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Kyoto University
Solutions of Ginzburg-Landau type systems with Higher-dimensional Zero Sets

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1 Introduction

In this paper, we consider the following elliptic system of diagonal type:

$$\Delta V + \lambda (1 - |V|^2) V = 0$$  \hspace{1cm} (1)

where $V = (V^1, \ldots, V^n)$ is defined on some domain in $\mathbb{R}^{n+k}$, $n \geq 2$, $k \geq 1$, and $\lambda \in \mathbb{R}$ is a parameter.

Here, we construct some solutions of (1) on certain domains in $\mathbb{R}^{n+k}$, with boundary values, invariant under the action of a $k$-parameter group of isometries of $\mathbb{R}^{n+k}$, and having nontrivial $k$-dimensional zero sets.

When $n = 2$, the equation (1) is the Ginzburg-Landau system (GLS), which is used as a mathematical model for many physical phenomena, such as super-conductivity and super-fluidity. In the theory of super-conductivity, the unknown $V$ represents an order parameter which has two degrees of freedom, and its zero set, called vortices, corresponds to the region of the normal state in super-conductors. So, especially our result produces an example of solutions of the GLS in $\mathbb{R}^3$ with curved vortex lines.

Some results concerning the isolated zeros of solutions of the GLS in $\mathbb{R}^2$ are known([1], [2]), however there seems to be no explicit example of solutions with higher-dimensional nontrivial zero sets.

Our proof is based on the "equivariant construction" method due to N.Smale [9], in which the examples of minimal hypersurfaces in Euclidean spaces with higher-dimensional singularities are shown. Later, the same method was used to construct examples with higher-dimensional singularities, of harmonic maps [4], and of solutions of a certain non-linear elliptic equation [6].

Main result of this paper can be extended to equations with other type of nonlinearities, but we do not pursue here for simplicity of description.

2 Notations and statement of the main result

We follow the setting of "equivariant construction" method described in the papers [9],[4] and [6]: Let $n \geq 2$, $k \geq 1$ be two integers. Let $\mathcal{U} \subset \mathbb{R}^k$ be an open set containing $\{0\} \in \mathbb{R}^k$ and assume that there is a $C^\infty$ group action

$$\Phi : t \in \mathcal{U} \longrightarrow \Phi(t) \in \text{Isom}(\mathbb{R}^{n+k}),$$
here $\text{Isom}(\mathbb{R}^{n+k})$ means the group of isometries of $\mathbb{R}^{n+k}$. We will denote $\Phi(t)$ by $G_t$.

We define
\[
\begin{align*}
\Gamma &= \{ G_t(0) : t \in \mathcal{U} \}, \\
\tilde{B}^n &= B^n_1(0) \times \{ 0 \} \times = \{ (x, 0) \in \mathbb{R}^n \times \mathbb{R}^k, |x| < 1 \}, \\
\Omega &= \{ G_t(\mathbb{B}^n) : t \in \mathcal{U} \}.
\end{align*}
\]

So, $\Gamma$ is the orbit of $\{ 0 \} \in \mathbb{R}^{n+k}$ of the group action $\Phi$, and $\Omega$ is the unit $n$-disc bundle over $\Gamma$ obtained by moving $\tilde{B}^n$ along $\Gamma$ by $G_t$, $t \in \mathcal{U}$. On the group action $\Phi$, we make the following assumptions: $\Gamma$ is a properly embedded $k$-dimensional submanifold in $\mathbb{R}^{n+k}$ and whenever $G_t(0) = 0$, we must have $G_t(\tilde{B}^n) = \mathbb{B}^n$ for any $t \in \mathcal{U}$, that is, the isotropy group of 0 is the same as the one of $\mathbb{B}^n$. Furthermore, when $G_t = O(t) + v_t$ is the decomposition of the element of $\text{Isom}(\mathbb{R}^{n+k})$, where $O(t) \in O(n + k)$, the orthogonal group of $\mathbb{R}^{n+k}$, and $v_t \in \mathbb{R}^{n+k}$, we define the group action
\[
\Phi_\epsilon : t \in \mathcal{U} \longrightarrow G_t^\epsilon \in \text{Isom}(\mathbb{R}^{n+k}),
\]
and
\[
\begin{align*}
\Gamma_\epsilon &= \{ G_t^\epsilon(0) : t \in \mathcal{U} \} = (\frac{1}{\epsilon}) \Gamma, \\
\Omega_\epsilon &= \{ G_t^\epsilon(\mathbb{B}^n) : t \in \mathcal{U} \},
\end{align*}
\]
where $G_t^\epsilon = O(t) + \frac{1}{\epsilon} v_t$. Note that under the assumption of the group action $\Phi$, $\Omega_\epsilon$ is well-defined and then $\Omega_\epsilon$ is the unit $n$-disc bundle over $\Gamma_\epsilon$ obtained by moving $\tilde{B}^n$ along $\Gamma_\epsilon$ by $G_t^\epsilon$, $t \in \mathcal{U}$. Note also that when $\epsilon > 0$ is sufficiently small, $\Omega_\epsilon$ is close locally the trivial product bundle $B^n_1(0) \times \mathbb{R}^k$ over $\{ 0 \} \times \mathbb{R}^k$. Finally, for a map $U : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$, we denote by $\Gamma(U)$ the set of zeros of $U$, namely, $\Gamma(U) = \{ x : U(x) = 0 \in \mathbb{R}^n \}$.

Now we state the main result of this paper.

**Theorem.** For any $\lambda \in \mathbb{R}$, there exists an open domain $\tilde{\Omega} \subset \mathbb{R}^{n+k}$ containing $\Gamma$, on which there are infinitely many solutions of (1) with boundary values, whose zero set is $\Gamma$.

In the proof of the theorem, we will show that there exists $\epsilon > 0$ sufficiently small, such that for any $0 < \epsilon < \tilde{\epsilon}$, there is a solution $U$ of
\[
\Delta U + \lambda \epsilon^2 (1 - |U|^2) U = 0 \quad \text{in} \quad \Omega_\epsilon,
\]
\[
\Gamma(U) = \Gamma_\epsilon
\]
with a boundary data fixed up to a finite dimensional space, and $U$ is invariant under the action $\Phi_\epsilon$, i.e., $U(G_t^\epsilon(\tilde{x})) = U(\tilde{x})$ for all $\tilde{x} \in \tilde{B}^n$ and $t \in \mathcal{U}$.

We will find a solution $U$ of (2) by solving the appropriate fixed point problem. We make essential use of the invariant condition of $U$, thanks to which, we can think of (2) as a PDE on each fibers of the disc bundle $\Omega_\epsilon$, especially on $\tilde{B}^n$ for $t = 0$. Note that the nonlinear term of (2) is well controlled when $\epsilon$ is small enough, so we can get a solution as a perturbation of the $\mathbb{R}^n$-valued harmonic function $v_0 : B^n_1(0) \rightarrow \mathbb{R}^n, v_0(x) = x$. Taking $\tilde{\Omega} = \epsilon \cdot \Omega_\epsilon$, and $V(y) = U(\frac{y}{\epsilon})$ for $y \in \tilde{\Omega}$ will give the desired result. The domain $\tilde{\Omega}$ so obtained, is the bundle over $\Gamma$ of the $n$-dimensional discs of radius $\epsilon$, so looks like locally a thin perturbed tube of radius $\epsilon$ with center axis $\Gamma$.

Now we describe our coordinate of $\Omega_\epsilon$: For $y \in \Omega_\epsilon$, there exists $x \in B^n_1(0)$ and $t \in \mathcal{U}$ such that $y = G_t^\epsilon(\tilde{x})$, then let us denote $F : B^n_1(0) \times \mathcal{U} \rightarrow \Omega_\epsilon$, $F(x, t) = G_t^\epsilon(\tilde{x})$. we will introduce the local coordinate system by this map, and identify $y$ with $(r, \theta, t)$ where $(r, \theta)$ are polar coordinates for $x \in B^n_1(0)$. So, functions defined on $\Omega_\epsilon$ can naturally be
considered as functions on $B^n_1(0) \times \mathcal{U}$ by $F$. Note for $\varepsilon > 0$ sufficiently small, $r$ also is the distance to $\Gamma_\varepsilon$.

In the sequel we use the following function spaces: For $\nu \in \mathbb{R}, \alpha \in (0,1), m = 0, 1, 2$, define
\[
C^{m,\alpha,\nu}(\Omega_\varepsilon; \mathbb{R}^n) = \{ u \in C^{m,\alpha}_{loc}(\Omega_\varepsilon \setminus \Gamma_\varepsilon; \mathbb{R}^n) : |u|_{m,\alpha,\nu} < +\infty \},
\]
where $| \cdot |_{m,\alpha,\nu}$ is the norm
\[
|u|_{m,\alpha,\nu} = \sup_{0 < r \leq 1/2} \left( \sum_{j=0}^{m} |\nabla^j u|_{0,s} s^{-\nu} + \sum_{j=0}^{m} |\nabla^j u|_{(\alpha),s} s^{j+\alpha-\nu} \right).
\]
Here, $\nabla$ and $\nabla^2$ denote the gradient and Hessian respectively on $\Omega_\varepsilon$, and $|\eta|_{0,s,2s}$ and $|\eta|_{(\alpha),s,2s}$ are the sup norm and the $\alpha$-th Hölder seminorm of a function (or a section) $\eta$ on $\Omega_\varepsilon$ over the set $\{ y = y(r, \theta, t) \in \Omega_\varepsilon : s \leq r \leq 2s \}$. These are Banach spaces under the norm $| \cdot |_{m,\alpha,\nu}$, and if $u \in C^{m,\alpha,\nu}(\Omega_\varepsilon; \mathbb{R}^n)$, then $|u|$ decays like $r^\nu$ near $\Gamma_\varepsilon$.

Furthermore, let us define the closed subspace of $C^{m,\alpha,\nu}(\Omega_\varepsilon; \mathbb{R}^n)$ as
\[
C^{m,\alpha,\nu}_G(\Omega_\varepsilon; \mathbb{R}^n) = \{ u \in C^{m,\alpha,\nu}(\Omega_\varepsilon; \mathbb{R}^n) : u(G_\varepsilon^t(\tilde{x})) = u(\tilde{x}) \quad \text{for all } \tilde{x} \in B^n_1(0), \ t \in \mathcal{U} \},
\]
that is, maps in $C^{m,\alpha,\nu}$ which are $\Phi_\varepsilon$-invariant. We also denote $C^{m,\alpha}(\partial \Omega_\varepsilon; \mathbb{R}^n)$ for the space of $\Phi_\varepsilon$-invariant boundary data in $C^{m,\alpha}(\partial \Omega_\varepsilon; \mathbb{R}^n)$.

Weighted Hölder spaces like above are now widely used for other nonlinear problems, see [9], [10], [4], [6], [8], [5], [3].

3 Proof of the Theorem

In this section, we seek for a solution of (2) satisfying (3) by the same technique as in [9], [4], [6]: linearization and solving the appropriate fixed point problem. First, we construct the approximate solution. We fix $\varepsilon > 0$. Let $v_0 : B^n_1(0) \to \mathbb{R}^n$ be the identity map $v_0(x) = x$; so evidently $\Gamma(v_0) = \{ 0 \} \subset \mathbb{R}^n$ and $\Delta_{B^n} v_0 = 0$, where $\Delta_{B^n}$ means the Laplace operator on $B^n_1(0)$. Now we define the approximate solution $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}^n$ by
\[
u(\nu^t(\tilde{x})) = v_0(\tilde{x}) \quad \text{for } \tilde{x} \in B^n_1(0), \ t \in \mathcal{U}
\]
where $\tilde{x} = (x, 0) \in \tilde{B}^n$. By definition of $\Omega_\varepsilon$ and by our assumption on the group action $\Phi$, $u_\varepsilon$ is well-defined and invariant under the action $\Phi_\varepsilon$. The zero set of $u_\varepsilon$ satisfies $\Gamma(u_\varepsilon) = \Gamma_\varepsilon$.

We wish to find a solution of (2) of the form
\[
U(u) = u_\varepsilon + u
\]
where the perturbation $u$ is assumed to be invariant under the action $\Phi_\varepsilon$ and to decay rapidly near $\Gamma_\varepsilon$, so as to ensure that $\Gamma(U(u)) = \Gamma_\varepsilon$.

Let $N(u)$ be the left hand side of (2) for $U(u)$, that is,
\[
N(u) = \Delta U(u) + \lambda \varepsilon^2 (1 - |U(u)|^2) U(u).
\]
We make a Taylor expansion of $N(u)$ about $u = 0$ to get
\[
N(u) = N(0) + Lu + Q(u),
\]
where $L$ is a linear operator determined by $\Delta U(u)$ and $Q(u)$ is a quadratic form determined by $\lambda \varepsilon^2 (1 - |u|^2)$. The linear operator $L$ is also strongly $\varepsilon$-invariant, which ensures $\Gamma(Lu) = \Gamma(Lu)$, as well as the quadratic form $Q(u)$ being strongly $\varepsilon$-invariant.
where
\[
N(0) = \Delta u_{\epsilon} + \lambda \epsilon^2 (1 - |u_{\epsilon}|^2) u_{\epsilon},
\]
\[
Lu = \frac{d}{dt} N(tu)|_{t=0} = \Delta u + \lambda \epsilon^2 \{ (1 - |u_{\epsilon}|^2) u - 2(u_{\epsilon} \cdot u) u_{\mathcal{E}} \},
\]
\[
Q(u) = \int_0^1 (1-t) \frac{d^2}{dt^2} N(tu) dt = (-2 \lambda \epsilon^2) \int_0^1 (1-t) \{|u|^2 u_{\zeta} + 2(u_{\epsilon} \cdot u) u + 3t|u|^2 u\} dt = (-\lambda \epsilon^2) \{|u|^2 u_{\epsilon} + 2(u_{\epsilon} \cdot u) u + |u|^2 u\},
\]
here \(\Delta\) means the Laplace operator on \(\Omega_{\epsilon}\). Now, if we define the linear operators
\[
R = \Delta - \Delta_{B^n}
\]
and
\[
\xi u = \lambda \epsilon^2 \{ (1 - |u_{\epsilon}|^2) u - 2(u_{\epsilon} \cdot u) u_{\mathcal{E}} \},
\]
then the equation \(N(u) = 0\) can be rewritten as
\[
\Delta_{B^n} u = -N(0) - Ru - \xi u - Q(u) \tag{4}
\]
which we solve by contraction mapping argument on some weighted Hölder space. Note that if \(u\) is invariant under the action \(\Phi_{\epsilon}\), all of the terms in (4) are also \(\Phi_{\epsilon}\)-invariant, so we can consider (4) as a PDE on the slice \(\tilde{B}^n\). This is crucial for our subsequent arguments.

To estimate the terms in the right hand side of (4), we need the following lemma due to R.Mazzeo and N.Smale [5].

**Lemma 1** Under the local coordination by \(F\), we have
\[
\Delta = \Delta_{B^n} + \Delta_{R^n} + e_1 \nabla^2 + e_2 \nabla, \tag{5}
\]
where \(\Delta\) and \(\nabla\) are the Laplace operator and gradient on \(\Omega_{\epsilon}\), \(e_1 \in C^\infty((\text{Sym}^2 \Omega_{\epsilon})^*)\), \(e_2 \in C^\infty(T^* \Omega_{\epsilon})\) are smooth sections and satisfy
\[
|e_1(x,t)| \leq C_0 \epsilon, \quad |e_2(x,t)| \leq C_0 \epsilon, \quad |e_1|_{(\alpha),[s,2s]} s^\alpha \leq C_0 s \epsilon, \quad |e_2|_{(\alpha),[s,2s]} s^\alpha \leq C_0 \epsilon
\]
for some constant \(C_0\) independent of \(\epsilon > 0\) and \(\alpha \in (0,1)\).

For functions \(u\) invariant under \(\Phi_{\epsilon}\), the factor \(\Delta_{R^n}\) in (5) drops out.

Using this lemma, we have

**Lemma 2** If \(\epsilon > 0, 1 < \nu < 2\), and \(u \in C^2_{G}(\Omega_{\epsilon}; \mathbb{R}^n)\), then \(N(0), Ru, \xi u, Q(u)\) are all in \(C^0_{G}(\Omega_{\epsilon}; \mathbb{R}^n)\) and the following estimates hold:
\[
|N(0)|_{0,\alpha,\nu-2} \leq C_1 \epsilon (1 + |\lambda| \epsilon), \quad |Ru|_{0,\alpha,\nu-2} \leq C_1 \epsilon |u|_{2,\alpha,\nu},
\]
\[
|\xi u|_{0,\alpha,\nu-2} \leq C_1 |\lambda| \epsilon^2 |u|_{2,\alpha,\nu}, \quad |Q(u)|_{0,\alpha,\nu-2} \leq C_1 |\lambda| \epsilon^2 (|u|_{2,\alpha,\nu}^2 + |u|_{2,\alpha,\nu}^3)
\]
for some constant \(C_1 > 0\) independent of \(\epsilon\) and \(\lambda\).
Proof Since $u_\epsilon$ and $u$ are $\Phi_\epsilon$-invariant, so are also all terms appeared in the right hand side of (4), and can be considered as functions of $B_r^\alpha(0)$. By definition, the map $u_\epsilon$ satisfies $\Delta_{B^\alpha} u_\epsilon = 0$, so we have

$$N(0) = \Delta u_\epsilon + \lambda \epsilon^2 (1 - |u_\epsilon|^2) u_\epsilon = (\Delta - \Delta_{B^\alpha}) u_\epsilon + \lambda \epsilon^2 (1 - |u_\epsilon|^2) u_\epsilon.$$ 

Then using Lemma1 and the fact that $|\nabla u_\epsilon(x)| + |\nabla^2 u_\epsilon(x)| \leq C$ and $|u_\epsilon(x)| \leq 1$ for some constant $C$ independent of $\epsilon$ and $x \in B_r^\alpha(0)$, we have

$$|N(0)(x)| \leq |e_1 \nabla^2 u_\epsilon(x)| + |e_2 \nabla u_\epsilon(x)| + |\lambda| \epsilon^2 (1 - |u_\epsilon|^2)|u_\epsilon|$$

for $s \leq |x| \leq 2s$. Taking the supremum over the set $\{x : s \leq |x| \leq 2s\}$ and multiplying $s^{2-\nu}$, we get

$$|N(0)|_{0,[s,2s]}^{s^{2-\nu}} \leq s^{2-\nu} \cdot C \epsilon(1 + |\lambda| \epsilon),$$

since $1 < \nu < 2$ and $0 < s \leq 1/2$. H"older seminorm estimate for $N(0)$ has the same form, then by taking the supremum over $s \leq 1/2$, we have the first estimate of the lemma.

Similarly by Lemma1,

$$Ru = (\Delta - \Delta_{B^\alpha}) u = e_1 \nabla^2 u + e_2 \nabla u,$$

so we have

$$|Ru|_{0,[s,2s]}^{s^{2-\nu}} \leq C s \epsilon |\nabla^2 u|_{[s,2s]} + C |\nabla u|_{[s,2s]} s^{1-\nu}$$

for $0 < s \leq 1/2$. H"older seminorm estimate is also similar, then taking the supremum over $s \leq 1/2$ yields the estimate for $Ru$.

As for the estimates for $\xi u$ and $Q(u)$, by using the basic properties of the H"older seminorm

$$|\mu + \eta|_{(\alpha)} \leq |\mu|_{(\alpha)} + |\eta|_{(\alpha)}$$

and

$$|\mu \eta|_{(\alpha)} \leq |\mu|_{(0)}|\eta|_{(\alpha)} + |\mu|_{(\alpha)}|\eta|_{(0)},$$

as in the above computation, we can derive the following bounds:

$$|\xi u|_{[s,2s]} \leq C |\lambda| \epsilon^2 |u(x)|, \quad x \in B_r^\alpha(0) \quad (6)$$

$$|\xi u|_{[s,2s]} \leq C |\lambda| \epsilon^2 \left( |u|_{[s,2s]} + |u|_{(\alpha),[s,2s]} \right), \quad (7)$$

$$|Q(u)|_{[s,2s]} \leq C |\lambda| \epsilon^2 \left( |u(x)|^2 + |u(x)|^3 \right), \quad x \in B_r^\alpha(0) \quad (8)$$

$$|Q(u)|_{[s,2s]} \leq C |\lambda| \epsilon^2 \left( |u|_{[s,2s]} + |u|_{[s,2s]} + |u|_{[s,2s]} \right), \quad (9)$$

If we multiply both sides of (6) and (8) by $s^{2-\nu}$, or of (7) and (9) by $s^{2-\nu+\alpha}$ and take the supremum over $s \leq 1/2$, we immediately have

$$\sup_{0 < s \leq 1/2} (|\xi u|_{[s,2s]} s^{2-\nu} + |Q(u)|_{[s,2s]} s^{2-\nu+\alpha}) \leq C |\lambda| \epsilon^2 |u|_{[0,\alpha,\nu]}.$$
\[
\sup_{0<s\leq 1/2} (|Q(u)|_{s}, [s, 2s] s^{2} - \nu + |Q(u)|_{[s, 2s]} s^{2} - \nu + \alpha) \leq C|\lambda| \epsilon^{2} (|u|_{0}^{2}, |\lambda|^{\alpha} |y + |u|_{0, \alpha}^{3}, |u|_{0, \alpha})^{\nu}
\]

which complete the proof of the lemma. \(\square\)

Now, to find solutions of (4), we first recall the unique solvability result for the linear problem \(\Delta_{B^n} u = f\) on \(B_1^n(0)\), for \(f \in C^{0, \alpha, \nu-2}_{G}(\Omega_{\epsilon}; \mathbb{R}^{n})\) with some appropriate boundary conditions.

Let us take the sequence of eigenvalues of \(\Delta_{S^{n-1}}\) acting on \(C^{\infty}(S^{n-1}; \mathbb{R}^{n})\), \(\mu_{j}, 0 = \mu_{1} \leq \mu_{2} \leq \cdots\) (counting multiplicity), \(\mu_{j} \rightarrow \infty\), and corresponding sequence of \(L^{2}\) normalized eigenmaps \(\phi_{j} \in C^{\infty}(S^{n-1}; \mathbb{R}^{n})\) such that \(\Delta_{S^{n-1}} \phi_{j} + \mu_{j} \phi_{j} = 0, j = 1, 2, \cdots\).

Let \(\lambda_{j}\) and \(\lambda_{j}(-)\) be two real solutions of the equation \(\lambda^{2} + (n-2)\lambda - \mu_{j} = 0\), that is \(\lambda_{j} = \frac{2-n}{2} + \sqrt{\frac{(n-2)^{2}}{4} + \mu_{j}}\) and \(\lambda_{j}(-) = \frac{2-n}{2} - \sqrt{\frac{(n-2)^{2}}{4} + \mu_{j}}\).

We now fix \(\nu\) so that \(1 < \nu < 2\) and choose an positive integer \(J\) such that \(\lambda_{J} < \nu < \lambda_{J+1}\).

For this \(J\), we define \(\Pi_{J} : L^{2}(S^{n-1}; \mathbb{R}^{n}) \rightarrow \{\phi_{1}, \phi_{2}, \cdots, \phi_{J}\}^{\perp}\) be the orthogonal projection.

Then we have:

**Lemma 3** If \(f \in C^{0, \alpha, \nu-2}_{G}(\Omega_{\epsilon}; \mathbb{R}^{n})\) and \(\psi \in C^{2}_{G}(\partial \Omega_{\epsilon}; \mathbb{R})\) with \(0 < \alpha < 1\), then there exists a unique \(u \in C^{2, \alpha, \nu}_{G}(\Omega_{\epsilon}; \mathbb{R}^{n})\) such that

\[
\begin{align*}
\Delta_{B^n} u &= f \quad \text{on } \Omega_{\epsilon} \setminus \Gamma_{\epsilon}, \\
\Pi_{J}(u|_{\partial \Omega_{\epsilon}}) &= \Pi_{J}(\psi).
\end{align*}
\]

Furthermore, we have the estimate

\[
|u|_{2, \alpha, \nu} \leq C_{2} (|f|_{0, \alpha, \nu-2} + |\psi|_{2, \alpha})
\]

for some constant \(C_{2}\) depending only on \(\alpha\).

**Proof** The proof of this is done by separation of variables and now quite standard (see [3], [9], [4], [6]), so we make only few comments.

If we write

\[
\begin{align*}
u(r, \theta) &= \sum_{j=1}^{\infty} u_{j}(r) \phi_{j}(\theta), \\
f(r, \theta) &= \sum_{j=1}^{\infty} f_{j}(r) \phi_{j}(\theta), \\
\psi(\theta) &= \sum_{j=1}^{\infty} \psi_{j} \phi_{j}(\theta),
\end{align*}
\]

then each \(u_{j}\) must be the solution of the following ODE with boundary conditions:

\[
\begin{cases}
\frac{d^{2}u_{j}(r)}{dr^{2}} + \frac{n-1}{r} \frac{du_{j}(r)}{dr} - \frac{\mu_{j}}{r^{2}} = f_{j}(r), \\
a^{\nu}(r) = \psi_{j} \quad \text{for} \quad j > J, \\
|a(r)| \leq C r^{\nu}.
\end{cases}
\]
By elementary ODE argument, Caffarelli, Hardt and Simon [3] showed that

\[
  u_{j}(r) = r^{\lambda_{j}} \int_{0}^{r} s^{1-n-2\lambda_{j}} \int_{0}^{s} \tau^{n-1+\lambda_{j}} f_{j}(\tau) d\tau ds, \quad (j = 1, 2, \ldots, J)
\]

\[
  u_{j}(r) = \psi_{j} r^{\lambda_{j}} - r^{\lambda_{j}} \int_{r}^{1} s^{1-n-2\lambda_{j}} \int_{0}^{s} \tau^{n-1+\lambda_{j}} f_{j}(\tau) d\tau ds,
\]

\[
  (j = 1, 2, \cdots, J)
\]

are the unique solutions. Thus the map \( \sum_{j=1}^{\infty} u_{j} \phi_{j} \) formally solves the equation \( \Delta_{B^{n}} u = f \) on \( B_{1}^{n}(0) \) with \( \Pi_{J}(u|_{\partial \Omega}) = \Pi_{J}(\psi) \), and in fact \( C^{2} \) classical sense on \( B_{1}^{n}(0) \setminus \{0\} \).

To prove the estimate, note that we are dealing with the system of PDE, but in the same situation this was done in [4] using the local supremum estimates of [8] and the standard Schauder estimates in [7].

We now apply Lemma 2 and Lemma 3 to find fixed points of (4). Fix \( \alpha \in (0, 1) \) and \( \nu \in (1, 2) \) as before. For \( K > 0 \) and \( \epsilon > 0 \), let us define

\[
  B_{K\epsilon, \alpha, \nu} = \{ u \in C^{2, \alpha}_{G}(\Omega_{\epsilon}; \mathbb{R}^{n}) : |u|_{2, \alpha, \nu} \leq K\epsilon \}.
\]

Then we prove

**Lemma 4** For any \( \lambda \in \mathbb{R} \), there exists \( K > 0 \) and \( 0 < \bar{\epsilon} < 1 \) such that if \( \epsilon < \bar{\epsilon} \), \( v \in B_{K\epsilon, \alpha, \nu} \) and \( \psi \in C^{2, \alpha}_{G}(\partial \Omega_{\epsilon}; \mathbb{R}^{n}) \) satisfying \( |\psi|_{2, \alpha} \leq \epsilon \), then the problem: to find \( u \in B_{K\epsilon, \alpha, \nu} \) such that

\[
  \begin{align*}
  \Delta_{B^{n}} u &= -N(0) - Rv - \xi v - Q(v) \\
  \Pi_{J}(u|_{\partial \Omega}) &= \Pi_{J}(\psi)
  \end{align*}
\]

has a unique solution.

**Proof** The problem above has a unique solution \( u \in C^{2, \alpha}_{G}(\Omega_{\epsilon}; \mathbb{R}^{n}) \) by Lemma 2 and Lemma 3. Furthermore according to Lemma 2, Lemma 3 and \( |v|_{2, \alpha, \nu} \leq K\epsilon \), we have

\[
  |u|_{2, \alpha, \nu} \leq C_{2} (|\psi|_{2, \alpha} + |N(0)|_{0, \alpha, \nu} - 2 + |Rv|_{0, \alpha, \nu} - 2 + |\xi v|_{0, \alpha, \nu} - 2 + |Q(v)|_{0, \alpha, \nu} - 2) \\
  \leq C_{2} (\epsilon + C_{1} \epsilon (1 + |\lambda|\epsilon) + C_{1}\epsilon \cdot K\epsilon + |\lambda|\epsilon^{2} \cdot K\epsilon + |\lambda|\epsilon^{2} (K^{2}\epsilon^{2} + K^{3}\epsilon^{3})) \\
  \leq C_{3} (\epsilon + |\lambda|\epsilon^{2} + K\epsilon^{2} + |\lambda|\epsilon^{2} (K\epsilon + K^{2}\epsilon^{2} + K^{3}\epsilon^{3}))
\]

for some constant \( C_{3} > 0 \).

So, if we can take \( K \) and \( \epsilon \) such that

\[
  C_{3} \left( \frac{1 + |\lambda|\epsilon}{K} + \epsilon + |\lambda|\epsilon^{2} (1 + K\epsilon + K^{2}\epsilon^{2}) \right) \leq 1,
\]

then the proof will be completed. This can be done as follows: First, fix \( K > 0 \) sufficiently large so that

\[
  \frac{(1 + |\lambda|)}{K} < \frac{1}{2C_{3}},
\]

and then, fix \( \bar{\epsilon} \in (0, 1) \) sufficiently small so that

\[
  \bar{\epsilon} + |\lambda|\epsilon^{2} (1 + K\bar{\epsilon} + K^{2}\epsilon^{2}) < \frac{1}{2C_{3}},
\]

\[\square\]
Now, fix \( \psi \in C^{2,\alpha}_{G}(\partial \Omega_{\varepsilon} ; \mathbb{R}^{n}) \) so that \( |\psi|_{2,\alpha} \leq \varepsilon(\varepsilon) \). Let us denote \( T(v) \) the unique solution of (11) for \( v \in B_{K\varepsilon,\alpha,\nu} \). Then, by Lemma 4, \( T \) defines a self-map of \( B_{K\varepsilon,\alpha,\nu} \). To show that \( T \) is indeed a contraction, we need

**Lemma 5** There is a constant \( C_{4} > 0 \) independent of \( u, v \in C^{2,\alpha,\nu}_{G}(\Omega_{\varepsilon} ; \mathbb{R}^{n}) \), \( \varepsilon \) and \( \lambda \) such that

\[
|Q(u) - Q(v)|_{0,\alpha,\nu-2} \leq C_{4}\varepsilon^{2}|u - v|_{2,\alpha,\nu} + |v|_{2,\alpha,\nu} + (|u|_{2,\alpha,\nu} + |v|_{2,\alpha,\nu})^{2}
\]

holds.

**Proof** This is obtained quite easily by elementary computation and basic property of Hölder seminorms, if we write

\[
Q(u) - Q(v) = (-\lambda \varepsilon^{2}) (I_{1} + 2I_{2} + I_{3}),
\]

where

\[
I_{1} = [(u - v) \cdot (u + v)] u_{\varepsilon},
I_{2} = (u_{\varepsilon} \cdot u)(u - v) + [u_{\varepsilon} \cdot (u - v)] v,
I_{3} = |u|^{2}(u - v) + [(u - v) \cdot (u + v)] v.
\]

Let \( u_{1} = T(v_{1}) \) and \( u_{2} = T(v_{2}) \) are the unique solution of (11) for fixed \( \psi \), given by Lemma 4. Then by Lemma 3, Lemma 2 and Lemma 5, we have

\[
|T(v_{1}) - T(v_{2})|_{2,\alpha,\nu} \leq C_{2}|R(v_{1} - v_{2})|_{0,\alpha,\nu-2} + C_{3}|\varepsilon(v_{1} - v_{2})|_{0,\alpha,\nu-2} + C_{2}|Q(v_{1}) - Q(v_{2})|_{0,\alpha,\nu-2}
\]

\[
\leq C_{2}C_{1}\varepsilon|v_{1} - v_{2}|_{2,\alpha,\nu} + C_{2}C_{1}|\varepsilon^{2}|v_{1} - v_{2}|_{2,\alpha,\nu}
\]

\[
+ C_{2}C_{4}|\varepsilon^{2}|v_{1} - v_{2}|_{2,\alpha,\nu} + |v_{2}|_{2,\alpha,\nu} + (|v_{1}|_{2,\alpha,\nu} + |v_{2}|_{2,\alpha,\nu})^{2}
\]

\[
\leq C_{5} \varepsilon + |\varepsilon|^{2} + |\lambda|^{2} (K\varepsilon + K^{2}\varepsilon^{2}) |v_{1} - v_{2}|_{2,\alpha,\nu}.
\]

So if we retake \( \varepsilon \) small enough so that \( C_{5} \varepsilon + |\varepsilon|^{2} + |\lambda|^{2} (K\varepsilon + K^{2}\varepsilon^{2}) < 1 \), the map \( T \) defines a contraction on the closed subset of a complete metric space, then it has a fixed point \( u \). Thus we have found a map \( U = u + u_{\varepsilon} \) satisfying (2), at least in \( \Omega_{\varepsilon} \cap \Gamma_{\varepsilon} \).

Note that when \( 1 < \nu < 2 \), we can extend \( u \in C^{2,\alpha,\nu}_{G}(\Omega_{\varepsilon} ; \mathbb{R}^{n}) \) (as thought of a map defined on \( B_{1}^{n}(0) \setminus \{0\} \) to 0 in \( B_{1}^{n}(0) \) so that \( u(0) = 0 \) and \( |\nabla u(0)| = 0 \), then the map \( U \) is indeed a smooth solution of (2) on each fibers of \( \Omega_{\varepsilon} \). Moreover if we require \( \varepsilon \) small enough such that \( K\varepsilon \leq 1/2 \), then \( |U(x)| \geq |u_{\varepsilon}(x)| - |u(x)| \geq (1/2)|x| \) for any \( x \in B_{1}^{n}(0) \), so \( \Gamma(U) = \Gamma(u_{\varepsilon}) = \Gamma_{\varepsilon} \). As noted earlier, simple rescaling by a factor of \( \varepsilon \) completes the proof of Theorem.

**References**


