gamma(\gamma-\beta) < 0$. Then Lemma 5 show that there exists a mobius transformation h of H such that

$\text{fix}(hfg^{-1}h^{-1}) = \{z_1, z_2\}$ and $\text{fix}(hg^{-1}fh^{-1}) = \{w_1, w_2\}$ where $q(z_1, z_2) = q(w_1, w_2) = 2$. Then $2\beta(hfg^{-1}h^{-1}) = m(hfg^{-1}h^{-1})^2$, $2\beta(hg^{-1}fh^{-1}) = m(hg^{-1}fh^{-1})^2$ and $hfg^{-1}h^{-1}$ and $hg^{-1}fh^{-1}$ are both hyperbolic with antipodal fixed points. Therefore we have the result

$$\begin{aligned} d(f,g) &= d(fg^{-1}, id) = 2\left(\frac{\cos(\frac{2\pi}{7}) + \cos(\frac{\pi}{7}) - 1}{\cos(\frac{2\pi}{7}) + \cos(\frac{\pi}{7}) + 1}\right)^{\frac{1}{2}} \\ d(f^{-1},g^{-1}) &= d(g^{-1}f, id) = 2\left(\frac{\cos(\frac{2\pi}{7}) + \cos(\frac{\pi}{7}) - 1}{\cos(\frac{2\pi}{7}) + \cos(\frac{\pi}{7}) + 1}\right)^{\frac{1}{2}} \end{aligned}$$

from Lemma 2 and hence that $k_1 \leq 0.911$.

Theorem 7 Suppose that f and g are elements of a Kleinian subgroup G of $M\ddot{o}b$ which are not both of order 2,3,4,or 6. Then f and g commute or

$$\max\{d(f,g), d(f^{-1},g^{-1})\} \geq k_1 \tag{8}$$

Proof. It suffices to consider the case where $\gamma(f,g) = 0$ and $fg \neq gf$,then

$$fix(f) \cap fix(g) \neq \phi, fix(f) \neq fix(g)$$

Then $< f,g >$ is elementary and $h = [f,g]$ is parabolic. We complete the proof by showing that fg^{-1} is elliptic of order $n (= 2,3,4 or 6)$ and $d(f,g) = d(fg^{-1}, id) \geq 1$.

Suppose that $fix(h) = \{\infty\}$. If f or g, say f, is parabolic, then g is elliptic of order 2,3,4 or 6 and an elementary calculation shows that the same is true of fg^{-1}.If f and g are both elliptic, then

$$f(z) = \nu z + a, g(z) = \mu z + a, fg^{-1}(z) = \frac{\nu}{\mu}z + c$$

where $\nu^p = \mu^q = (\frac{\nu}{\mu})^r = 1$, $p, q \in \{2,3,4,6\}$ and $r \in \{1,2,3,4,6\}$. By hypothesis $p \neq q$, $\nu \neq \mu$and fg^{-1} is of order 2,3,4,6.

Example. If $f(z) = \lambda z$ and $g(z) = \lambda z - c$ where $\lambda^p = 1$,$0 < |c| \leq 2$ and $p = 2,3,4 or 6$, then $< f,g >$ is discrete while

$$d(f,g) = d(f^{-1},g^{-1}) = \frac{8|c|}{4 + |c|^2} \to 0$$

as $c \to 0$. It is necessary to make the hypothesis that f and g are not both of oder 2,3,4 or 6 in Theorem 7.

Remark. For each $1 < b < a < \infty$ let $f = f_0 g_0$ and $g = g_0$ where

$$f_0(z) = a^2 z, g_0(z) = \frac{(b^2+1)z + 2b}{2bz + (b^2+1)}$$

Then $< f, g >$ is nonelementary Kleinian while

(9) $$d(f,g) = d(f_0, id) = 2\left(\frac{a^2-1}{a^2+1}\right) \to 0$$

as $a \to 1$. Hence there exists no universal lower bound for $d(f,g)$ and $d(f^{-1}, g^{-1})$.

These theorems give a geometric estimate of how different two mobius transformations must be in order to generate a nonelementary Kleinian group.

Theorem 8 Suppose that $< f, g >$ is a Kleinian group and f and g have no common fixed point and are not both of order 2. If fg is also not of order 2, then

(10) $$\max\{d(fg, gf), d((fg)^{-1}, (gf)^{-1})\} \geq k_1.$$

Proof. Suppose that g is not of order 2 and let $\gamma = \gamma(f,g)$ and $\beta = \beta(fg)$. If $\gamma = \beta$, then $\beta([f,g]) = \gamma(\gamma+4) = -3, -4$ and thus that $[f,g]$ is elliptic of order 2 or 3. Hence

$$d(fg, gf) = d([f,g], id) \geq \sqrt{3}.$$

Otherwise $< fg, gf >=< fg, g(fg)g^{-1} >$ is Kleinian with

$$\gamma(fg, gf) = \gamma(fg, g)\{\gamma(fg, g) - \beta(fg)\} = \gamma(\gamma - \beta) \neq 0,$$

then $k_1 \geq 0.853$ from Theorem 6. Next let $< f, g >$ be the group which (6) holds with equality in Theorem 6. Then $f = \phi\psi$ and $g = \psi\phi$ where $< \phi, \psi >$ is the triangle group with $\phi^2 = \psi^3 = (\phi\psi)^7 = id$ and we obtain

$$d(f,g) = d(f^{-1}, g^{-1}) = 0.911.$$

from Theorem 5. Hence the group $< \phi, \psi >$ shows that $k_1 \leq 0.911$.

Gehring and Martin showed that if $< f, g >$ is a nonelementary Kleinian subgroup of $Möb$, then

(11) $$m(f)m(g) \geq 4(\sqrt{2}-1) = 1.656..,$$

follows from J ϕ rgensen's inequality and the proof of Lemma 4. In the proof of Lemma 4, we have $16|\gamma(f,g)| \leq m(f)^2 m(g)^2$ and if $< f, g >$ is a nonelementary Kleinian group then , $m(f)m(g) \geq 4\sqrt{|\gamma(f,g)|} \geq 1.780$.

The following result shows that the average of the chordal norms of the generators f and g is always bounded below by a constant k_1 and $d(g, id) \to 2$ as $d(f, id) \to 0$ uniformly in the collection of all nonelementary Kleinian groups $< f, g >$.

Theorem 9 Suppose that $< f, g >$ is nonelementary Kleinian group of $Möb$. Then

(12) $$d(f, id) + d(g, id) \geq 2k_1, 2d(f, id) + d(g, id) \geq 2$$

Proof. We may assume that $2a = d(f,id) + d(g,id) \leq 2$ in the proof for both parts of (12) and that $d(g,id) \leq d(f,id)$ in the proof for the first inequality in (12). Next above assumption together with the first inequality in (12) imply $2 < \frac{3}{2}\{d(f,id) + d(g,id)\} \leq 2d(f,id) + d(g,id)$ whenever $d(f,id) \geq d(g,id)$, hence we may also assume that $d(f,id) \leq d(g,id)$ in the proof for the second inequality in (12). Then $d(f,id) = a-x$ and $d(g,id) = a+x$ where $0 \leq x < a$ and we obtain

$$16|\gamma(f,g)| \leq m(f)^2m(g)^2 \leq \frac{8d(f,id)^2}{4-d(f,id)^2}\frac{8d(g,id)^2}{4-d(g,id)^2}$$

from Lemma 4. Let $\phi(x) = \{4(a-x)^{-2}-1\}\{4(a+x)^{-2}-1\}$ and $\phi(x) \leq 4\{2-2\cos(\pi/7)\}^{-1}$ by Cao. Since $\phi(x)$ is increasing with respect to $[0,a)$ where $0 < a \leq 1$ and we have

$$a \geq 2\left(\frac{\sqrt{|\gamma(f,g)|}}{2+\sqrt{|\gamma(f,g)|}}\right)^{\frac{1}{2}}$$

This establishes the first part of (12) with $k_1 \geq 0.853...$

If $2 - d(g,id) > 2d(f,id)$, then $a + x < 2\{1-(a-x)\}$ and

$$\psi(y) = (4y^{-2}-1)\{(1-y)^{-2}-1\} < \phi(x)$$

where $y = a - x$. By elementary calculus,

$$\begin{aligned} \psi'(y) &= -8y^{-2}(1-y)^{-3}(5y-2)(y-2) \\ \psi(y) &\geq \psi(2/5) > 40 > 4\{2-2\cos(\pi/7)\}^{-1} \end{aligned}$$

for $0 < y < 1$ and we have a contradiction. This establishes the second part of (12).

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