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A Rohlin Type Theorem for Automorphisms of Certain Purely Infinite $C^*$-Algebras

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1 Introduction

A noncommutative Rohlin type theorem is a fundamental tool for the classification theory of actions of operator algebras. This theorem was first introduced by A. Connes for single automorphisms (i.e. actions of $\mathbb{Z}$) of finite von Neumann algebras [3]. Subsequently it was extended for actions of more general groups [19, 20]. Also in the framework of $C^*$-algebras this type of theorem was established first for the UHF algebras [1, 8, 9] and recently for some AF, AT algebras and some purely infinite simple $C^*$-algebras [12, 13, 14]. In particular A. Kishimoto showed the Rohlin type theorem for automorphisms of the Cuntz algebras $O_n$ with $n$ finite [12]. Our first motivation is to obtain a similar result for the Cuntz algebra $O_\infty$. When $n$ is finite, the Rohlin property of the unital one-sided shift on the UHF algebra $M_{n\infty}$ plays a crucial role to derive Rohlin projections from outer automorphisms of $O_n$. However for $O_\infty$, there does not seem to be a similar mechanism at work. But fortunately by the progress of the classification theory of purely infinite simple $C^*$-algebras due to E. Kirchberg, N.C. Phillips and M. Rørdam, the Cuntz algebras $O_n$, $n = 2, 3, \ldots, \infty$ (or more generally the purely infinite unital simple $C^*$-algebras which are in the bootstrap category $\mathcal{N}$ and have trivial $K_1$-groups) can be decomposed as the crossed products of unital AF algebras by proper (i.e. non-unital) corner endomorphisms [10, 23, 24]. Moreover these non-unital endomorphisms also have the Rohlin property like the unital one-sided shift on $M_{n\infty}$ [24]. We shall use these endomorphisms to derive Rohlin projections.
2 Crossed product decomposition

We start our argument with some definitions of key words which we use throughout this paper. For details we refer to [21, 24].

Definition 1 Let $\alpha$ be a (unital or non-unital) endomorphism on a unital $C^*$-algebra $A$. Then $\alpha$ is said to have the Rohlin property if for any $M \in \mathbb{N}$, finite subset $F$ of $A$ and $\varepsilon > 0$, there exist projections $e_0, \ldots, e_{M-1}, f_0, \ldots, f_M$ in $A$ such that

$$\sum_{i=0}^{M-1} e_i + \sum_{j=0}^{M} f_j = 1,$$

$$e_i \alpha(1) = \alpha(1)e_i, \quad f_j \alpha(1) = \alpha(1)f_j,$$

$$\|e_i x - xe_i\| < \varepsilon, \quad \|f_j x - xf_j\| < \varepsilon,$$

$$\|\alpha(e_i) - e_{i+1}\alpha(1)\| < \varepsilon, \quad \|\alpha(f_j) - f_{j+1}\alpha(1)\| < \varepsilon$$

for $i = 0, \ldots, M - 1$, $j = 0, \ldots, M$ and all $x \in F$, where $e_M \equiv e_0$, $f_{M+1} \equiv f_0$.

Definition 2 An endomorphism $\rho$ on a unital $C^*$-algebra $B$ is called a corner endomorphism if $\rho$ is an isomorphism from $B$ onto $\rho(1)B\rho(1)$. A corner endomorphism $\rho$ is called a proper corner endomorphism if $\rho$ is non-unital. Let $\rho$ be a corner endomorphism on $B$. Then the crossed product $B \rtimes_{\rho} \mathbb{N}$ is defined to be the universal $C^*$-algebra generated by a copy of $B$ and an isometry $s$ which implements $\rho$, that is, $\rho(b) = sbs^*$ for all $b \in B$.

Let $\mathcal{N}$ be the smallest full subcategory of the separable nuclear $C^*$-algebras which contains the separable Type I $C^*$-algebras and is closed under strong Morita equivalence, inductive limits, extensions, and crossed products by $\mathbb{R}$ and by $\mathbb{Z}$ [25]. A simple unital $C^*$-algebra $A$, which has at least dimension two, is said to be purely infinite if for any nonzero positive element $a \in A$ there exists $x \in A$ such that $xax^* = 1$. For convenience let $\mathcal{A}$ denote the purely infinite unital simple $C^*$-algebras which are in the bootstrap category $\mathcal{N}$ and have trivial $K_1$-groups. According to Theorem 3.1, Proposition 3.7, Corollary 4.6 in [24] and to Theorem 4.2.4 in [23] we have the following theorem immediately.

Theorem 3 For any $C^*$-algebra $A$ in $\mathcal{A}$ there exist a unital simple AF algebra $B$ with a unique tracial state, unital finite-dimensional $C^*$-subalgebras $(B_N | N \in \mathbb{N})$ of $B$ and a proper corner endomorphism $\rho$ on $B$ with the Rohlin property such that

$$A \cong B \rtimes_{\rho} \mathbb{N},$$

$$B_N \subseteq B_{N+1}, \quad \bigcup_{N \in \mathbb{N}} B_N \text{ is dense in } B,$$

$$\rho(B_N) \subseteq B_{N+1}, \quad \rho B_N \rho \subseteq \rho(B_{N+1})$$
for all \(N \in \mathbb{N}\), where \(p \equiv \rho(1) \neq 1\) and that \(p\) is full in \(B_1\), i.e. \(p \in B_1\) and the linear hull of \(B_1pB_1\) is \(B_1\). Conversely every \(C^*\)-algebra arising as a crossed product algebra described above and having the trivial \(K_1\)-group is in \(A\).

Henceforth we let \(A\) denote a \(C^*\)-algebra in \(A\) and let \(B, (B_N), \rho, p\) be as in the statement of Theorem 3. Finally in this section we state some technical lemma needed later. Since \(p\) is full in \(B_1\) we have elements \(a_1, \ldots, a_r\) in \(B_1\) such that
\[
\sum_{i=1}^{r} a_i p a_i^* = 1, \quad a_i p = a_i.
\]
Let \(s\) be an isometry in \(A \cong B \rtimes_{\rho} \mathbb{N}\) which implements \(\rho\). Define \(\sigma(x) = \sum_{i=1}^{r} a_i x s a_i^*x\) for \(x \in A\), then \(\sigma\) has the following properties ([24, Lemma 6.3.]):

**Lemma 4**

1. \(\sigma \upharpoonright A \cap B_2'\) is a unital \(*\)-homomorphism.
2. \(\sigma(A \cap B_{N+1}') \subseteq A \cap B_N'\) for all \(N \in \mathbb{N}\).
3. \(s^j x s^j = \sigma^j(x) s^j s^j = s^j s^j \sigma^j(x)\) for all \(j \in \mathbb{N}\), and \(x \in A \cap B_{j+1}'\).

### 3 Rohlin type theorem

**Theorem 5** Let \(A\) be a \(C^*\)-algebra in the class \(A\). For any approximately inner automorphism \(\alpha\) of \(A\) the following conditions are equivalent:

1. \(\alpha^k\) is outer for any nonzero integer \(k\).
2. \(\alpha\) has the Rohlin property.

Here an automorphism of a \(C^*\)-algebra is said to be approximately inner if it can be approximated pointwise by inner automorphisms. It is clear that (2) implies (1). To show the converse we take several steps. Since \(A\) is in \(A\) we use the notation appeared in the previous section. Suppose that (1) in Theorem 5 holds. The next three lemmas follow by the methods used in [6, 12]

**Lemma 6** Let \(q\) be a projection in \(A \cap B_2'\). Then
\[
c(\alpha^k \sigma(q)) = c(\alpha^k(q))
\]
for any \(k \in \mathbb{Z}\), where \(c(\cdot)\) denotes the central support in the enveloping von Neumann algebra \(A^{**}\) of \(A\).

Let \(\text{Proj}(A)\) denote the projections of a \(C^*\)-algebra \(A\).
Lemma 7 Let $l, m$ and $N$ be nonnegative integers with $N \geq l + m + 2$ and let $k$ be a nonzero integer. Then for any nonzero projection $e$ in $A \cap B_{N'}$,

$$\inf\{ \|q\alpha^k\sigma^l(q)\| \mid q \in \text{Proj} \sigma^m(e(A \cap B_{N'})e) \setminus \{0\} \} = 0.$$ 

Lemma 8 Let $K, L$ and $N$ be positive integers with $N \geq K + L + 2$ and let $\varepsilon > 0$. Then there exists a nonzero projection $e$ in $A \cap B_{N'}$ such that

$$[e] = 0 \quad \text{in} \quad K_0(A \cap B_{N'})$$

$$\|\alpha^{k_1}\sigma^{l_1}(e) \cdot \alpha^{k_2}\sigma^{l_2}(e)\| < \varepsilon$$

for $k_1, k_2 = 0, \ldots, K$ and $l_1, l_2 = 0, \ldots, L$ with $(k_1, l_1) \neq (k_2, l_2)$.

From Lemma 8 we have the next lemma, which says we almost find Rohlin projections if we drop the condition that the sum of the projections is 1.

Lemma 9 Let $M, N$ be positive integers and let $\varepsilon > 0$. Then there exist mutually orthogonal nonzero projections $e_0, \ldots, e_{M-1}$ in $A$ such that

$$\|\alpha(e_i) - e_{i+1}\| < \varepsilon,$$

$$e_i \in B_{N'}, \quad \|e_i s - se_i\| < \varepsilon$$

for $i = 0, \ldots, M - 1$, where $e_M = e_0$.

To derive genuine Rohlin projections, we can exactly follow the method of Proof of Theorem 3.1 in [12], replacing almost $\Phi$-invariance there by almost commutativity with $B_N \cup \{s, s^*\}$. In this process the number of towers of projections increases from one to two as in Definition 1.

4 Examples

We present several examples of automorphisms which have the Rohlin property. Let $A$ be a $C^*$-algebra in $A$ and let $B \rtimes_{\rho} N$ be a crossed product decomposition of $A$ as in Section 2. By the universality of the crossed product we have the dual action $\hat{\rho}$ of $\mathbb{T}$ on $A$, that is, we define $\hat{\rho}$ by the formulas: $\hat{\rho}(b) = b$, $\hat{\rho}_\lambda(s) = \lambda s$ for all $b \in B$, $\lambda \in \mathbb{T}$. Using the universality similarly for an automorphism $\alpha$ of $B$ with $\alpha \circ \rho = \rho \circ \alpha$, we define an automorphism $\tilde{\alpha}$ of $B \rtimes_{\rho} N$ by $\tilde{\alpha}(b) = \alpha(b)$ for all $b \in B$ and by $\tilde{\alpha}(s) = s$. Clearly $\tilde{\alpha}$ commutes with each $\hat{\rho}_\lambda$ from the definition. Then we have

Proposition 10 An automorphism $\tilde{\alpha} \circ \hat{\rho}_\lambda$ of $A \cong B \rtimes_{\rho} N$ is approximately inner for any $\lambda \in \mathbb{T}$, and one has the following:
(1) If $\alpha$ is the identity mapping on $A$ then $\tilde{\alpha} \circ \hat{\rho}_\lambda = \hat{\rho}_\lambda$ is outer for any $\lambda \in T \setminus \{0\}$.

(2) If $\alpha$ is outer (as an automorphism of $B$) then $\tilde{\alpha} \circ \hat{\rho}_\lambda$ is outer for any $\lambda \in T$.

(3) If $\alpha$ is inner then $\tilde{\alpha} \circ \hat{\rho}_\lambda$ are inner for at most a countable number of $\lambda \in T$.

Therefore in any case $\tilde{\alpha} \circ \hat{\rho}_\lambda$ have the Rohlin property for an uncountable number of $\lambda \in T$.

References


