CONSTRUCTION OF RIEMANN SURFACES
BY PARALLEL TRANSFORMATIONS

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1. INTRODUCTION

In this paper we introduce a new method of constructing once punctured Riemann surfaces. In our construction we use line segments in the complex plane \(\mathbb{C}\) and parallel transformations: For a pair of disjoint parallel line segments with the same length in \(\mathbb{C}\), we first cut \(\mathbb{C}\) along the segments and paste each side of one segment and the opposite side of the other segment by a parallel transformation obtaining a once punctured elliptic curve. The puncture is at infinity. (See §2, Figure 1.) We shall call such a pair an Igeta. (Igeta is a Japanese word coming from a technical term "Igeta-kuzushi" used in a Japanese martial art.) Putting \(g\) disjoint pieces of Igeta on \(\mathbb{C}\), we obtain a once punctured Riemann surface of genus \(g\) in the same way. We denote a set of \(g\) disjoint Igeta by \(\Gamma\) and the resulting once punctured Riemann surface by \((R(\Gamma), p_\infty)\). Moreover when we move the position of \(g\) Igeta, there appears a family of once punctured Riemann surfaces of genus \(g\). All the possible configurations of \(g\) disjoint Igeta up to the affine automorphisms of \(\mathbb{C}\) form a \(3g - 2\)-dimensional complex \(V\)-manifold and this dimension is the same as the dimension of the moduli space \(\mathcal{M}_{g,1}\) of once punctured Riemann surfaces of genus \(g\). We thus expect to have a visual image of the moduli space by using this construction.

We first consider the Kodaira-Spencer maps of the family. Let \(I_g\eta\) be the collection of \(\Gamma\)'s, and let \(I_g\eta_0\) be the subset of \(I_g\eta\) consisting of those \(\Gamma\) having \([0, 1]\) as one of its \(2g\) line segments. \(I_g\eta\) turns out to be a \(3g\)-dimensional complex manifold and \(I_g\eta_0\) a \(3g - 2\)-dimensional complex manifold. Our first main result is as follows:

**Theorem 1.** The Kodaira-Spencer map

\[
\rho_{\Gamma}[-3] : T(I_g\eta_0)_{\Gamma} \to H^1(R(\Gamma); \Theta(-3p_\infty))
\]

is an isomorphism for any \(\Gamma \in I_g\eta_0\), where \(T(I_g\eta)_{\Gamma}\) is the holomorphic tangent space of \(I_g\eta\) at \(\Gamma\) and \(\Theta(-3p_\infty)\) is the sheaf of germs of holomorphic vector fields on \(R(\Gamma)\) having zero at \(p_\infty\) of order at least 3.

**Corollary 1.** The Kodaira-Spencer map

\[
\rho_{\Gamma,0} : T(I_g\eta_0)_{\Gamma} \to H^1(R(\Gamma); \Theta(-p_\infty))
\]

is an isomorphism for any \(\Gamma \in I_g\eta_0\).

For a closed Riemann surface \(R\) of genus \(g\) we define a Lagrangian sublattice \(\Lambda\) of \(R\) to be a subgroup of \(H_1(R; \mathbb{Z})\) which coincides its orthogonal complement with respect to the intersection form on \(H_1(R; \mathbb{Z})\), i.e. a subgroup isomorphic to \(\mathbb{Z}^g\) such that the quotient \(H_1(R; \mathbb{Z})/\Lambda\) is also isomorphic to \(\mathbb{Z}^g\) and the intersection number of any two elements in \(\Lambda\) equals zero. Moreover, for any once punctured Riemann surface \((R, p)\) of genus \(g\), a Lagrangian sublattice \(\Lambda\) of \(H_1(R; \mathbb{Z})\) and the puncture \(p\) determine a certain Abelian differential \(\omega_{\Lambda}\) of the second kind on the surface unique up to scalars. When we construct
a once punctured Riemann surface \((R(\Gamma),p_\infty)\) from \(\Gamma\), \(R(\Gamma)\) has a natural Lagrangian sublattice \(\Lambda_\Gamma\). On the other hand if we denote by \(\zeta\) the standard coordinate of \(\mathbb{C}\), \(R(\Gamma)\) has a natural Abelian differential \(\omega_\Gamma\) of the second kind induced by \(d\zeta\). It turns out that \(\omega_\Gamma\) is equal to \(\omega_{\Lambda_\Gamma}\) up to scalars. We use \(\omega_\Gamma\) to prove Theorem 1.

Furthermore, using \(\omega_\Lambda\) of \((R,p,\Lambda)\) we obtain the following result:

**Corollary 2.** For an arbitrary once punctured Riemann surface with a Lagrangian sublattice \((R,p,\Lambda)\), \((R,\omega_\Lambda)\) and \((\mathbb{CP}_1,d\zeta)\) are piecewise parallel.

We call two Riemann surfaces \((R,\omega)\) and \((R',\omega')\) with Abelian differentials of the second kind piecewise parallel if after decomposing \((R,\omega)\) into small pieces having line-segment-boundaries we can obtain \((R',\omega')\) by pasting them together using parallel transformations in another way. This operation turns out to be reversible. (See §3.)

Corollary 2 indicates that any once punctured Riemann surface can be obtained from \(\mathbb{C}\) by cutting along line segments and pasting by parallel transformations. We have to remark here that this corollary does not imply that any Riemann surface can be obtained by Igeta-construction. Nevertheless from this result we expect that any once punctured Riemann surface with a Lagrangian sublattice would appear in some natural extension of our family.

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2. **Igeta-construction and the Kodaira-Spencer maps**

The Gauss plane is the complex affine line \(A^1\) with a fixed global coordinate \(\zeta : A^1 \rightarrow \mathbb{C}\). Consider the set \(\eta\) consisting of unordered pairs \((\sigma^+,\sigma^-)\) of disjoint line segments in the Gauss plane \((A^1,\zeta)\) such that \(\sigma^+\) and \(\sigma^-\) are parallel and equilateral. We denote by \(I_\eta\) the collection of unordered sets \(\Gamma\) of \(g\) elements of \(\eta\) where

\[
\Gamma = ((\sigma_j^+,\sigma_j^-) \in \eta ; j = 1, \ldots, g)
\]

such that \(\sigma_j^\pm\) are pairwise disjoint. Let \(\phi_j^\pm = \phi_j^\pm[\Gamma]\) be the affine map from the Gauss plane \((A^1,\zeta)\) to \((A^1,\zeta)\) given by

\[
\zeta = a_j\xi + b_j^\pm, \quad a_j \in \mathbb{C}^\times, \quad b_j^\pm \in \mathbb{C}
\]

such that \(\sigma_j^\pm = \phi_j^\pm([-1,1])\). The space \(I_\eta\) is a 3\(g\)-dimensional open complex manifold with local coordinates \((a_j,b_j^\pm ; j = 1, \ldots, g)\) for a fixed order of line segments.

We construct a holomorphic family of once punctured Riemann surfaces of genus \(g\) over \(I_\eta\) as follows. Let \(B\) be an open and relatively compact subset of \(I_\eta\). Set

\[
E_\Gamma^0 = A^1 - \bigcup_{j=1}^g (\sigma_j^+ \cup \sigma_j^-)
\]

for \(\Gamma = (\sigma_j^\pm) \in I_\eta\) and set

\[
E^0 = \bigcup_{\Gamma \in I_\eta} E_\Gamma^0 \subset I_\eta \times A^1,
\]

\[
E_B^0 = E^0 \cap (B \times A^1).
\]
Let $U_\infty$ be the disk \( \{ w \in \mathbb{C} ; |w| < \epsilon \} \) and $V_j$ \((j = 1, \ldots, g)\) copies of the annulus \( \{ z \in \mathbb{C} ; (1 + \epsilon)^{-1} < |z| < 1 + \epsilon \} \) for $\epsilon > 0$ and let

\[ V_j^+ = \{ z \in V_j ; |z| > 1 \}, \quad V_j^- = \{ z \in V_j ; |z| < 1 \}. \]

Note that the Joukowski transform

\[ J(z) = \frac{1}{2}(z + \frac{1}{z} - 1) \]

maps the unit circle in $\mathbb{C}$ onto the interval $[-1, 1]$. For sufficiently small $\epsilon > 0$, we paste the patches

\[ E^0_B, \quad B \times U_\infty, \quad B \times V_j \, (j = 1, \ldots, g) \]

by the attaching maps

\begin{align*}
B \times (U_\infty - \{0\}) \ni (\Gamma, w) & \mapsto (\Gamma, w^{-1}) \in E^0_B, \\
B \times V_j^\pm \ni (\Gamma, z) & \mapsto (\Gamma, \phi_j^\pm[\Gamma] \circ J(z)) \in E^0_B 
\end{align*}

and obtain a complex manifold $E_B$, which is the total space of a holomorphic family of once punctured Riemann surfaces of genus $g$ over $B$. As $I_g \eta$ is locally compact, we can construct the holomorphic family $\pi : E \to I_g \eta$ such that $\pi^{-1}(B) = E_B$ for any open and relatively compact subset $B$ of $I_g \eta$. For a point $\Gamma$ of $I_g \eta$ the Riemann surface $R(\Gamma) = \pi^{-1}(\Gamma)$ is constructed by pasting the patches

\[ E^0_\Gamma, \quad U_\infty, \quad V_j \, (j = 1, \ldots, g) \]

through the attaching map

\begin{align*}
U_\infty \ni w & \mapsto \zeta = w^{-1} \in E^0_\Gamma, \\
V_j^\pm \ni z & \mapsto \zeta = \phi_j^\pm \circ J(z) \in E^0_\Gamma.
\end{align*}

Denote by $p_\infty$ the puncture on $R(\Gamma)$ corresponding to $0 \in U_\infty$.

We call such a pair $(\sigma^+, \sigma^-)$ Igeta and the construction mentioned above Igeta-construction. Roughly speaking, the Igeta-construction is to cut the Gauss plane along the $g$ pairs of line segments $(\sigma_j^\pm ; j = 1, \ldots, g)$ and to paste each side of $\sigma_j^+$ and the opposite side of $\sigma_j^-$ by a parallel transformation. (Figure 1. The numbers $(1), \ldots, (6)$ in Figure 1 indicate where to paste.)

We investigate the infinitesimal deformation for this family $\pi : E \to I_g \eta$. We differentiate at a point $\Gamma$ of $I_g \eta$ the coordinate transformation for the Riemann surface $R(\Gamma)$ by the parameters of $I_g \eta$, and obtain the Kodaira-Spencer map ([K]). We consider the Kodaira-Spencer map with respect to the deformation fixing $p_\infty$ to order $n$:

\[ \rho_\Gamma[-n] : T(I_g \eta)_{\Gamma} \to H^1(R(\Gamma), \Theta(-np_\infty)) \]

where $\Theta$ denotes the sheaf of holomorphic vector fields on $R(\Gamma)$. Now we state the following:

**Theorem 1.** The Kodaira-Spencer map

\[ \rho_\Gamma[-3] : T(I_g \eta)_{\Gamma} \to H^1(R(\Gamma), \Theta(-3p_\infty)) \]

is an isomorphism.
We first calculate the dimension of $H^1(\Theta(-3p_{\infty}))$. (Note that we omit $R(\Gamma)$ in $H^*(R(\Gamma),*)$.)

By the Riemann-Roch formula,

$$
\dim H^0(\Theta(-3p_{\infty})) - \dim H^1(\Theta(-3p_{\infty})) = 1 - g + c_1(\Theta(-3p_{\infty})) = -3g.
$$

As $c_1(\Theta(-3p_{\infty})) = -(2g+1)$ is negative, $H^0(\Theta(-3p_{\infty})) = 0$. So $\dim H^1(\Theta(-3p_{\infty})) = 3g$, which coincides with the dimension of $I_g \eta$. Thus we show the surjectivity of $\rho = \rho_{\Gamma}[-3]$.

The pairing

$$
\langle , \rangle : H^0(\Omega^1 \otimes \Omega^1(3p_{\infty})) \times H^1(\Theta(-3p_{\infty})) \to H^1(\Omega^1) \cong \mathbb{C}
$$

is nondegenerate because of the Serre duality. Note that the 1-form $d\zeta$ on $E^0_{\Gamma}$ extends to a meromorphic 1-form $\omega = \omega_{\Gamma}$ on $R(\Gamma)$, which has a 2-pole at $p_{\infty}$ and $2g$ zeros at $q_j^{\pm}$ ($j = 1, \ldots, g$) corresponding to $\pm 1 \in V_j$. The multiplication by $\omega$ induces the isomorphism

$$
N := H^0(\Omega^1(p_{\infty} + \sum_{j=1}^g (q_j^+ + q_j^-))) \to H^0(\Omega^1 \otimes \Omega^1(3p_{\infty})).
$$

Hence it is sufficient to show the following:

$$
\chi \in N \text{ vanishes if } \langle \chi \omega, \rho(v) \rangle = 0 \text{ for any } v \in T(I_g \eta)_{\Gamma}.
$$

The calculation of the pairing is based on the following lemma:

**Lemma 1.** Let $R = \bigcup_{U \in S} U$ be a Riemann surface with an open covering. A holomorphic 1-form $\alpha$ on $U_1 \cap U_2$ for $U_1, U_2 \in S$ induces an element $[\alpha]$ of $H^1(R, \Omega^1)$ (the cocycle
vanishes on the other intersections of the coordinate neighborhoods). If $U_1 \cap U_2$ is an annulus bounded by two circles $C_1 \subset U_2$, $C_2 \subset U_1$, then the evaluation of $[\alpha]$ is given by $\langle \alpha \rangle = \pm \int_{C_1} \alpha$.

Proof. We introduce a cut-off function $\psi$ on $U_1 \cap U_2$ such that

$$\text{supp } \psi \subset U_2, \quad \text{supp } (1 - \psi) \subset U_1.$$  

Then the $(1,1)$-form $\bar{\partial}(\psi \alpha)$ extends by 0 outside $U_1 \cap U_2$ and the evaluation of $[\alpha]$ is given by

$$\int_{R} \bar{\partial}(\psi \alpha) = \int_{R} d(\psi \alpha) = \int_{C_1 + C_2} \psi \alpha = \int_{C_1} \alpha$$

up to sign. \hfill \square

Now we differentiate the coordinate transformations of $R(\Gamma)$. The map

$$U_\infty - \{0\} \ni w \mapsto \zeta = w^{-1} \in E^0_t$$

is independent of the parameters of $I_g \eta$, and

$$V_j^\pm \ni z \mapsto \zeta = \phi_j^\pm \circ J(z) \in E^0_t$$

depends only on $(a_j, b_j^\pm)$ and

$$\frac{\partial}{\partial a_j} (\phi_j^\pm \circ J(z)) = \frac{1}{2} (z + z^{-1}),$$

$$\frac{\partial}{\partial b_j^\pm} (\phi_j^\pm \circ J(z)) = 1.$$

Thus

$$\langle \chi \omega, \rho(\frac{\partial}{\partial a_j}) \rangle = \langle \chi \omega, [\frac{\partial}{\partial \zeta} \text{ on } \phi_j^\pm \circ J(V_j^\pm)] \rangle$$

$$= \langle \chi \text{ on } \phi_j^\pm \circ J(V_j^\pm) \rangle$$

$$= \int_{C_j^+} \chi$$

where $C_j^\pm = \{z \in V_j ; |z| = (1 + \frac{\epsilon}{2})^{\pm 1}\}$ oriented appropriately. Since the left-hand side vanishes by the assumption,

$$\int_{C_j^+} \chi = 0.$$

Hence $\text{Res}_{q_j^+} \chi + \text{Res}_{q_j^-} \chi = \int_{C_j^+ + C_j^-} \chi = 0$. Further,

$$\langle \chi \omega, \rho(\frac{\partial}{\partial a_j}) \rangle = \langle \chi \omega, [\frac{\partial}{\partial \zeta} \text{ on } \phi_j^\pm \circ J(V_j^\pm) \cup \phi_j^\pm \circ J(V_j^-)] \rangle$$

$$= \frac{1}{2} (z + z^{-1}) \chi \text{ on } \phi_j^\pm \circ J(V_j^\pm) \cup \phi_j^- \circ J(V_j^-))$$

$$= \int_{C_j^+ + C_j^-} \frac{1}{2} (z + z^{-1}) \chi$$

$$= \text{Res}_{q_j^+} \chi - \text{Res}_{q_j^-} \chi$$
and it vanishes, so we get
\[ \operatorname{Res}_{q_j^+} \chi = 0 \]
and
\[ \chi \in H^0(\Omega^1(p_{\infty})) = H^0(\Omega^1) \]
(by the residue theorem). Finally
\[ \int_{C_j^+} \chi = 0, \quad j = 1, \ldots, g \]
yields \( \chi = 0 \) by the bilinear relations of Riemann.

The group \( \operatorname{Aut}(\mathbb{C}) \) of automorphisms of \( \mathbb{C} \) acts on \( I_g \eta \) as
\[ (a_j, b_j^\pm) \mapsto (aa_j, b_j^\pm + b) \quad a \in \mathbb{C}^\times, b \in \mathbb{C} \]
and it is implied that the family of punctured Riemann surfaces of genus \( g \) by Igeta-construction is complete and effectively parametrized at any point for each \( g \).

**Corollary 1.** The Kodaira-Spencer map
\[ \rho_{\Gamma, 0} : T(I_g \eta_0)_\Gamma \longrightarrow H^1(R(\Gamma); \Theta(-p_{\infty})) \]
is an isomorphism where
\[ I_g \eta_0 = \{ \Gamma \in I_g \eta : \Gamma \text{ has } [0, 1] \text{ as one of its } 2g \text{ line segments.} \} \]

Note that the submanifold \( I_g \eta_0 \) gives a local manifold cover of the \( V \)-manifold \( I_g \eta / \operatorname{Aut}(\mathbb{C}) \) at any point.

**3. CUTTING AND PASTING OF RIEMANN POLYGONS**

In the previous section, given a once punctured Riemann surface \( (R(\Gamma), p_{\infty}) \) constructed from Igeta \( \Gamma \), we used the Abelian differential \( \omega_{\Gamma} \) of the second kind on \( R(\Gamma) \) in order to prove Theorem 1. Let \( \Lambda_{\Gamma} \) be the subgroup of \( H_1(R(\Gamma); \mathbb{Z}) \) generated by \( C_j^+ (j = 1, \ldots, g) \).

The integral of \( \omega_{\Gamma} \) on any element \( \lambda \) of \( \Lambda_{\Gamma} \) vanishes, and the orthogonal complement of \( \Lambda_{\Gamma} \) with respect to the intersection form coincides with \( \Lambda_{\Gamma} \) itself.

We first give a definition of Lagrangian sublattice, which is deduced from the properties of \( \Lambda_{\Gamma} \), for any closed Riemann surface as follows:

**Definition 3.1.** Let \( R \) be a closed Riemann surface. A Lagrangian sublattice is defined to be a subgroup \( \Lambda \) of \( H_1(R; \mathbb{Z}) \) satisfying \( \Lambda = \Lambda^\perp \), where \( \Lambda^\perp \) denotes the orthogonal complement with respect to the intersection form on \( H_1(R; \mathbb{Z}) \).

Let \( (R, p) \) be a once punctured Riemann surface of genus \( g \) and \( \Lambda \) a Lagrangian sublattice of \( H_1(R; \mathbb{Z}) \). The kernel \( Z_\Lambda \) of the homomorphism given by Abelian integrals
\[ H^0(R; \Omega^1(2p)) \longrightarrow \operatorname{Hom}(\Lambda, \mathbb{C}) (\cong \mathbb{C}^g) \]
is always one-dimensional because it holds that
\[ \dim H^0(R; \Omega^1(2p)) = g + 1 \]
from the Riemann-Roch formula and the surjectivity is implied by the bilinear relations of Riemann. Accordingly, a Lagrangian sublattice and a point on the surface determine an Abelian differential up to scalars. (Note that \( \Lambda \cong \mathbb{Z}^g \).)
Each Igeta $\Gamma$ is associated to a once punctured Riemann surface $(R(\Gamma), p_{\infty})$ together with the Lagrangian sublattice $\Lambda_{\Gamma}$. In the case of $(R, p, \Lambda) = (R(\Gamma), p_{\infty}, \Lambda_{\Gamma})$, the kernel $Z_{A_{\Gamma}}$ is generated by $\omega_{\Gamma}$. Now the problem is the following:

**Problem.** *Is it possible to construct any $(R, p, \Lambda)$ by cutting and pasting the Gauss plane using line segments and parallel transformations as in the Igeta-construction?*

We next introduce a concept “Riemann polygon” so that we can consider cutting and pasting of Riemann surfaces with Abelian differentials using “line segments” and “parallel transformations”.

Let $R$ be a Riemann surface and $\omega$ an Abelian differential on it. We call a simple path or simple loop $\gamma : [0, 1] \rightarrow R$ *$\omega$-line-segment* if its image contains no poles of $\omega$ and the integral

$$\int_{\gamma(0)}^{\gamma(t)} \omega$$

depends on $t \in [0, 1]$ linearly. We also call its image *$\omega$-line-segment*. The 2-form $\frac{i}{2} \omega \wedge \bar{\omega}$ induces a metric $g_{\omega}$ on $R$ which has conical singularities at the zeros of $\omega$. $\omega$-line-segments are geodesics for this metric $g_{\omega}$. Let us cut $R$ along $\omega$-line-segments and separate into finitely many pieces. We call a collection of such pieces *Riemann polygon*:

**Definition 3.2.** A Riemann polygon $(F, \omega)$ is defined to be a pair consisting of a compact, not necessarily connected Riemann surface $F$ and an Abelian differential $\omega$ on $F$ such that the boundary of $F$, if not empty, consists of $\omega$-line-segments.

A Riemann surface with boundary in our understanding is a 2-dimensional topological manifold $F$ with boundary which is embedded in a Riemann surface $R$ and whose interior inherits its complex structure from $R$; in addition an Abelian differential on $F$ is a restriction of some Abelian differential on $R$. For example, any polygon $P$ in the real plane $\mathbb{R}^2$ is considered as a Riemann polygon $(P, d\zeta|_P)$ when $\mathbb{R}^2$ is identified with $\mathbb{C}$. If we compactify $E_{\Gamma}^0 \cup U_{\infty}$ in §2 by attaching line segments to both sides of $\sigma_f^\pm$, the pair consisting of the compactification $\overline{E_{\Gamma}^0}$ and the 1-form $d\zeta$ is a Riemann polygon. We can also consider a closed Riemann surface with an Abelian differential as a Riemann polygon with empty boundary.

We can easily generalize the concept of “the translation scissors congruence” (see [Mo], [S]) to the case of Riemann polygons. In order to do so, we introduce two operations “$P$-cutting” and “$P$-pasting” for getting a Riemann polygon from another Riemann polygon.

**$P$-cutting:** Let $(F, \omega)$ be a Riemann polygon, and let $\gamma$ be an $\omega$-line-segment on it such that the image $\gamma((0, 1))$ is in the interior of $F$. We first remove the $\omega$-line-segment $\gamma([0, 1])$ from $F$, and then compactify by attaching one copy of $\gamma([0, 1])$ to each side of $\gamma([0, 1])$ obtaining a new Riemann polygon $(F', \omega')$, where $\omega'$ is induced by $\omega$ naturally. (If $\gamma(0)$ (resp. $\gamma(1)$) is in the interior of $F$ and is different from $\gamma(1)$ (resp. $\gamma(0)$), $\gamma(0)$ (resp. $\gamma(1)$) in one of the copies of $\gamma([0, 1])$ should be identified with $\gamma(0)$ (resp. $\gamma(1)$) in the other.)

**$P$-pasting:** Let $(F, \omega)$ be a Riemann polygon. If there are $\omega$-line-segments $\gamma$ and $\gamma'$ on the boundary $\partial F$ such that

$$\int_{\gamma(0)}^{\gamma(t)} \omega = \int_{\gamma'(0)}^{\gamma'(t)} \omega$$
for any \( t \in [0, 1] \) and the interior of \( F \) sits on opposite sides of the paths, then we can paste \( \gamma([0,1]) \) and \( \gamma'([0,1]) \) by identifying \( \gamma(t) \) with \( \gamma'(t) \) and obtain a new Riemann polygon \((F', \omega')\), where \( \omega' \) is induced by \( \omega \) naturally.

For Riemann polygons in \( \mathbb{C} \), P-cutting indicates cutting along line segments, and P-pasting indicates pasting by parallel transformations. Igeta-construction is a special way of P-cutting and P-pasting.

It is obvious that P-pasting is the inverse operation of P-cutting. So these operators give rise to an equivalence relation between Riemann polygons: We call Riemann polygons \((F, \omega)\) and \((F', \omega')\) piecewise parallel, if \((F, \omega)\) is obtained from \((F', \omega')\) by finitely many P-cuttings and P-pastings.

We shall consider the special case where \( F \) is a closed Riemann surface of genus \( g \). Let \( R \) be a closed Riemann surface of genus \( g \). For an Abelian differential \( \omega \) of the second kind on \( R \), we define a sequence \( PT(\omega) \) and a real number \( S(\omega) \) as follows:

**Definition 3.3.**

\[
PT(\omega) := \{n_i\}_{i \in \mathbb{Z}_+} \quad (n_i \text{ is the number of poles of order } i),
\]

\[
S(\omega) := \text{Im}\left( \sum_{j=1}^{g} \int_{\alpha_j} \omega \int_{\beta_j} \omega \right), \quad ((\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) \text{ is a symplectic basis of } H_1(R, \mathbb{Z}))
\]

The value of \( S \) will be shown not to depend on the choice of the basis \((\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)\) in the proof of Theorem 2 below.

**Theorem 2.** Let \( \omega \) and \( \omega' \) be Abelian differentials of the second kind on closed Riemann surfaces \( R \) and \( R' \) such that \( PT(\omega) = PT(\omega') \). Then the Riemann polygons \((R, \omega)\) and \((R', \omega')\) are piecewise parallel if and only if \( S(\omega) = S(\omega') \).

Before proving Theorem 2 we recall a result of Hadwiger-Glur about polygons in \( \mathbb{R}^2 \).

Let \( M \) be a finite set of polygons in \( \mathbb{R}^2 \), and let \( v \) be a unit vector in \( \mathbb{R}^2 \). (By our convention, \( M \) is a Riemann polygon.) Assume that the boundary of each polygon in \( M \) is oriented counterclockwise; we consider each boundary segment as a vector, and call them “boundary vectors”. We denote by \( A(M) \) the sum of the area of polygons in \( M \), and define \( J_v(M) \) to be the algebraic sum of the boundary vectors of \( M \) which are parallel to \( v \).

**Hadwiger-Glur’s Theorem** [Mo], [S]. Let \( M \) and \( M' \) be finite sets of polygons in \( \mathbb{R}^2 \) such that \( A(M) = A(M') \). \( M \) and \( M' \) are piecewise parallel if and only if \( J_v(M) = J_v(M') \) for any unit vector \( v \).

Obviously, the invariant \( J_v(M) \) can be extended as an invariant \( J_v(R, \omega) \) of any Riemann polygon \((R, \omega)\) with respect to P-cuttings and P-pastings.

**Proof of Theorem 2.** It is obvious that \( J_v(R, \omega) = J_v(R', \omega') = 0 \) for any unit vector \( v \) because both \( R \) and \( R' \) are closed Riemann surfaces.

Fix a one-to-one correspondence between the poles of \( \omega \) and the ones of \( \omega' \) such that the orders of corresponding poles are equal, and fix for each pole \( p_j \) a local biholomorphism \( h_j \) which maps a neighborhood of \( p_j \) onto a neighborhood of the corresponding pole \( p'_j \) transforming \( \omega \) into \( \omega' \). Let \( C_j \) be a small simple loop consisting of \( \omega \)-line-segments around \( p_j \). (We shall call such a loop simple polygonal loop.) Then \( h_j(C_j) \) is a polygonal loop around \( p'_j \). Cut off from \( R \) those components of \( R - \bigcup_j C_j \) (resp. \( R' - \bigcup_j h_j(C_j) \)) which
contain the poles. We then obtain a Riemann polygon \((R_0, \omega)\) (resp. \((R'_0, \omega')\)) with no poles. Furthermore, fix a symplectic basis \((\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)\) and \(2g\) simple polygonal loops \((a_1, b_1, \ldots, a_g, b_g)\) representing them such that their intersection is only one point on \(R\), and that they have no intersection with \(C_j\)'s. Let \((\tilde{R}_0, \omega)\) be a Riemann polygon obtained from \((R_0, \omega)\) by cutting along \(a_1, b_1, \ldots, a_g, b_g\). We do the same with \((R'_0, \omega')\) and denote by \((\tilde{R}'_0, \omega')\) the resulting Riemann polygon. It is sufficient for proving Theorem 2 to show that \((\tilde{R}_0, \omega)\) and \((\tilde{R}'_0, \omega')\) are piecewise parallel.

Now we can define a holomorphic function \(h\) on \(\tilde{R}_0\) such that \(dh = \omega\) because \(\omega\) has no periods on \(\tilde{R}_0\). Let \(g_\omega\) be the metric on \(\tilde{R}_0\) induced by the 2-form \(\frac{i}{2} \omega \wedge \bar{\omega}\). The map \(h\) between \((\tilde{R}_0, g_\omega)\) and \((\mathbb{C}, g)\) is a local isometry at any point except zeros of \(\omega\), where \(g\) is the standard metric on \(\mathbb{C}\). We deduce from Stokes formula

\[
\int_{\tilde{R}_0} \frac{i}{2} \omega \wedge \bar{\omega} = S(\omega) + \int_{\Sigma_j C_j} \frac{i}{2} h \bar{\omega}
\]

where \(C_j\)'s are oriented counterclockwise.

The left-hand side of the equation above is the area of \(\tilde{R}_0\) with respect to \(g_\omega\) and the second term of the right hand side depends only on the behavior of the differential around its poles. So \(S(\omega)\) is independent of the choice of the symplectic basis. We can decompose \(\tilde{R}_0\) into small pieces each of which can be mapped to some polygon in \(\mathbb{C}\) isometrically. Hence the conclusion follows from the theorem of Hadwiger-Glur. □

Now we return to the case of once punctured Riemann surfaces with Lagrangian sublattices. For the standard global coordinate \(\zeta\) of \(\mathbb{C}\), the differential \(d\zeta\) uniquely extends to the complex projective line \(\mathbb{C}P_1\) as an Abelian differential, which has a pole of order 2 at \(\infty\). We also denote it by \(d\zeta\).

**Corollary 2.** Let \((R, p)\) be a once punctured Riemann surface and \(\Lambda\) be a Lagrangian sublattice of \(H_1(R; \mathbb{Z})\). For a nonzero element \(\omega \in Z_\Lambda\) the Riemann polygon \((R, \omega)\) is piecewise parallel to \((\mathbb{C}P_1, d\zeta)\).

**Proof.** The assumption \(\omega \in Z_\Lambda\) implies that \(S(\omega) = 0\) and \(PT(\omega) = PT(d\zeta)\). □

Corollary 2 indicates that any once punctured Riemann surface can be obtained from \(\mathbb{C}\) by cutting along line segments and pasting by parallel transformations; the triple \((R, p, \Lambda)\) is represented by a set of line segments on the complex plane plus pasting-data.

**Remark 1.** When we consider Riemann surfaces together with Abelian differentials of the first kind or holomorphic 1-forms, a result similar to Corollary 2 holds; any closed Riemann surface can be obtained from a fixed elliptic curve by cutting along line segments and pasting by parallel transformations. We first fix an elliptic curve and an Abelian differential on it. For instance, let \(E\) be the quotient \(\mathbb{C}/L\) where \(L\) is the lattice generated by 1 and \(i\), and we consider the standard Abelian differential \(d\zeta\) of the first kind on \(E\) where \(\zeta\) is the coordinate of \(\mathbb{C}\). We next choose an Abelian differential \(\omega\) of the first kind on any closed Riemann surface \(R\) so that \(S(\omega) = S(d\zeta) = 1\). (It is easy to find such Abelian differentials.) We can show in the same way that \((R, \omega)\) and \((E, d\zeta)\) are piecewise parallel.

**Remark 2.** Igeta-construction leads us to consider the moduli space of once punctured Riemann surfaces with Lagrangian sublattices. In [H-O1] and [H-O2], we considered the case of genus 1, and described the moduli space using a natural extension of Igeta-construction,
that is, we made a complete list of once punctured elliptic curves with Lagrangian sublattices.

REFERENCES


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