数理解析研究所講究録
1025 巻 1998 年 35-43

Ginzburg-Landau Equation with a Variable Coefficient in a Disk

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1 Introduction

We are concerned with the Ginzburg-Landau equation with a variable coefficient in a disk of $\mathbb{R}^2$ subject to Neumann boundary condition:

$$\begin{cases} a(x)^{-1}\mathrm{div}(a(x)\nabla\Phi)+\lambda(1-|\Phi|^2)\Phi=0, & x\in D:=\{|x|<1\} \\ \dfrac{\partial\Phi}{\partial\nu}=0, & x\in\partial D, \end{cases} \tag{1.1}$$

where $a(x)$ is a positive smooth function, $\partial/\partial\nu$ denotes the outer normal derivative on the boundary $\partial D=\{|x|=1\}$ and $\Phi(x)$ is a complex valued function, say $\Phi(x)=u(x)+iv(x)$. We always identify $\Phi(x)$ with the two-component real vector function $(u(x),v(x))$. This equation is the Euler-Lagrange equation for the next energy functional:

$$E(\Phi):=\int_D\{|\nabla\Phi|^2+\frac{\lambda}{2}(1-|\Phi|^2)^2\}a(x)dx \tag{1.2}$$

For the physical meaning of this type of equation (with the variable coefficient) refer to the introduction of the paper [4].

We say that a solution of (1.1) is stable if it is a local minimizer of (1.2). On the other hand we may regard (1.1) as the stationary equation of the parabolic equation:

$$\begin{cases} \dfrac{\partial\Phi}{\partial t}=\Delta\Phi+\lambda(1-|\Phi|^2)\Phi, & (x,t)\in D\times(0,\infty) \\ \dfrac{\partial\Phi}{\partial\nu}=0, & (x,t)\in\partial D\times(0,\infty) \\ \Phi(x,0)=\Phi_0 \end{cases} \tag{1.3}$$

where Φ_0 is chosen in an appropriate function space, for instance, $C^0(\overline{\Omega};\mathbb{C})$. Then the solutions generate a smooth semiflow there. Thanks to the result in [17] the Lyapunov's stability for an equilibrium solution of (1.3) coincides with the above stability for the energy functional (1.2). Indeed the nonliner term of (1.2) is real analytic (for the detail, see [17]).

In this article we discuss the existence of a stable solution of (1.1) with a zero, which is called a "vortex"; we simply call such a solution a "vortex solution" from now. Before stating the result, we observe some features of the equation (1.1). As a specific aspect for the Neumann condition case, it is definite that global minimizers of (1.2) are realized by the constant solution with modulus one to (1.1). Moreover applying the result in [10], we see any nonconstant solution is unstable when $a(x)$ is a constant function. By the previous work [4], however, we can obtain a stable vortex solution provided that we choose λ and $a(x)$ appropriately. More precisely for sufficiently large but fixed λ there is a function $a(x)$ admitting a stable vortex solution. Then, corresponding to the size of λ, we have to make up $a(x)$ carefully so that the vortices can be trapped around prescribed points. Indeed the profile of $a(x)$ has a sharp layer around each vortex.

Here we assume that $a(x)$ is radially symmetric and monotone increasing, that is,

$$a = a(r), \quad r = |x| \quad \text{and} \quad a'(r) \geq 0 \quad (0 \leq r \leq 1) \quad \text{where} \quad ' = d/dr$$

Under this condition with sufficiently large λ there is a solution in the form $\Phi = f(r)e^{i\theta}$ (or $f(r)e^{-i\theta}$) satisfying $f(0) = 0$. Putting this into (1.1) yields

$$f'' + \frac{(ar)'}{ar}f' - \frac{1}{r^2}f + \lambda(1 - f^2)f = 0, \quad r \in (0,1), \qquad f(0) = 0, \quad f'(1) = 0 \tag{1.4}$$

It can be proved that a positive solution of (1.4) is uniquely determined and it satisfies $f' > 0 (0 < r < 1)$ (see Lemma 2.1). This solution is actually a vortex solution (with vortex $x = 0$).

Our main task here is to give a sufficient condition for $a(x)$ to allow that the vortex solutions are stable for large λ (Theorem 2.2). Moreover, as an application, we show that even though the total variation of $a(r)$ is arbitrarily small, the vortex solutions can be stabilized for large λ when the variation is localized near the vortex (see Corollary 2.3 and Remark 2.4). Note that the total variation of $a(x)$ in this case is just the difference $\max a(x) - \min a(x)$ because of the monotonicity of the function.

Compared with the result in [4], one sees that the strong ristriction of $a(x)$ for the previous work is certainly relaxed for this specific situation.

In the next section we propose a theorem, in which the sufficient condition of $a(r)$ is stated, and §3 shows a sketch for the proof of the theorem.

2 Main theorem

Let $a(r)$ be a C^3 function in $r \in [0, 1]$ satisfying

$$\begin{cases} a(r) > 0 \qquad a'(r) > 0 \quad (0 \leq r \leq 1), \qquad a(1) = 1, \\ a'(0) = a''(0) = 0, \qquad a'(1) = a''(1) = 0, \\ a'(r) \text{ has at most a finite number of zeros}, \\ a''(r) \geq 0 \quad \text{in a neighborhood of } r = 0. \end{cases} \tag{2.1}$$

Lemma 2.1 *Assume the condition* (2.1). *Then there is a* $\lambda_1 > 0$ *such that for each* $\lambda > \lambda_1$ *Equation* (1.4) *has a unique positive solution* $f = f_\lambda(r)$, *thus Equation* (1.1) *has a pair of solutions*

$$\Phi = f_\lambda(r)e^{i\theta}, \qquad f_\lambda(r)e^{-i\theta} \tag{2.2}$$

for $\lambda > \lambda_1$.

Proof. In the case $a(r) \equiv 1$, the unique existance of the positive solution is known (for instance see [3]). Let f_0 be such a unique positive solution for $a = 1$ and let $\bar{f} \equiv 1$. We can easily check that $\bar{f}$ and f_0 are an upper and lower solutions to (1.4) respectively. Hence it guarantees the existence of a positive solution to (1.4). The uniqueness follows from the same argument as in the proof of Lemma 3.1 in [10]. ∎

The next theorem is the main result of this article.

Theorem 2.2 *In addition to the condition* (2.1) *if*

$$\int_0^1 \frac{a'(r)}{r} dr > 1, \tag{2.3}$$

then there is a $\lambda_0 (> \lambda_1)$ *such that for* $\lambda > \lambda_0$ *the solutions* (2.2) *are stable.*

Corollary 2.3 *Under the condition* (2.1) *suppose that there is a* $\beta \in (0, 1]$ *such that*

$$\frac{a(\beta) - a(0)}{\beta} > 1. \tag{2.4}$$

Then the same assertion of Theorem 2.2 *is true.*

This corollary immediately follows from Theorem 2.2 and the fact

$$\int_0^1 \frac{a'(r)}{r} dr \geq \int_0^\beta \frac{a'(r)}{r} dr \geq \frac{1}{\beta} \int_0^\beta a'(r) dr.$$

Remark 2.4 The condition (2.4) implies that any smallness of the total variation of $a(r)$ doesn't matter with vortex solution to be stable for large λ if the mean value in $[0, \beta]$ is larger than one. The following $a(r)$ is a simple case to enjoy the conditions (2.1) and (2.4).

$$\begin{aligned} & a'(r) > 0, \quad r \in (0, \beta), \quad a(r) = 1 \;\; r \in [\beta, 1]. \\ & \text{and} \quad a''(r) \geq 0 \;\; \text{in a neighborhood of } r = 0 \qquad (2.5) \\ & (1 - a(0))/\beta > 1 \qquad (2.6) \end{aligned}$$

3 Sketch for the proof

3.1 Decomposition of the second variation of the energy

First we note that the equation (1.1) is equivariant under the transformation:

$$\Phi(x) \longmapsto \Phi(x) e^{ic}$$

for an arbitrarily given real number c. Hence given a solution $\tilde{\Phi}(x)$ which is not identically zero, the set

$$\{\tilde{\Phi} e^{ic} : c \in \mathbb{R}\} \tag{3.1}$$

is a continuum of the solutions. The tangential direction of this continuum at $c = 0$ is given by $i\tilde{\Phi}$. Considering this fact, it is enough for the proof of Theorem 2.2 to show that there is a $\mu > 0$ such that

$$\frac{d^2}{ds^2}E(\Phi_\lambda + s\Psi)_{|s=0} \geq \mu \int_D |\Psi|^2 adx$$
$$\text{for any} \quad \Psi \in \{\Psi \in H^1(D;\mathbb{C}) : \ \mathrm{Re}\int_D \Psi(i\Phi_\lambda)^* dx = 0\}, \tag{3.2}$$

where we put $\Phi_\lambda = f_\lambda e^{i\theta}$ or $f_\lambda e^{-i\theta}$ and * denotes the complex conjugate. Remember that $\mathbb{C}$ is identified with $\mathbb{R}^2$.

We only consider the case $\Phi_\lambda = f_\lambda e^{i\theta}$ since the other case is also treated literally in the same way. Substituting $\Phi = \Phi_\lambda + \Psi$ and putting $\Psi = \psi e^{i\theta}$ yield

$$\begin{aligned} F(\psi) \ &:= E(\Phi_\lambda + \psi e^{i\cdot}) - E(\Phi_\lambda) \\ &= \int_D \left\{ |\nabla\psi|^2 + \frac{i}{r^2}\left(\psi\frac{\partial\Psi^*}{\partial\theta} - \psi^*\frac{\partial\psi}{\partial\theta}\right) + \frac{|\psi|^2}{r^2} \right. \\ &\quad \left. -\lambda(1-f_\lambda^2)|\psi|^2 + \frac{\lambda}{2}(|\psi|^2 + 2f_\lambda\mathrm{Re}\psi)^2 \right\} adx \end{aligned} \tag{3.3}$$

Using Fourier expansion

$$\psi = \sum_{n=-\infty}^{+\infty} \psi_n e^{in\theta}$$

we obtain

$$F(\psi) = 2\pi \sum_{n=-\infty}^{+\infty} \tilde{F}_n(\psi_n) + \frac{\lambda}{2}\int_D \{|\psi|^2 + f_\lambda(2\mathrm{Re}\psi)^2\}^2 adx \tag{3.4}$$

where

$$\tilde{F}_n(\psi_n) := \int_0^1 \{|\psi'|^2 + \frac{(n+1)^2}{r^2}|\psi_n|^2 - \lambda(1-f_\lambda^2)|\psi_n|^2\} ardr \tag{3.5}$$

Because of

$$2\mathrm{Re}\psi = \sum_{n=-\infty}^{+\infty} (\psi_n e^{in\theta} + \psi_n^* e^{-in\theta})$$

we have

$$\int_0^{2\pi} (2\mathrm{Re}\psi)^2 d\theta = 2\pi \sum_{n=-\infty}^{+\infty} 2\{\mathrm{Re}(\psi_n\psi_{-n}) + |\psi_n|^2\}.$$

Thus (3.4) can be written as

$$\begin{aligned} F(\psi) \ &= 2\pi \sum_{n=-\infty}^{+\infty} \left\{ \tilde{F}_n(\psi_n) + \lambda\int_0^1 \{\mathrm{Re}(\psi_n\psi_{-n}) + |\psi_n|^2\} f_\lambda^2 ardr \right\} \\ &\quad + \frac{\lambda}{2}\int_D (|\psi|^4 + 4f_\lambda \mathrm{Re}\psi|\psi|^2) adx \end{aligned} \tag{3.6}$$

To verify the inequality (3.2), we can drop the higher order terms than quadratic ones of (3.6). It is thereby reduced to solving the minimizing problem of the infinitely many energy functionals:

$$\begin{aligned} F_0(\psi_0) &:= \tilde{F}_0(\psi_0) + 2\lambda \int_0^1 (\mathrm{Re}\psi_0)^2 f_\lambda^2 ardr, \\ F_n(\psi_n, \psi_{-n}) &:= \tilde{F}_n(\psi_n) + \tilde{F}_{-n}(\psi_{-n}) + \lambda \int_0^1 \{(|\psi_n|^2 + |\psi_{-n}|^2) + 2\mathrm{Re}(\psi_n \psi_{-n})\} f_\lambda^2 ardr, \\ & \qquad n = 1, 2, \cdots \end{aligned} \tag{3.7}$$

(note that $\{\mathrm{Re}(\psi_0^2)\}^2 + |\psi_0|^2 = 2(\mathrm{Re}\psi_0)^2$). By virtue of the next lemma, however, it turns out that the functional F_1 determines the stability.

Lemma 3.1

(*i*) *Given λ, there is a poitive number μ_0 such that*

$$F_0(\varphi) \geq \mu_0 \int_0^1 |\varphi|^2 ardr, \qquad \varphi \in \{\varphi \in H_r^1(0,1) : \mathrm{Re} \int_0^1 \varphi(-if_\lambda) ardr = 0\},$$

where

$$H_r^1(0,1) := \{\varphi \in H^1((0,1); \mathbb{C}) : \int_0^1 (|\varphi|^2 + |\varphi'|^2) ardr < \infty\}.$$

(*ii*) *For any $n \geq 2$*

$$F_n(\varphi, \phi) > F_1(\varphi, \phi), \qquad \varphi, \phi \in H_r^1(0,1) \quad \textit{and} \quad \varphi, \phi \not\equiv 0$$

holds.

Proof. Since the proof of (ii) is straightforward, we only prove (i).
With the real form $\varphi = g + ih$ we can decouple F_0 as

$$F_0(\varphi) = F_{01}(g) + F_{02}(h),$$

$$F_{01}(g) := \int_0^1 \{(g')^2 + \frac{1}{r^2} g^2 - \lambda(1 - 3f_\lambda^2) g^2\} ardr$$

$$F_{02}(h) := \int_0^1 \{(h')^2 + \frac{1}{r^2} h^2 - \lambda(1 - f_\lambda^2) h^2\} ardr$$

Note that the minimizer of each energy functional can be realized by a positive function in $r \in (0,1]$. Indeed we can first exclude the case that the minimizer has a degenerate zero, because of the uniqueness of solutions to 2nd order ordinary differential equations. Next suppose that it changes the sign. Then the modulus of the minimizer retains the same energy level. It, however, is not C^1. This is a contradiction.

From the above decomposition it follows that the minimizing problem of F_0 is reduced to the next decoupled eigenvalue problems:

$$g'' + \frac{(ar)'}{ar}g' - \frac{1}{r^2}g + \lambda(1-3f_\lambda^2)g = -\mu g$$
$$h'' + \frac{(ar)'}{ar}h' - \frac{1}{r^2}h + \lambda(1-f_\lambda^2)h = -\mu h \tag{3.8}$$

Namely the minimum of F_{01} (resp. F_{02}) is the least eigenvalue μ of the first (resp. second) problem and a minimizer is realized by the corresponding eigenfunction.

We easily check that there is a zero eigenvalue and the corresponding eigenfunction is given by $(g,h) = (0, f_\lambda)$ (remenber $\psi = i\Phi_\lambda$). Since $f_\lambda > 0$ in $(0,1]$ the zero is the least eigenvalue of the second problem. Moreover $F_{01}(\varphi) > F_{02}(\varphi)$ for $\varphi \not\equiv 0$. Hence we obtain the assertion of the lemma. ∎

The next corollary immediately follows from the above lemma.

Corollary 3.2 *Suppose*

$$\min\left\{(\varphi,\phi) \in (H_r^1((0,1);\mathbb{R}))^2, \quad (\varphi,\phi) \not\equiv (0,0) \quad ; \quad \frac{F_1(\varphi,\phi)}{|\varphi|_{L_r^2}^2 + |\phi|_{L_r^2}^2}\right\} > 0$$

where

$$|\cdot|_{L_r^2} := \left\{\int_0^1 |\cdot|^2 ardr\right\}^{1/2}.$$

Then (3.2) *holds.*

3.2 A key lemma

Next putting $\varphi = g_1 + ih_1, \phi = g_2 + ih_2$, we write

$$F_1(\varphi,\phi) = \mathcal{E}(g_1, -g_2) + \mathcal{E}(h_1, h_2)$$

where

$$\mathcal{E}(v,w) := \int_0^1 \{(v')^2 + (w')^2 + \frac{4}{r^2}w^2 - \lambda(1-2f_\lambda^2)(v^2+w^2) - 2\lambda f_\lambda^2 vw\}ardr \tag{3.9}$$

Thus our probelm is reduced to the minimizing problem of $\mathcal{E}(v,w)$.Using the change of variables

$$p = (w-v)/\sqrt{2}, \qquad q = (v+w)/\sqrt{2} \tag{3.10}$$

we obtain

$$\mathcal{E}(v,w) = \mathcal{F}(p,q) := \int_0^1 \{(p')^2 + (q')^2 + \frac{2}{r^2}(p-q)^2$$
$$-\lambda(1-f_\lambda^2)(p^2+q^2) + \lambda f_\lambda^2 p^2\}ardr \tag{3.11}$$

The corresponding eigenvalue problem to the energy $\mathcal{F}$ is as follows:

$$-\mathcal{L}\begin{pmatrix} p \\ q \end{pmatrix} = \mu \begin{pmatrix} p \\ q \end{pmatrix} \tag{3.12}$$
$$\mathrm{Dom}(\mathcal{L}) = \{(p,q) \in (H_r^2(0,1);\mathbb{R})^2 : p'(1) = q'(1) = 0\}$$

where

$$\mathcal{L}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p'' + \dfrac{(ar)'}{ar}p' - \dfrac{2}{r^2}(p-q) + \lambda(1-3f_\lambda^2)p \\ q'' + \dfrac{(ar)'}{ar}q' - \dfrac{2}{r^2}(q-p) + \lambda(1-f_\lambda^2)q \end{pmatrix} \tag{3.13}$$

We use $(f_\lambda', f_\lambda/r)$ as test functions to investigate the least eigenvalue of $-\mathcal{L}$. Indeed differentiating (1.4) with respect to r, we can check

$$-\mathcal{L}\begin{pmatrix} f_\lambda' \\ \dfrac{f_\lambda}{r} \end{pmatrix} = \begin{pmatrix} \left(\dfrac{a'}{a}\right)' f_\lambda' \\ \left(\dfrac{a'}{a}\right)\dfrac{f_\lambda}{r^2} \end{pmatrix} \tag{3.14}$$

Multiplying $f_\lambda' a(r)r$ and $(f_\lambda/r)a(r)r$ with the first and the second components of (3.12) respectively and integrating from 0 to 1 by parts yield

$$\int_0^1 \left(\frac{a'}{a}\right)' f_\lambda' pardr + \int_0^1 \left(\frac{a'}{a}\right)\frac{f_\lambda}{r^2}qardr$$
$$+a(1)f_\lambda''(1)p(1) + a(1)\left(\frac{f_\lambda}{r}\right)'(1)q(1)$$
$$= \mu\left\{\int_0^1 f_\lambda' pardr + \int_0^1 \frac{f_\lambda}{r}qardr\right\}$$

where we used (3.14). Hence we obtain

$$\mu = \frac{f_\lambda''(1)p(1) - f_\lambda(1)q(1) + < (a'/a)'f_\lambda', p > + < (a'/a)f_\lambda/r^2, q >}{< f_\lambda', p > + < f_\lambda/r, q >} \tag{3.15}$$
$$< v, w > := \int_0^1 v(r)w(r)a(r)rdr$$

The next lemma will play a key role to evaluate the right hand side of (3.15).

Lemma 3.3 *Let $(p(r), q(r))$ be a pair of eigenfunctions corresponding to the least eigenvalue of $-\mathcal{L}$.*

(*i*) *Those eigenfunctions can be taken as*

$$q(r) > p(r) > 0, \qquad r \in (0,1].$$

(*ii*) *Let* μ *be the least eigenvalue of* $-\mathcal{L}$ *and assume* $\mu \leq 0$. *Then arbitrarily given* $\alpha, 0 < \alpha < 1$, *there are positive numbers* λ_3 *and* C_2 *such that for each* $\lambda > \lambda_3$

$$\left(\frac{C_2}{\lambda}p(r) + q(r)\right)' < 0, \quad r \in (\alpha, 1),$$

where λ_3 *and* C_2 *are independent of* $\mu(\leq 0)$.

To prove the above lemma, we use following properties on the solution f_λ to (1.4):

(*a*)

$$0 < f_\lambda(r) < 1 \quad \text{and} \quad f_\lambda'(r) > 0, \qquad r \in (0,1).$$

(*b*)

$$\frac{f_\lambda}{r} > f_\lambda'(r), \qquad r \in (0,1).$$

(*c*) For an arbitrarily given and fixed $\alpha > 0$ there are $\lambda_2 > 0$ and $C_1 > 0$ such that for each $\lambda > \lambda_2$

$$\|f_\lambda - 1\|_{C^2[\alpha,1]} \leq \frac{C_1}{\lambda}$$

holds. Thus

$$\sup_{\alpha \leq r \leq 1} \left| -\frac{1}{r^2} + \lambda(1 - f_\lambda^2) \right| \leq \frac{C_1}{\lambda}.$$

For the proof of this lemma and the positivity of the eigenvalue μ, see [13].

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