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A Rohlin Type Theorem for Automorphisms of Certain Purely Infinite C^*-Algebras

Hideki Nakamura
(中村英樹)
Department of Mathematics
Hokkaido University
Sapporo 060 Japan
h-nakamu@math.hokudai.ac.jp

1 Introduction

A noncommutative Rohlin type theorem is a fundamental tool for the classification theory of actions of operator algebras. This theorem was first introduced by A. Connes for single automorphisms (i.e. actions of $\mathbb{Z}$) of finite von Neumann algebras [3]. Subsequently it was extended for actions of more general groups [19, 20]. Also in the framework of C^*-algebras this type of theorem was established first for the UHF algebras [1, 8, 9] and recently for some AF, A$\mathbb{T}$ algebras and some purely infinite simple C^*-algebras [12, 13, 14]. In particular A. Kishimoto showed the Rohlin type theorem for automorphisms of the Cuntz algebras O_n with n finite [12]. Our first motivation is to obtain a similar result for the Cuntz algebra O_∞. When n is finite, the Rohlin property of the unital one-sided shift on the UHF algebra M_{n^∞} plays a crucial role to derive Rohlin projections from outer automorphisms of O_n. However for O_∞, there does not seem to be a similar mechanism at work. But fortunately by the progress of the classification theory of purely infinite simple C^*-algebras due to E. Kirchberg, N.C. Phillips and M. Rϕrdam, the Cuntz algebras O_n, $n=2,3\ldots,\infty$ (or more generally the purely infinite unital simple C^*-algebras which are in the bootstrap category $\mathcal{N}$ and have trivial K_1-groups) can be decomposed as the crossed products of unital AF algebras by proper (i.e. non-unital) corner endomorphisms [10, 23, 24]. Moreover these non-unital endomorphisms also have the Rohlin property like the unital one-sided shift on M_{n^∞} [24]. We shall use these endomorphisms to derive Rohlin projections.

2 Crossed product decomposition

We start our argument with some definitions of key words which we use throughout this paper. For details we refer to [21, 24].

Definition 1 Let α be a (unital or non-unital) endomorphism on a unital C^*-algebra A. Then α is said to have the *Rohlin property* if for any $M \in \mathbb{N}$, finite subset F of A and $\varepsilon > 0$, there exist projections $e_0, \ldots, e_{M-1}, f_0, \ldots, f_M$ in A such that

$$\sum_{i=0}^{M-1} e_i + \sum_{j=0}^{M} f_j = 1 \ ,$$

$$e_i\alpha(1) = \alpha(1)e_i, \quad f_j\alpha(1) = \alpha(1)f_j \ ,$$

$$\|e_i x - x e_i\| < \varepsilon, \quad \|f_j x - x f_j\| < \varepsilon \ ,$$

$$\|\alpha(e_i) - e_{i+1}\alpha(1)\| < \varepsilon, \quad \|\alpha(f_j) - f_{j+1}\alpha(1)\| < \varepsilon$$

for $i = 0, \ldots, M-1$, $j = 0, \ldots, M$ and all $x \in F$, where $e_M \equiv e_0$, $f_{M+1} \equiv f_0$.

Definition 2 An endomorphism ρ on a unital C^*-algebra B is called a *corner endomorphism* if ρ is an isomorphism from B onto $\rho(1)B\rho(1)$. A corner endomorphism ρ is called a *proper corner endomorphism* if ρ is non-unital. Let ρ be a corner endomorphism on B. Then the crossed product $B \rtimes_\rho \mathbb{N}$ is defined to be the universal C^*-algebra generated by a copy of B and an isometry s which implements ρ, that is, $\rho(b) = sbs^*$ for all $b \in B$.

Let $\mathcal{N}$ be the smallest full subcategory of the separable nuclear C^*-algebras which contains the separable Type I C^*-algebras and is closed under strong Morita equivalence, inductive limits, extensions, and crossed products by $\mathbb{R}$ and by $\mathbb{Z}$ [25]. A simple unital C^*-algebra A, which has at least dimension two, is said to be *purely infinite* if for any nonzero positive element $a \in A$ there exists $x \in A$ such that $xax^* = 1$. For convenience let $\mathcal{A}$ denote the purely infinite unital simple C^*-algebras which are in the bootstrap category $\mathcal{N}$ and have trivial K_1-groups. According to Theorem 3.1, Proposition 3.7, Corollary 4.6 in [24] and to Theorem 4.2.4 in [23] we have the following theorem immediately.

Theorem 3 *For any C^*-algebra A in $\mathcal{A}$ there exist a unital simple AF algebra B with a unique tracial state, unital finite-dimensional C^*-subalgebras $(B_N \,|\, N \in \mathbb{N})$ of B and a proper corner endomorphism ρ on B with the Rohlin property such that*

$$A \cong B \rtimes_\rho \mathbb{N} \ ,$$

$$B_N \subseteq B_{N+1}, \quad \bigcup_{N\in\mathbb{N}} B_N \ \textit{is dense}\ B,$$

$$\rho(B_N) \subseteq B_{N+1}, \quad pB_Np \subseteq \rho(B_{N+1})$$

for all $N \in \mathbb{N}$, where $p \equiv \rho(1) \neq 1$ and that p is full in B_1, i.e. $p \in B_1$ and the linear hull of B_1pB_1 is B_1. Conversely every C^*-algebra arising as a crossed product algebra described above and having the trivial K_1-group is in $\mathcal{A}$.

Henceforth we let A denote a C^*-algebra in $\mathcal{A}$ and let B, (B_N), ρ, p be as in the statement of Theorem 3. Finally in this section we state some technical lemma needed later. Since p is full in B_1 we have elements $a_1, \ldots, a_r$ in B_1 such that

$$\sum_{i=1}^{r} a_i p a_i^* = 1, \quad a_i p = a_i \ .$$

Let s be an isometry in $A \cong B \rtimes_\rho \mathbb{N}$ which implements ρ. Define $\sigma(x) = \sum_{i=1}^{r} a_i s x s^* a_i^*$ for $x \in A$, then σ has the following properties ([24, Lemma 6.3.]):

Lemma 4 (1) *$\sigma \restriction A \cap B_2{}'$ is a unital $*$-homomorphism.*

(2) *$\sigma(A \cap B_{N+1}{}') \subseteq A \cap B_N{}'$ for all $N \in \mathbb{N}$.*

(3) *$s^j x s^{*j} = \sigma^j(x) s^j s^{*j} = s^j s^{*j} \sigma^j(x)$ for all $j \in \mathbb{N}$, and $x \in A \cap B_{j+1}{}'$.*

3 Rohlin type theorem

Theorem 5 *Let A be a C^*-algebra in the class $\mathcal{A}$. For any approximately inner automorphism α of A the following conditions are equivalent:*

(1) *α^k is outer for any nonzero integer k.*

(2) *α has the Rohlin property.*

Here an automorphism of a C^*-algebra is said to be *approximately inner* if it can be approximated pointwise by inner automorphisms. It is clear that (2) implies (1). To show the converse we take several steps. Since A is in $\mathcal{A}$ we use the notation appeared in the previous section. Suppose that (1) in Theorem 5 holds. The next three lemmas follow by the methods used in [6, 12]

Lemma 6 *Let q be a projection in $A \cap B_2{}'$. Then*

$$c(\alpha^k \sigma(q)) = c(\alpha^k(q))$$

for any $k \in \mathbb{Z}$, where $c(\cdot)$ denotes the central support in the enveloping von Neumann algebra A^{**} of A.

Let $\mathrm{Proj}(A)$ denote the projections of a C^*-algebra A.

Lemma 7 *Let l, m and N be nonnegative integers with $N \geq l+m+2$ and let k be a nonzero integer. Then for any nonzero projection e in $A \cap B_N{}'$,*

$$\inf\{\, \|q\alpha^k\sigma^l(q)\| \mid q \in \mathrm{Proj}\sigma^m(e(A \cap B_N{}')e) \setminus \{0\}\,\} = 0\ .$$

Lemma 8 *Let K, L and N be positive integers with $N \geq K+L+2$ and let $\varepsilon > 0$. Then there exists a nonzero projection e in $A \cap B_N{}'$ such that*

$$[e] = 0 \quad \text{in } K_0(A \cap B_N{}')$$

$$\|\alpha^{k_1}\sigma^{l_1}(e) \cdot \alpha^{k_2}\sigma^{l_2}(e)\| < \varepsilon$$

for $k_1, k_2 = 0, \ldots, K$ and $l_1, l_2 = 0, \ldots, L$ with $(k_1, l_1) \neq (k_2, l_2)$.

From Lemma 8 we have the next lemma, which says we almost find Rohlin projections if we drop the condition that the sum of the projections is 1.

Lemma 9 *Let M, N be positive integers and let $\varepsilon > 0$. Then there exist mutually orthogonal nonzero projections $e_0, \ldots, e_{M-1}$ in A such that*

$$\|\alpha(e_i) - e_{i+1}\| < \varepsilon\ ,$$

$$e_i \in B_N{}'\ , \quad \|e_i s - s e_i\| < \varepsilon$$

for $i = 0, \ldots, M-1$, where $e_M = e_0$.

To derive genuine Rohlin projections, we can exactly follow the method of Proof of Theorem 3.1 in [12], replacing almost Φ-invariance there by almost commutativity with $B_N \cup \{s, s^*\}$. In this process the number of towers of projections increases from one to two as in Definition 1.

4 Examples

We present several examples of automorphisms which have the Rohlin property. Let A be a C^*-algebra in $\mathcal{A}$ and let $B \rtimes_\rho \mathbb{N}$ be a crossed product decomposition of A as in Section 2. By the universality of the crossed product we have the dual action $\hat{\rho}$ of $\mathbb{T}$ on A, that is, we define $\hat{\rho}$ by the formulas: $\hat{\rho}(b) = b$, $\hat{\rho}_\lambda(s) = \lambda s$ for all $b \in B$, $\lambda \in \mathbb{T}$. Using the universality similarly for an automorphism α of B with $\alpha \circ \rho = \rho \circ \alpha$, we define an automorphism $\tilde{\alpha}$ of $B \rtimes_\rho \mathbb{N}$ by $\tilde{\alpha}(b) = \alpha(b)$ for all $b \in B$ and by $\tilde{\alpha}(s) = s$. Clearly $\tilde{\alpha}$ commutes with each $\hat{\rho}_\lambda$ from the definition. Then we have

Proposition 10 *An automorphism $\tilde{\alpha} \circ \hat{\rho}_\lambda$ of $A \cong B \rtimes_\rho \mathbb{N}$ is approximately inner for any $\lambda \in \mathbb{T}$, and one has the following:*

(1) *If* α *is the identity mapping on* A *then* $\tilde{\alpha} \circ \hat{\rho}_\lambda = \hat{\rho}_\lambda$ *is outer for any* $\lambda \in \mathbb{T} \setminus \{0\}$.

(2) *If* α *is outer (as an automorphism of* B*) then* $\tilde{\alpha} \circ \hat{\rho}_\lambda$ *is outer for any* $\lambda \in \mathbb{T}$.

(3) *If* α *is inner then* $\tilde{\alpha} \circ \hat{\rho}_\lambda$ *are inner for at most a countable number of* $\lambda \in \mathbb{T}$.

Therefore in any case $\tilde{\alpha} \circ \hat{\rho}_\lambda$ *have the Rohlin property for an uncountable number of* $\lambda \in \mathbb{T}$.

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