数理解析研究所講究録
1024 巻 1998 年 18-31

Extremal richness of C*-algebras

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1 Definition

Definition 1.1 (Brown-Pedersen [4]) *Let A be a unital C^*-algebra. A is said to have extremal richness if $GL(A)ex(A)GL(A)$ is dense in A, where*

- *$GL(A)$: the set of all invertible elements in A.*
- *$ex(A)$: the set of extremal points in the closed unit ball A_1 of A.*

If $1 \notin A$, we say A is extremally rich if the untization $\tilde{A}$ of A is extremally rich.

An element in $GL(A)ex(A)GL(A)$ is said to be quasi-invertible.

The reason why A is called extremally rich is the following:

Theorem 1.2 (Brown-Pedersen [4]) *Let A be a unital C^*-algebra. The following conditions are equivalent:*

1. *$GL(A)ex(A)GL(A)$ is dense in A.*
2. *$co(ex(A)) = A_1$.*
3. *For any $x \in A_1$, $0 < \forall\varepsilon < 1/2$, there are extremal points u_1, u_2, u_3 in $ex(A)$ such that*

$$x = \frac{1}{2}(1-\varepsilon)u_1 + \frac{1}{2}(1-\varepsilon)u_2 + \varepsilon u_3.$$

Remarks 1.3 *1. From Russo-Dye's classical theorem ([23]),* $A_1 = \overline{co(ex(A))}$ *is guaranteed always. In the above (2), however, the closure of* $co(ex(A))$ *is not needed if* A *is extremally rich.*

2. If A *is finite, we can take extremal points in the above (3) as unitaries , that is,*

$GL(A)$ *is dense in* A *(*$sr(A) = 1$*) if and only if for any* $x \in A_1$ *and* $0 < \forall\varepsilon < 1/2$ *there are unitaries* u_1, u_2, u_3 *in* $ex(A)$ *such that*

$$x = \frac{1}{2}(1-\varepsilon)u_1 + \frac{1}{2}(1-\varepsilon)u_2 + \varepsilon u_3.$$

3. Let A *be a unital abelian* C^**-algebra* $C(X)$*. Then,* $GL(A)$ *is dense in* A *if and only if* $dimX \leq 1$.

4. Suppose that A *is a prime unital* C^**-algebra. Then, an elements in* $ex(A)$ *is either isometry or co-isometry from Kadison's characterization ([12]). So, in this case* A *is extremally rich if and only if the set of one-sided invertible elements is dense in* A*.*

Examples 1.4 *1. AF* C^**-algebras (*$sr(A) = 1$ *,* $RR(A) = 0$*). Simple AT-algebras. More generaly, simple direct limit of real rank zero of subhomogeneous* C^**-algebras with Hausdorff spectrums and slow dimension growth ([6, 17]).*

2. von Neumann algebras ([20]).

3. Cuntz algebras $O_n (n \geq 2)$ *(*$sr(O_n) = \infty$ *,* $RR(A) = 0$*). More generally, purely infinite simple unital* C^**-algebras ([20, 22]).*

Related to real rank,

Theorem 1.5 (Pedersen [20]) *Let* A *be a unital* C^**-algebra with extremal* *richness. Then,* $RR(A) \leq 1$.

Picture:

1) $M(A \otimes K)$ (A: separable infinite dimensional simple AF C*-algebras).

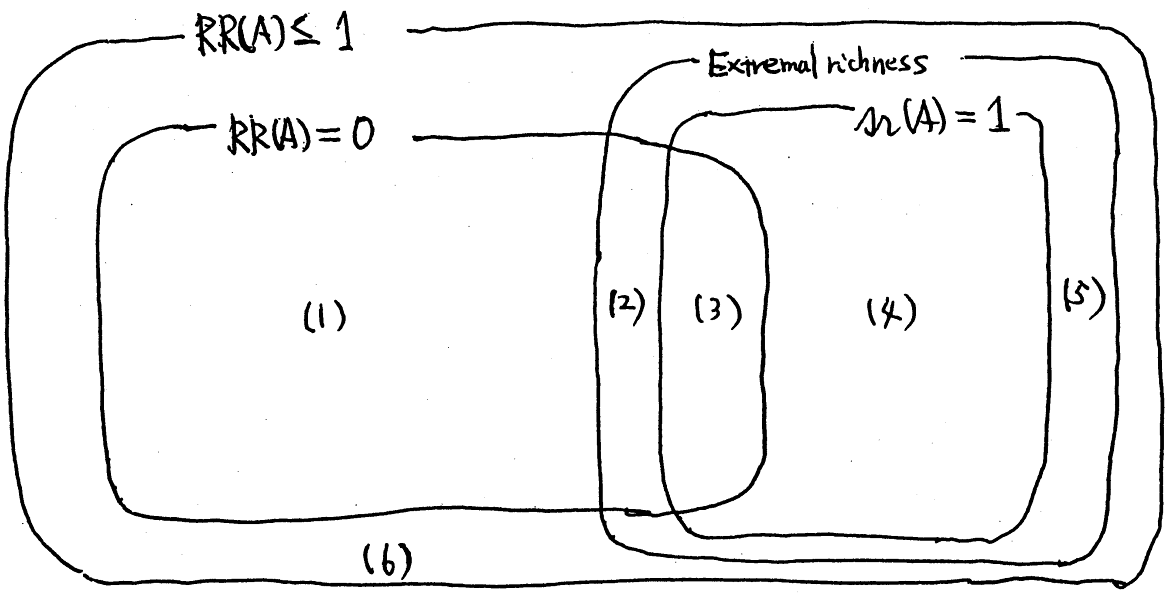
2) O_n $(n \geq 2)$, $B(H)$ $(\dim H = \infty)$.

3) AF C*-algebras, A_θ.

4) $C[0,1] \otimes A$ (A: AF algebras, etc.)

5) $O_n \otimes O_n \otimes E_n$ $(n \geq 3)$.

6) $C[0,1] \otimes A_\theta$.



2 Algebras of C*-valued continuous functions on a compact Hausdorff space

Proposition 2.1 (Osaka) *Let A be a unital C^*-algebra and let X be a compact Hausdorff space with $\dim X \geq 1$. Then, the followings are equivalent:*

1. *$C(X) \otimes A$ is extremally rich.*
2. *$sr(C(X) \otimes A) = 1$.*

Theorem 2.2 (Nagisa-Phillips-Osaka [18]) *Let A be a unital C^*-algebra. Then, we have*

1. *$sr(C[0,1] \otimes A) \leq sr(A) + 1$.*
2. *$RR(C[0,1] \otimes A) \leq RR(A) + 1$.*
3. *$sr(C[0,1]^2 \otimes A) \geq 2$.*
4. *$RR(C[0,1] \otimes A) \geq 1$.*
5. *If $RR(A) = 0$, $sr(A) = 1$, and $K_1(A) = 0$, then $sr(C[0,1] \otimes A) = 1$.*

Examples 2.3 1. *$C[0,1] \otimes A_\theta$ (θ is irrational) is not extremally rich., that is, $sr(C[0,1] \otimes A_\theta) = 2$. Note that its real rank is one.*

Proof. Take a unitary u in A_θ so that $u \notin GL_0(A_\theta)$. Set $f(t) = t + (1-t)u \in C[0,1] \otimes A_\theta$. Then, f can not be approximated by an invertible element. □

2. *$C[0,1] \otimes O_n$ (more generally, $C[0,1] \otimes$purely infinite simple C^*-algebra) is not extremally rich. In fact $sr(C[0,1] \otimes O_n) = \infty$. On the contrary, its real rank is one.*

3. *Let B be a free product C^*-algebra $C[0.1] *_{\nu_1,\nu_2} C[0,1]$, where ν_i is a*

Lebesgue measure on $[0,1]$. *Set* $A = B \otimes UHF$. *Then,* $RR(A) = 0, sr(A) = 1, K_1(A) = 0$, *and* $K_0(A) = K_0(UHF)$ *([22, 25]).*

Question 2.4 *If* $\dim X \geq 2$, *then*

$$sr(C(X) \otimes A) \geq 2$$

for any unital C^**-algebra* A *?*

3 Infinite C*-algebras

Theorem 3.1 (Pedersen[20], Rørdam[22], Larsen-Osaka[16]) *Let* A *be a (not necessarily* σ*-unital) purely infinite simple* C^**-algebra. Then,* A *is extremally rich.*

Theorem 3.2 (Osaka [19]) *Let* A *be a unital* C^**-algebra and* J *be a closed two-sided ideal in* A. *Consider the following* C^**-exact sequence:*

$$0 \to J \to A \to A/J \to 0.$$

Suppose that J *and* A/J *are purely infinite simple* C^**-algebras. Then,* A *is extremally rich, that is, the set of one-sided invertible elments is dense in* A.

Remark 3.3 *As for the behaviour of extremal richness under the above extension it is not enough to have that the ideal and quotient are extremally rich and that extremal points are lifted from the quotient ([4]).*

Corollary 3.4 (Nuclear case) *Let* $n \geq 2$ *and let* $E_n = C^*(s_1, \cdots . s_n)$ *be a* C^**-subalgebra of* $O_{n+1} = C^*(s_1, \cdots, s_n, s_{n+1})$. *Then,* $O_n \otimes O_n \otimes E_n$ *is* *extremally rich. In particular,*

$$RR(O_n \otimes O_n \otimes E_n) = \begin{cases} 0 & \text{if} \quad n = 2 \\ 1 & \text{if} \quad n \geq 3. \end{cases}$$

Corollary 3.5 (Non-nuclear case) *Let $k \geq 2$ and let $C^*_\gamma(F_k)$ be the reduced group C^*-algebra of the free group F_k with k generators. Then, $C^*_\gamma(F_k) \otimes O_n \otimes E_n$ is extremally rich. In particular,*

$$RR(C^*_\gamma(F_k) \otimes O_n \otimes E_n) = \begin{cases} 0 & \text{if} \quad n = 2 \\ 1 & \text{if} \quad n \geq 3. \end{cases}$$

Remark 3.6 $RR(E_n) = RR(O_n \otimes O_n) = RR(C^*_\gamma(F_k) \otimes O_n) = 0$. *But,* $RR(O_n \otimes O_n \otimes E_n) \neq 0$ *and* $RR(C^*_\gamma(F_k) \otimes O_n \otimes E_n) \neq 0$ *if* $n \geq 3$. *(Kodaka-Osaka [15]).*

However, extremal richness is not always stable under the minimal C*-tensor products.

Proposition 3.7 (Osaka [19]) *Let H be a separable infinite dimensional Hilbert space. Then, $B(H) \otimes B(H)$ is not extremally richness.*

Note that $RR(B(H) \otimes B(H)) \neq 0$.

Question 3.8 *Let A and B be simple unital C^*-algebras with extremal richness. Then, $A \otimes B$ is extremally rich ?*

Remark 3.9 *In the case of non-simple C^*-algebras, this question is false. Think $A = B = C[0,1]$.*

Here are some informations to the above question.

Theorem 3.10 (Brown-Pedersen [4]) *Let A be a simple unital C^*-algebra with extremal richness. Then, either A is purely infinite or $sr(A) = 1$.*

Unfortunately, there are many simple C*-algebras which are not extremally rich.

Theorem 3.11 (Villadsen [24]) *For any $n \in N$ there is a simple AH-algebra (i.e. direct limit of direct sums of homogeneous C^*-algebras $C(X_i) \otimes M_{n_i}$) with stable rank n.*

Here is an affirmative information.

Theorem 3.12 (Kirchberg [13], Winter[26]) *Let A and B be a unital simple C^*-algebras. If A is not stably finite, then $A \otimes B$ is purely infinite.*

4 C*-crossed products

Theorem 4.1 (Jeong-Kodaka-Osaka [9], Kishimoto-Kumjian [14]) *Let A be a unital purely infinite simple C^*-algebra and $\alpha : G \to Aut(A)$ be an action of a discrete group G. Suppose that α is outer, that is, $\alpha_g (g \neq e)$ is outer for any $g \in G$. Then, $A \times_\alpha G$ is purely infinite simple C^*-algebra.*

Remark 4.2 *If α is not outer, then $A \times_\alpha Z$ is not always extremally rich. In fact, if α is an n periodic automorphism on A, then $A \times_\alpha Z$ is not extremally rich.*

Theorem 4.3 (Jeong-Osaka [10], Izumi [8]) *Let A be a unital purely infinite simple C^*-algebra and $\alpha : G \to Aut(A)$ be an action of a finite group G. Then, $A \times_\alpha G$ has a decomposition $C_1 \oplus \cdots \oplus C_l$ into central summands of purely infnite simple C^*-algebras. That is, the crossed product is extremally rich.*

It is natural to ask that the stable rank one property is stable under crossed products by finite groups. We stress that the simplicity of a given C*-algebra is needed.

Examples 4.4 (Blackadar [2]) *There is a period two automorphism α on CAR algebra such that $CAR \times_\alpha Z_2$ is isomorphic to the tensor product of the Bunce-Deddens algebra of type 2^∞ $(= C)$ and CAR algebra. So, if $B = C[0,1] \otimes CAR$, then $B \times_{id \otimes \alpha} Z_2$ is isomorphic to $C[0,1] \otimes C \otimes CAR$. As the same argument in the case of A_θ (see Example 2.3 and Theorem 2.2), this C*-algebra has stable rank two, so not extremally rich.*

Question 4.5 *Let A be a simple unital C*-algebra with stable rank one and let $\alpha : G \to Aut(A)$ be an action of a finite group G. Then, does $A \times_\alpha G$ have stable rank one ?*

5 Multiplier algebras of simple C*-algebras of real rank zero

In sections 5 and 6 we will explain some conditions on a separable , simple infinite dimensional C*-algebra A of real rank zero under which we can deduce whether the multiplier algebras $M(A)$, $M(A \otimes K)$, and the corona algebras $Q(A)$, and $Q(A \otimes K)$. Those results came from a joint work with Nadia S. Larsen ([16]).

Theorem 5.1 ([16]) *Let A be a separable unital simple C*-algebra of real rank zero with a finite trace. Then, $M(A \otimes K)$ is not extremally rich.*

The point of the proof : We find a proper isometry in $Q(A \otimes K)$ which can not be lifted to an isometry in $M(A \otimes K)$. This is a little technical. □

Theorem 5.2 ([16]) *Let A be a σ-unital C^*-algebra of real rank zero. Assume that A has a finite trace. Then $M(A)$ is not extremally rich.*

The point of the proof: Since $M(A)$ is finite, extremal richness would imply that $sr(M(A)) = 1$. But, since $M(A)/A$ is purely infinite (not necesarily simple) ([27]), $M(A)/A$ has a higher stable rank. So, this is a contradiction. □

Proposition 5.3 (see Theorem 3.1) *Let A be a σ-unital non-unital purely infinite simple C^*-algebra. Then, $M(A)$ is extremally rich.*

6 Corona algebra of simple C*-algebra of real rank zero

Theorem 6.1 ([16]) *Let A be a separable, simple C^*-algebra with real rank zero such that $M(A \otimes K)$ has exactly one proper closed two-sided ideal J strictly containig $A \otimes K$. Then, $Q(A \otimes K)$ is extremally rich.*

The point of the proof: Let π be the canonical quotient map from $M(A \otimes K)$ to $Q(A \otimes K)$. Then, $\pi(J)$ and $Q(A \otimes K)/\pi(J)$ are purely infinite simple C*-algebras ([28]). So, the statement comes from Theorem 3.1. □

Examples 6.2 ([1]) *The following C^*-algebras satisfy the conditions in Theorem 6.1.*

1. *Finite matroid C^*-algebras.*
2. *A_θ (θ is irrational).*
3. *Bunce-Deddens algebras.*

However, even if A is an AF algebra we cannot be certain that its corona algebra is always extremally rich.

(stable case)

Proposition 6.3 (Brown [5]) *Let D be a simple AF C^*-algebra with exact two extremal traces. Then, $Q(D \otimes K)$ is not extremally rich.*

Using the ideal structure of the multiplier algebra of a simple AF algebra due to G. A. Elliott ([7]) and H. Lin ([11]) we get

Theorem 6.4 ([16]) *Let A be a separable, simple unital AF algebra with at least two extremal points in the set of semi-finite traces on A. Then, $Q(A \otimes K)$ is not extremally rich.*

The point of the proof: Take τ_1, τ_2 be distinct extremal semi-finie traces on A. Using a result in [7] the closure J_{τ_i} $(= J_i)$ of the set $\{x \in M(A \otimes K) | \tau_i(x^*x) < \infty\}$ ($i = 1, 2$) are proper maximal ideals in $M(A \otimes K)$. Then, $M(A \otimes K)/J_i$ is a purely infinite simple C*-algebra, which is isomorphic to $Q(A \otimes K)/\pi(J_i)$ $(i = 1, 2)$, where π is the canonical quotient map from $M(A \otimes K)$ to $Q(A \otimes K)$.

Consider the C*-exact sequence:

$$0 \to J_1 \cap J_2/A \otimes K \to Q(A \otimes K) \to J_1/J_1 \cap J_2 \oplus J_2/J_1 \cap J_2 \to 0.$$

Then, there is an extremal point in the closed unit ball of $J_1/J_1 \cap J_2 \oplus J_2/J_1 \cap J_2$ which is neither an isometry nor co-isometry. So, it can not be lifted to an extremal point in the closed unit ball of $Q(A \otimes K)$. Here we get the assertion from Remark 3.2. $\square$

(Non-stable case)

For a separable, simple , non-elementsry AF algebra with a dimension group G the set $S = S_u(G)$ represents the homomorphism $\tau : G \to R$ such that $\tau(G_+) \geq 0$ and $\tau(u) = 1$ for some fixed element u in G_+. The set of extremal points of the convex compact set S is denoted by $E(S)$. With $Aff(S)$ the set of affine , real continuous functions on S one has a positive homomorphism $\theta : G \to Aff(S)$ which sends a to $\hat{a}$ defined by $\hat{a}(\tau) = \tau(a)$. The image of G under θ is dense additive semigroup in $Aff(S)$. Let $F = \{\tau \in S | \tau(1) = \infty\}$, where 1 is the unit of $M(A)$ and every τ in S is extended to a trace still denoted by τ on $M(A)_+$.

Proposition 6.5 ([16]) *Let A be a separable, simple , non-unital AF algebra. Suppose that $E(S)$ has only finitely many points and $F \cap E(S)$ has at least two points . Then, $Q(A)$ is not extremally rich.*

The point of the proof : Through some steps we know that there are proper maximal ideal J_1, J_2 in $M(A)$ such that $M(A)/J_i$ $(i = 1, 2)$ are purely infinite simple C*-algebras. So, as the same argument in Theorem 6.4 we get the assertion. □

Remark 6.6 *If A is a separable simple non-unital AF-algebra with many extremal traces, then we don't know whether J_τ is maximal for an infinite extremal trace τ.*

Proposition 6.7 ([16]) *If A is a separable simple non-unital AF algebra with finitely many extremal traces of which exactly one is infinite, then $Q(A)$ is extremally rich.*

Proposition 6.8 ([11]) *If A is a separable simple AF algebra with finitely many extremal traces of which no one is infinite, then $Q(A)$ is extremally rich.*

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