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Kyoto University
Solutions of Ginzburg-Landau type systems with Higher-dimensional Zero Sets

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1 Introduction

In this paper, we consider the following elliptic system of diagonal type:

\[ \Delta V + \lambda (1 - |V|^2) V = 0 \]

(1)

where \( V = (V^1, \ldots, V^n) \) is defined on some domain in \( \mathbb{R}^{n+k}, n \geq 2, k \geq 1 \), and \( \lambda \in \mathbb{R} \) is a parameter.

Here, we construct some solutions of (1) on certain domains in \( \mathbb{R}^{n+k} \), with boundary values, invariant under the action of a \( k \)-parameter group of isometries of \( \mathbb{R}^{n+k} \), and having nontrivial \( k \)-dimensional zero sets.

When \( n = 2 \), the equation (1) is the Ginzburg-Landau system (GLS), which is used as a mathematical model for many physical phenomena, such as super-conductivity and super-fluidity. In the theory of super-conductivity, the unknown \( V \) represents an order parameter which has two degrees of freedom, and its zero set, called vortices, corresponds to the region of the normal state in super-conductors. So, especially our result produces an example of solutions of the GLS in \( \mathbb{R}^3 \) with curved vortex lines.

Some results concerning the isolated zeros of solutions of the GLS in \( \mathbb{R}^2 \) are known ([1], [2]), however there seems to be no explicit example of solutions with higher-dimensional nontrivial zero sets.

Our proof is based on the "equivariant construction" method due to N.Smål [9], in which the examples of minimal hypersurfaces in Euclidean spaces with higher-dimensional singularities are shown. Later, the same method was used to construct examples with higher-dimensional singularities, of harmonic maps [4], and of solutions of a certain non-linear elliptic equation [6].

Main result of this paper can be extended to equations with other type of nonlinearities, but we do not pursue here for simplicity of description.

2 Notations and statement of the main result

We follow the setting of "equivariant construction" method described in the papers [9],[4] and [6]: Let \( n \geq 2, k \geq 1 \) be two integers. Let \( \mathcal{U} \subset \mathbb{R}^k \) be an open set containing \( \{0\} \in \mathbb{R}^k \) and assume that there is a \( C^\infty \) group action

\[ \Phi : t \in \mathcal{U} \longrightarrow \Phi(t) \in \text{Isom}(\mathbb{R}^{n+k}) \]
here $\text{Isom}(\mathbb{R}^{n+k})$ means the group of isometries of $\mathbb{R}^{n+k}$. We will denote $\Phi(t)$ by $G_t$.

We define

\[
\Gamma = \{G_t(0) : t \in \mathcal{U}\},
\]
\[
\tilde{B}^n = B^n_t(0) \times \{0\}_k = \{\tilde{x} = (x, 0) \in \mathbb{R}^n \times \mathbb{R}^k, |x| < 1\},
\]
\[
\Omega = \{G_t(B^n) : t \in \mathcal{U}\}.
\]

So, $\Gamma$ is the orbit of $\{0\} \in \mathbb{R}^{n+k}$ of the group action $\Phi$, and $\Omega$ is the unit $n$-disc bundle over $\Gamma$ obtained by moving $\tilde{B}^n$ along $\Gamma$ by $G_t$, $t \in \mathcal{U}$. On the group action $\Phi$, we make the following assumptions: $\Gamma$ is a properly embedded $k$-dimensional submanifold in $\mathbb{R}^{n+k}$ and whenever $G_t(0) = 0$, we must have $G_t(\tilde{B}^n) = \tilde{B}^n$ for any $t \in \mathcal{U}$, that is, the isotropy group of 0 is the same as the one of $\tilde{B}^n$. Furthermore, when $G_t = O(t) + v_t$ is the decomposition of the element of $\text{Isom}(\mathbb{R}^{n+k})$, where $O(t) \in O(n + k)$, the orthogonal group of $\mathbb{R}^{n+k}$, and $v_t \in \mathbb{R}^{n+k}$, we define the group action

\[
\Phi_t : t \in \mathcal{U} \longmapsto G_t^\epsilon \in \text{Isom}(\mathbb{R}^{n+k}),
\]

and

\[
\Gamma^\epsilon = \{G_t^\epsilon(0) : t \in \mathcal{U}\} = (\frac{1}{\epsilon}) \Gamma,
\]
\[
\Omega^\epsilon = \{G_t^\epsilon(Id) : t \in \mathcal{U}\},
\]

where $G^\epsilon_t = O(t) + \frac{1}{\epsilon}v_t$. Note that under the assumption of the group action $\Phi$, $\Omega^\epsilon$ is well-defined and then $\Omega^\epsilon$ is the unit $n$-disc bundle over $\Gamma^\epsilon$ obtained by moving $\tilde{B}^n$ along $\Gamma^\epsilon$ by $G^\epsilon_t$, $t \in \mathcal{U}$. Note also that when $\epsilon > 0$ is sufficiently small, $\Omega^\epsilon$ is close locally the trivial product bundle $B^n_t(0) \times \mathbb{R}^k$ over $\{0\}_n \times \mathbb{R}^k$. Finally, for a map $U : \mathbb{R}^{n+k} \to \mathbb{R}^n$, we denote by $\Gamma(U)$ the set of zeros of $U$, namely, $\Gamma(U) = \{x : U(x) = 0 \in \mathbb{R}^n\}$.

Now we state the main result of this paper.

**Theorem** For any $\lambda \in \mathbb{R}$, there exists an open domain $\tilde{\Omega} \subset \mathbb{R}^{n+k}$ containing $\Gamma$, on which there are infinitely many solutions of (1) with boundary values, whose zero set is $\Gamma$.

In the proof of the theorem, we will show that there exists $\bar{\epsilon} > 0$ sufficiently small, such that for any $0 < \epsilon < \bar{\epsilon}$, there is a solution $U$ of

\[
\Delta U + \lambda \epsilon^2 (1 - |U|^2)U = 0 \quad \text{in} \quad \Omega^\epsilon,
\]
\[
\Gamma(U) = \Gamma^\epsilon
\]

with a boundary data fixed up to a finite dimensional space, and $U$ is invariant under the action $\Phi^\epsilon$, i.e., $U(G^\epsilon_t(\tilde{x})) = U(\tilde{x})$ for all $\tilde{x} \in \tilde{B}^n$ and $t \in \mathcal{U}$.

We will find a solution $U$ of (2) by solving the appropriate fixed point problem. We make essential use of the invariant condition of $U$, thanks to which, we can think of (2) as a PDE on each fibers of the disc bundle $\Omega^\epsilon$, especially on $\tilde{B}^n$ for $t = 0$. Note that the nonlinear term of (2) is well controlled when $\epsilon$ is small enough, so we can get a solution as a perturbation of the $\mathbb{R}^n$-valued harmonic function $v_0 : B^n_t(0) \to \mathbb{R}^n$, $v_0(x) = x$. Taking $\tilde{\Omega} = \epsilon \cdot \Omega^\epsilon$, and $V(y) = U(y_{\epsilon})$ for $y \in \tilde{\Omega}$ will give the desired result. The domain $\tilde{\Omega}$ so obtained, is the bundle over $\Gamma$ of the $n$-dimensional discs of radius $\epsilon$, so looks like locally a thin perturbed tube of radius $\epsilon$ with center axis $\Gamma$.

Now we describe our coordinate of $\Omega^\epsilon$: For $y \in \Omega^\epsilon$, there exists $x \in B^n_t(0)$ and $t \in \mathcal{U}$ such that $y = G_t^\epsilon(x)$, then let us denote $F : B^n_t(0) \times \mathcal{U} \to \Omega^\epsilon$, $F(x, t) = G^\epsilon_t(x)$. we will introduce the local coordinate system by this map, and identify $y$ with $(r, \theta, t)$ where $(r, \theta)$ are polar coordinates for $x \in B^n_t(0)$. So, functions defined on $\Omega^\epsilon$ can naturally be
considered as functions on $B_1^n(0) \times \mathcal{U}$ by $F$. Note for $\varepsilon > 0$ sufficiently small, $r$ also is the distance to $\Gamma_\varepsilon$.

In the sequel we use the following function spaces: For $\nu \in \mathbb{R}, \alpha \in (0,1), m = 0,1,2$, define

$$C^{m,\alpha,\nu}(\Omega_\varepsilon; \mathbb{R}^n) = \{ u \in C^{m,\alpha}_{\text{loc}}(\Omega_\varepsilon \setminus \Gamma_\varepsilon; \mathbb{R}^n) : |u|_{m,\alpha,\nu} < +\infty \},$$

where $| \cdot |_{m,\alpha,\nu}$ is the norm

$$|u|_{m,\alpha,\nu} = \sup_{0 < r \leq 1/2} \left( \sum_{j=0}^{m} |\nabla^j u|_{0,[s,2s]} s^{-\nu} + \sum_{j=0}^{m} |\nabla^j u|_{[(\alpha),[s,2s]} s^{j+\alpha-\nu} \right).$$

Here, $\nabla$ and $\nabla^2$ denote the gradient and Hessian respectively on $\Omega_\varepsilon$, and $|\eta|_{[0,2s]}$ and $|\eta|_{[(\alpha),[s,2s]}$ are the sup norm and the $\alpha$-th Hölder seminorm of a function (or section) $\eta$ on $\Omega_\varepsilon$ over the set $\{ y = y(r, \theta, t) \in \Omega_\varepsilon : s \leq r \leq 2s \}$. These are Banach spaces under the norm $| \cdot |_{m,\alpha,\nu}$, and if $u \in C^{m,\alpha,\nu}(\Omega_\varepsilon; \mathbb{R}^n)$, then $|u|$ decays like $r^\nu$ near $\Gamma_\varepsilon$.

Furthermore, let us define the closed subspace of $C^{m,\alpha,\nu}(\Omega_\varepsilon; \mathbb{R}^n)$ as

$$C^{m,\alpha,\nu}_G(\Omega_\varepsilon; \mathbb{R}^n) = \{ u \in C^{m,\alpha,\nu}(\Omega_\varepsilon; \mathbb{R}^n) : u(\Phi_{\varepsilon}(x)) = u(x) \text{ for all } x \in B_1^n(0), t \in \mathcal{U} \},$$

that is, maps in $C^{m,\alpha,\nu}$ which are $\Phi_\varepsilon$-invariant. We also denote $C^{m,\alpha}\Gamma(\partial \Omega_\varepsilon; \mathbb{R}^n)$ for the space of $\Phi_\varepsilon$-invariant boundary data in $C^{m,\alpha}(\partial \Omega_\varepsilon; \mathbb{R}^n)$.

Weighted Hölder spaces like above are now widely used for other nonlinear problems, see [9], [10], [4], [6], [8], [5], [3].

## 3 Proof of the Theorem

In this section, we seek for a solution of (2) satisfying (3) by the same technique as in [9], [4], [6]: linearization and solving the appropriate fixed point problem. First, we construct the approximate solution. We fix $\varepsilon > 0$. Let $v_0 : B_1^n(0) \rightarrow \mathbb{R}^n$ be the identity map $v_0(x) = x$; so evidently $\Gamma(v_0) = \{0\} \in \mathbb{R}^n$ and $\Delta_B v_0 = 0$, where $\Delta_B$ means the Laplace operator on $B_1^n(0)$. Now we define the approximate solution $u_\varepsilon : \Omega_\varepsilon \rightarrow \mathbb{R}^n$ by

$$u_\varepsilon(\Phi_{\varepsilon}(x)) = v_0(x) \text{ for } x \in B_1^n(0), t \in \mathcal{U}$$

where $\Phi_{\varepsilon} = (x_0) \in \hat{B}^n$. By definition of $\Omega_\varepsilon$ and by our assumption on the group action $\Phi$, $u_\varepsilon$ is well-defined and invariant under the action $\Phi_{\varepsilon}$. The zero set of $u_\varepsilon$ satisfies $\Gamma(u_\varepsilon) = \Gamma_\varepsilon$.

We wish to find a solution of (2) of the form

$$U(u) = u_\varepsilon + u$$

where the perturbation $u$ is assumed to be invariant under the action $\Phi_\varepsilon$ and to decay rapidly near $\Gamma_\varepsilon$, so as to ensure that $\Gamma(U(u)) = \Gamma_\varepsilon$.

Let $N(u)$ be the left hand side of (2) for $U(u)$, that is,

$$N(u) = \Delta U(u) + \lambda \varepsilon^2 (1 - |U(u)|^2) U(u).$$

We make a Taylor expansion of $N(u)$ about $u = 0$ to get

$$N(u) = N(0) + Lu + Q(u),$$
where

\[
\begin{align*}
N(0) &= \Delta u_{\varepsilon} + \lambda \varepsilon^2 (1 - |u_{\varepsilon}|^2) u_{\varepsilon}, \\
Lu &= \frac{d}{dt} N(tu)|_{t=0} \\
&= \Delta u + \lambda \varepsilon^2 \{(1 - |u_{\varepsilon}|^2) u - 2(u_{\varepsilon} \cdot u) u_{\varepsilon}\}, \\
Q(u) &= \int_{0}^{1} (1-t) \frac{d^2}{dt^2} N(tu) dt \\
&= \frac{-2 \lambda_{E}^2}{\mathcal{E}} \int_{0}^{1} (1-t) \{|u|^{2} u_{\zeta} + 2(u_{\varepsilon} \cdot u) u + 3t|u|^{2} u\} dt \\
&= \frac{-\lambda_{E}^2}{\mathcal{E}} \{|u|^{2} u_{\epsilon} + 2(u_{\epsilon} \cdot u) u + |u|^{2} u\},
\end{align*}
\]

here $\Delta$ means the Laplace operator on $\Omega_{\varepsilon}$. Now, if we define the linear operators

\[
R = \Delta - \Delta_{B^{n}}
\]

and

\[
\xi u = \lambda \varepsilon^2 \{(1 - |u_{\varepsilon}|^2) u - 2(u_{\xi} \cdot u) u_{\epsilon}\},
\]

then the equation $N(u) = 0$ can be rewritten as

\[
\Delta_{B^{n}} u = -N(0) - Ru - \xi u - Q(u) \tag{4}
\]

which we solve by contraction mapping argument on some weighted Hölder space. Note that if $u$ is invariant under the action $\Phi_{\varepsilon}$, all of the terms in (4) are also $\Phi_{\varepsilon}$-invariant, so we can consider (4) as a PDE on the slice $\tilde{B}^{n}$. This is crucial for our subsequent arguments.

To estimate the terms in the right hand side of (4), we need the following lemma due to R.Mazzeo and N.Smale [5].

**Lemma 1** Under the local coordination by $F$, we have

\[
\Delta = \Delta_{B^{n}} + \Delta_{R^{*}} + e_{1} \nabla^2 + e_{2} \nabla,
\]

where $\Delta$ and $\nabla$ are the Laplace operator and gradient on $\Omega_{\varepsilon}$, $e_{1} \in C^{\infty}((Sym^2 \Omega_{\varepsilon})^{*})$, $e_{2} \in C^{\infty}(T^{*}\Omega_{\varepsilon})$ are smooth sections and satisfy

\[
|e_{1}(x, t)| \leq C_{0} r_{\varepsilon}, \quad |e_{2}(x, t)| \leq C_{0} \varepsilon,
\]

\[
|e_{1}|_{(\alpha),[t, 2s]} \leq C_{0} s \varepsilon, \quad |e_{2}|_{(\alpha),[s, 2s]} \leq C_{0} \varepsilon
\]

for some constant $C_{0}$ independent of $\varepsilon > 0$ and $\alpha \in (0, 1)$.

For functions $u$ invariant under $\Phi_{\varepsilon}$, the factor $\Delta_{R^{*}}$ in (5) drops out. Using this lemma, we have

**Lemma 2** If $\varepsilon > 0$, $1 < \nu < 2$, and $u \in C_{G}^{2, \alpha, \nu}(\Omega_{\varepsilon}; R^n)$, then $N(0), Ru, \xi u, Q(u)$ are all in $C_{G}^{0, \alpha, \nu - 2}(\Omega_{\varepsilon}; R^n)$ and the following estimates hold:

\[
\begin{align*}
|N(0)|_{0, \alpha, \nu - 2} &\leq C_{1} \varepsilon (1 + |\lambda|) \varepsilon, \\
|R u|_{0, \alpha, \nu - 2} &\leq C_{1} \varepsilon |u|_{2, \alpha, \nu}, \\
|\xi u|_{0, \alpha, \nu - 2} &\leq C_{1} |\lambda| \varepsilon^2 |u|_{2, \alpha, \nu}, \\
|Q(u)|_{0, \alpha, \nu - 2} &\leq C_{1} |\lambda| \varepsilon^2 (|u|_{2, \alpha, \nu} + |u|_{2, \alpha, \nu}^3)
\end{align*}
\]

for some constant $C_{1} > 0$ independent of $\varepsilon$ and $\lambda$. 
Since $\Phi_\epsilon$-invariant, so are also all terms appeared in the right hand side of (4), and can be considered as functions of $B_1^\alpha(0)$. By definition, the map $u_\epsilon$ satisfies $\Delta_{B} u_\epsilon = 0$, so we have

$$N(0) = \Delta u_\epsilon + \lambda \epsilon^2 (1 - |u_\epsilon|^2) u_\epsilon = (\Delta - \Delta_{B}) u_\epsilon + \lambda \epsilon^2 (1 - |u_\epsilon|^2) u_\epsilon.$$  

Then using Lemma 1 and the fact that $|\nabla u_\epsilon(x)| + |\nabla^2 u_\epsilon(x)| \leq C$ and $|u_\epsilon(x)| \leq 1$ for some constant $C$ independent of $\epsilon$ and $x \in B_1^\alpha(0)$, we have

$$|N(0)(x)| \leq |e_1 \nabla^2 u_\epsilon(x)| + |e_2 \nabla u_\epsilon(x)| + |\Delta - \Delta_{B}| u_\epsilon + \lambda \epsilon^2 (1 - |u_\epsilon|^2) |u_\epsilon|$$

for $s \leq |x| \leq 2s$. Taking the supremum over the set $\{x : s \leq |x| \leq 2s\}$ and multiplying $s^{2-\nu}$, we get

$$|N(0)|_{0, \alpha, [s, 2s]} \lesssim s^{2-\nu} C \epsilon (1 + |\lambda|)$$

since $1 < \nu < 2$ and $0 < s \leq 1/2$. Hölder seminorm estimate for $N(0)$ has the same form, then by taking the supremum over $s \leq 1/2$, we have the first estimate of the lemma.

Similarly by Lemma 1,

$$Ru = (\Delta - \Delta_{B}) u = e_1 \nabla^2 u + e_2 \nabla u,$$

so we have

$$|Ru|_{0, \alpha, [s, 2s]} \lesssim \frac{C \epsilon (1 + |\lambda|)}{s^{2-\nu}}.$$

for $0 < s \leq 1/2$. Hölder seminorm estimate is also similar, then taking the supremum over $s \leq 1/2$ yields the estimate for $Ru$.

As for the estimates for $\xi u$ and $Q(u)$, by using the basic properties of the Hölder seminorm

$$|\mu + \eta|_{(\alpha)} \leq |\mu|_{(\alpha)} + |\eta|_{(\alpha)},$$

and

$$|\mu \eta|_{(\alpha)} \leq |\mu|_{(0)} |\eta|_{(\alpha)} + |\mu|_{(\alpha)} |\eta|_{(0)},$$

as in the above computation, we can derive the following bounds:

$$|\xi u(x)| \leq C |\lambda| |\xi|^2 |u(x)|, \quad x \in B_1^\alpha(0) \quad (6)$$

$$|\xi u|_{(\alpha), [s, 2s]} \leq C |\lambda| |\xi|^2 (|u|_{0, [s, 2s]} + |u|_{(\alpha), [s, 2s]}), \quad (7)$$

$$|Q(u)(x)| \leq C |\lambda| |\xi|^2 (|u(x)| + |u(x)|^3), \quad x \in B_1^\alpha(0) \quad (8)$$

$$|Q(u)|_{(\alpha), [s, 2s]} \leq C |\lambda| |\xi|^2 \left( |u|_{0, [s, 2s]} |u|_{(\alpha), [s, 2s]} + |u|_{0, [s, 2s]} + |u|_{[s, 2s]} |u|_{(\alpha), [s, 2s]} \right). \quad (9)$$

If we multiply both sides of (6) and (8) by $s^{2-\nu}$, or of (7) and (9) by $s^{2-\nu + \alpha}$ and take the supremum over $s \leq 1/2$, we immediately have

$$\sup_{0 < s \leq 1/2} \left( (\xi u|_{0, [s, 2s]} s^{2-\nu} + |\xi u|_{(\alpha), [s, 2s]} s^{2-\nu + \alpha}) \right) \leq C |\lambda| |\xi|^2 |u|_{0, \alpha, \nu}.$$
\[ \sup_{0<s\leq 1/2} (|Q(u)|_{0,[s,2s]}s^{2-\nu} + |Q(u)|_{[s,2s]}^{2-\nu+\alpha}) \leq C|\lambda|\epsilon^{2}(|u|_{0}^{2} + |\alpha|_{y+|u|_{0}}^{3})\nu \]

which complete the proof of the lemma.

Now, to find solutions of (4), we first recall the unique solvability result for the linear problem \( \Delta_{B^{n}}u = f \) on \( B_{1}^{n}(0) \), for \( f \in C_{G}^{0,\alpha,\nu-2}(\Omega_{\epsilon};\mathbb{R}^{n}) \) with some appropriate boundary conditions.

Let us take the sequence of eigenvalues of \( \Delta_{S^{n-1}} \) acting on \( C_{G}^\infty(S^{n-1};\mathbb{R}^{n}) \), \( \mu_{j}, 0 = \mu_{1} \leq \mu_{2} \leq \cdots \) (counting multiplicity), \( \mu_{j} \to \infty \), and corresponding sequence of \( L^{2} \)-normalized eigenmaps \( \phi_{j} \in C^{\infty}(S^{n-1};\mathbb{R}^{n}) \) such that \( \Delta_{S^{n-1}}\phi_{j} + \mu_{j}\phi_{j} = 0 \), \( j = 1,2,\cdots \). Let \( \lambda_{j} \) and \( \lambda_{j}(-) \) be two real solutions of the equation \( \lambda^{2} + (n-2)\lambda - \mu_{j} = 0 \), that is

\[ \lambda_{j} = \frac{2-n}{2} + \sqrt{\frac{(n-2)^{2}}{4} + \mu_{j}} \quad \text{and} \quad \lambda_{j}(-) = \frac{2-n}{2} - \sqrt{\frac{(n-2)^{2}}{4} + \mu_{j}}. \]

We now fix \( \nu \) so that \( 1 < \nu < 2 \) and choose an positive integer \( J \) such that \( \lambda_{J} < \nu < \lambda_{J+1} \).

For this \( J \), we define

\[ \Pi_{J} : L^{2}(S^{n-1};\mathbb{R}^{n}) \to \{\phi_{1}, \phi_{2}, \cdots, \phi_{J}\}^{\perp} \]

be the orthogonal projection.

Then we have:

**Lemma 3** If \( f \in C_{G}^{0,\alpha,\nu-2}(\Omega_{\epsilon};\mathbb{R}^{n}) \) and \( \psi \in C_{G}^{2,\alpha,\nu}(\partial\Omega_{\epsilon};\mathbb{R}) \) with \( 0 < \alpha < 1 \), then there exists a unique \( u \in C_{G}^{2,\alpha,\nu}(\Omega_{\epsilon};\mathbb{R}^{n}) \) such that

\[ \begin{cases} 
\Delta_{B^{n}}u = f & \text{on } \Omega_{\epsilon} \setminus \Gamma_{\epsilon}, \\
\Pi_{J}(u|_{\partial\Omega_{\epsilon}}) = \Pi_{J}(\psi). 
\end{cases} \tag{10} \]

Furthermore, we have the estimate

\[ |u|_{2,\alpha,\nu} \leq C_{2}(|f|_{0,\alpha,\nu-2} + |\psi|_{2,\alpha}) \]

for some constant \( C_{2} \) depending only on \( \alpha \).

**Proof** The proof of this is done by separation of variables and now quite standard (see [3], [9], [4], [6]), so we make only few comments.

If we write

\[ u(r, \theta) = \sum_{j=1}^{\infty} u_{j}(r)\phi_{j}(\theta), \quad u_{j}(r) = \langle u(r, \cdot), \phi_{j}(\cdot) \rangle_{L^{2}(S^{n-1};\mathbb{R}^{n})}, \]

\[ f(r, \theta) = \sum_{j=1}^{\infty} f_{j}(r)\phi_{j}(\theta), \quad f_{j}(r) = \langle f(r, \cdot), \phi_{j}(\cdot) \rangle_{L^{2}(S^{n-1};\mathbb{R}^{n})}, \]

\[ \psi(\theta) = \sum_{j=1}^{\infty} \psi_{j}\phi_{j}(\theta), \quad \psi_{j} = \langle \psi, \phi_{j} \rangle_{L^{2}(S^{n-1};\mathbb{R}^{n})}, \]

then each \( u_{j} \) must be the solution of the following ODE with boundary conditions:

\[ \begin{cases} 
a''(r) + \frac{n-1}{r}a'(r) - \frac{\mu_{j}}{r^{2}} = f_{j}(r), \\
a(1) = \psi_{j} \quad \text{for } j > J, \\
|a(r)| \leq Cr^{\nu}. \end{cases} \]
By elementary ODE argument, Caffarelli, Hardt and Simon [3] showed that
\[ u_j(r) = r^{\lambda_j} \int_0^r s^{1-n-2\lambda_j} \int_0^s \tau^{n-1+\lambda_j} f_j(\tau) d\tau ds, \quad (j = 1, 2, \ldots, J) \]
\[ u_j(r) = \psi_j r^{\lambda_j} - r^{\lambda_j} \int_r^1 s^{1-n-2\lambda_j} \int_0^s \tau^{n-1+\lambda_j} f_j(\tau) d\tau ds, \quad (j \geq J + 1) \]
are the unique solutions. Thus the map \( \sum_{j=1}^\infty u_j \phi_j \) formally solves the equation \( \Delta_{B^n} u = f \) on \( B_1^n(0) \) with \( \Pi_I(u|_{\partial \Omega}) = \Pi_I(\psi) \), and in fact \( C^2 \) classical sense on \( B_1^n(0) \setminus \{0\} \).

We now apply Lemma 2 and Lemma 3 to find fixed points of (4). Fix \( \alpha \in (0, 1) \) and \( \nu \in (1, 2) \) as before. For \( K > 0 \) and \( \varepsilon > 0 \), let us define \( B_{K\varepsilon, \alpha, \nu} = \{ u \in C^2_G(\Omega_\varepsilon; \mathbb{R}^n) : |u|_{2, \alpha, \nu} \leq K\varepsilon \} \).

Then we prove

**Lemma 4** For any \( \lambda \in \mathbb{R} \), there exists \( K > 0 \) and \( 0 < \varepsilon < 1 \) such that if \( \varepsilon < \varepsilon \), \( \nu \in B_{K\varepsilon, \alpha, \nu} \) and \( \psi \in C^2_G(\partial \Omega_\varepsilon; \mathbb{R}^n) \) satisfying \( |\psi|_{2, \alpha} \leq \varepsilon \), then the problem: to find \( u \in B_{K\varepsilon, \alpha, \nu} \) such that

\[
\begin{align*}
\Delta_{B^n} u &= -N(0) - R\nu - \xi v - Q(v) \\
\Pi_I(u|_{\partial \Omega}) &= \Pi_I(\psi)
\end{align*}
\]

has a unique solution.

**Proof** The problem above has a unique solution \( u \in C^2_G(\Omega_\varepsilon; \mathbb{R}^n) \) by Lemma 2 and Lemma 3. Furthermore according to Lemma 2, Lemma 3 and \( |v|_{2, \alpha, \nu} \leq K\varepsilon \), we have

\[
|u|_{2, \alpha, \nu} \leq C_2 (|\psi|_{2, \alpha} + |N(0)|_{0, \alpha, \nu-2} + |R\nu|_{0, \alpha, \nu-2} + |\xi v|_{0, \alpha, \nu-2} + |Q(v)|_{0, \alpha, \nu-2})
\leq C_2 (\varepsilon \cdot C_1 \varepsilon + |\lambda| \varepsilon + K\varepsilon + |\lambda| \varepsilon^2 + K\varepsilon + K^2 \varepsilon^2)
\leq C_3 (\varepsilon + |\lambda| \varepsilon^2 + K^2 \varepsilon^2 + |\lambda| \varepsilon^2 (K\varepsilon + K^2 \varepsilon^2 + K^3 \varepsilon^3))
\]

for some constant \( C_3 > 0 \).

So, if we can take \( K \) and \( \varepsilon \) such that

\[
C_3 \left( \frac{1 + |\lambda| \varepsilon}{K} + \varepsilon + |\lambda| \varepsilon^2 (1 + K\varepsilon + K^2 \varepsilon^2) \right) \leq 1,
\]

then the proof will be completed. This can be done as follows: First, fix \( K > 0 \) sufficiently large so that

\[
\frac{1 + |\lambda|}{K} < \frac{1}{2C_3},
\]

and then, fix \( \varepsilon \in (0, 1) \) sufficiently small so that

\[
\varepsilon + |\lambda| \varepsilon^2 (1 + K\varepsilon + K^2 \varepsilon^2) < \frac{1}{2C_3}.
\]
Now, fix $\psi \in C^2_G(\partial \Omega_\epsilon; \mathbb{R}^n)$ so that $|\psi|_{2,\alpha} \leq \epsilon < \epsilon$. Let us denote $T(v)$ the unique solution of (11) for $v \in B_{K\epsilon, \alpha, \nu}$. Then, by Lemma 4, $T$ defines a self-map of $B_{K\epsilon, \alpha, \nu}$. To show that $T$ is indeed a contraction, we need

**Lemma 5** There is a constant $C_4 > 0$ independent of $u, v \in C^2_G(\Omega_\epsilon; \mathbb{R}^n)$, $\epsilon$ and $\lambda$ such that

$$|Q(u) - Q(v)|_{0, \alpha, \nu - 2} \leq C_4|\epsilon^2|u - v|_{2, \alpha, \nu} + |v|_{2, \alpha, \nu} + (|u|_{2, \alpha, \nu} + |v|_{2, \alpha, \nu})^2$$

holds.

**Proof** This is obtained quite easily by elementary computation and basic property of Hölder seminorms, if we write

$$Q(u) - Q(v) = (-\lambda \epsilon^2) (I_1 + 2I_2 + I_3),$$

where

$$I_1 = [(u - v) \cdot (u + v)] u_\epsilon,$$

$$I_2 = (u_\epsilon \cdot u)(u - v) + [u_\epsilon \cdot (u - v)] v_\epsilon,$$

$$I_3 = |u|^2 (u - v) + [(u - v) \cdot (u + v)] v.$$

Let $u_1 = T(v_1)$ and $u_2 = T(v_2)$ are the unique solution of (11) for fixed $\psi$, given by Lemma 4. Then by Lemma 3, Lemma 2 and Lemma 5, we have

$$|T(v_1) - T(v_2)|_{2, \alpha, \nu} \leq C_2|R(v_1 - v_2)|_{0, \alpha, \nu - 2} + C_2|\epsilon(v_1 - v_2)|_{0, \alpha, \nu - 2} + C_2|Q(v_1) - Q(v_2)|_{0, \alpha, \nu - 2}$$

$$\leq C_2^2\epsilon|v_1 - v_2|_{2, \alpha, \nu} + C_2C_1|\epsilon^2|v_1 - v_2|_{2, \alpha, \nu}$$

$$\leq C_5 [\epsilon + |\lambda| \epsilon^2 + |\lambda| \epsilon^2 (K \epsilon + K^2 \epsilon^2)] |v_1 - v_2|_{2, \alpha, \nu}.$$

So if we retake $\epsilon$ small enough so that $C_5 [\epsilon + |\lambda| \epsilon^2 + |\lambda| \epsilon^2 (K \epsilon + K^2 \epsilon^2)] < 1$, the map $T$ defines a contraction on the closed subset of a complete metric space, then it has a fixed point $u$. Thus we have found a map $U = u + u_\epsilon$ satisfying (2), at least in $\Omega_\epsilon \setminus \Gamma_\epsilon$.

Note that when $1 < \nu < 2$, we can extend $u \in C^2_G(\Omega_\epsilon; \mathbb{R}^n)$ (as thought of a map defined on $B_1^0(0) \setminus \{0\}$ ) to $0 \in B_1^0(0)$ so that $u(0) = 0$ and $|\nabla u(0)| = 0$, then the map $U$ is indeed a smooth solution of (2) on each fibers of $\Omega_\epsilon$. Moreover if we require $\epsilon$ small enough such that $K \epsilon \leq 1/2$, then $|U(x)| \geq |u_\epsilon(x)| - |u(x)| \geq (1/2)|x|$ for any $x \in B_1^0(0)$, so $\Gamma(U) = \Gamma(u_\epsilon) = \Gamma_\epsilon$. As noted earlier, simple rescaling by a factor of $\epsilon$ completes the proof of Theorem.

**References**


