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Berger-Shaw's theorem for p-hyponormal operators

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For a n-multicyclic p-hyponormal operator T, we shall show that $|T|^{2p} - |T^*|^{2p}$ belongs to the Schatten $\frac{1}{p}$-class $\mathcal{C}_{\frac{1}{p}}$ and that $tr\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{p}}\right) \leq \frac{n}{\pi} \mathrm{A}rea(\sigma(T))$.

1. **Introduction** For a bounded linear operator T on Hilbert space $\mathcal{H}$, $\mathcal{R}(\sigma(T))$ denotes the set of all rational functions analytic on $\sigma(T)$, where $\sigma(T)$ is the spectrum of T. The operator T is said to be n-multicyclic if there are n vectors $x_1, \ldots, x_n \in \mathcal{H}$, called generating vectors, such that $\vee\{g(T)x_i;\ i = 1, \ldots, n,\ g \in \mathcal{R}(\sigma(T))\} = \mathcal{H}$. For a p such as $0 < p \leq 1$, T is said to be p-hyponormal, if $(T^*T)^p \geq (TT^*)^p$. In particular, 1-hyponormal is called hyponormal and $\frac{1}{2}$-hyponormal is called semihyponormal. Xia([7]) gave an example which is not hyponormal but semihyponormal. Thus, the class of p-hyponormal operators properly contains 1- hyponormal operators. Putnam([6]) obtained the norm estimation for the self-commutator of a hyponormal operator, so called Putnam's inequality. This inequality is extended for a p-hyponormal operator by Xia([8]) and Cho-Itoh([3]). Berger-Shaw([2]) showed the trace norm estimation for the self-commutator of n-multicyclic hyponormal operator, so called Berger-Shaw's inequality. In this paper we shall extend this inequality to the case of a n-multicyclic p-hyponormal operator.

2. Preliminary lemmas

For p-hyponormal operator T with its polar decomposition $T = U|T|$, the operator $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is said to be the Aluthge transform.

It is known, by Aluthge([1]), that $\widetilde{T}$ is hyponormal if $\frac{1}{2} \le p \le 1$ and $(p+\frac{1}{2})$-hyponormal if $0 < p \le \frac{1}{2}$ and if U is unitary. In this paper, we deal with the operator $\widehat{T} = |\,T\,|^{\frac{1}{2^m}-p}U|\,T\,|^{1-\frac{1}{2^m}+p}$ for $\frac{1}{2^{m+1}} \le p \le \frac{1}{2^m}$, where m is non-negative integer.

Lemma 1. *If T is p-hyponormal operator for a p such as $\frac{1}{2^{m+1}} \le p \le \frac{1}{2^m}$, then*

$$\begin{aligned}(\widehat{T}\widehat{T}^*)^{\frac{1}{2^m}} &\le |\,T\,|^{\frac{1}{2^m}-p}|\,T^*\,|^{2p}|\,T\,|^{\frac{1}{2^m}-p}\\ &\le |\,T\,|^{\frac{1}{2^{m-1}}} \le (\widehat{T}^*\widehat{T})^{\frac{1}{2^m}},\end{aligned}$$

and hence $\widehat{T}$ is $\frac{1}{2^m}$-hyponormal.

Proof. Since $\frac{1}{2^{m+1}} \le p \le \frac{1}{2^m}$, $\frac{1}{2^m}-p \le p$ and T is $(\frac{1}{2^m}-p)$-hyponormal by Heinz's inequality and hence

$$\begin{aligned}\widehat{T}^*\widehat{T} &= |\,T\,|^{1-\frac{1}{2^m}+p}U^*|\,T\,|^{2(\frac{1}{2^m}-p)}U|\,T\,|^{1-\frac{1}{2^m}+p}\\ &\ge |\,T\,|^{1-\frac{1}{2^m}+p}U^*|\,T^*\,|^{2(\frac{1}{2^m}-p)}U|\,T\,|^{1-\frac{1}{2^m}+p}\\ &= |\,T\,|^{1-\frac{1}{2^m}+p}|\,T\,|^{2(\frac{1}{2^m}-p)}|\,T\,|^{1-\frac{1}{2^m}+p} = |\,T\,|^2.\end{aligned}$$

Thus, by Heinz's inequality, we have the inequality,

$$|\,\widehat{T}\,|^s \ge |\,T\,|^s \quad \forall s \in (0,2].$$

Since, by the $(\frac{1}{2^m}-p)$-hyponormality of T,

$$\begin{aligned}\widehat{T}\widehat{T}^* &= |\,T\,|^{\frac{1}{2^m}-p}U|\,T\,|^{2-\frac{1}{2^{m-1}}+2p}U^*|\,T\,|^{\frac{1}{2^m}-p}\\ &= |\,T\,|^{\frac{1}{2^m}-p}|\,T^*\,|^{2-\frac{1}{2^{m-1}}+2p}|\,T\,|^{\frac{1}{2^m}-p}\\ &= |\,T\,|^{\frac{1}{2^m}-p}|\,T^*\,|^{1-\frac{1}{2^{m-1}}+2p}|\,T^*\,|^{2(\frac{1}{2^m}-p)}|\,T^*\,|^{1-\frac{1}{2^{m-1}}+2p}|\,T\,|^{\frac{1}{2^m}-p}\\ &\le |\,T\,|^{\frac{1}{2^m}-p}|\,T^*\,|^{1-\frac{1}{2^{m-1}}+2p}|\,T\,|^{2(\frac{1}{2^m}-p)}|\,T^*\,|^{1-\frac{1}{2^{m-1}}+2p}|\,T\,|^{\frac{1}{2^m}-p}\\ &= (|\,T\,|^{\frac{1}{2^m}-p}|\,T^*\,|^{1-\frac{1}{2^{m-1}}+2p}|\,T\,|^{\frac{1}{2^m}-p})^2,\end{aligned}$$

We have, by Heinz's inequality and by $(\frac{1}{2^m}-p)$-hyponormality of T,

$$
\begin{aligned}
(\widehat{T}\widehat{T}^*)^{\frac{1}{2}} &\leq | T |^{\frac{1}{2^m}-p} | T^* |^{1-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p} \\
&= | T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} | T^* |^{2(\frac{1}{2^m}-p)} | T^* |^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p} \\
&\leq | T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} | T |^{2(\frac{1}{2^m}-p)} | T^* |^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p} \\
&= (| T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p})^2,
\end{aligned}
$$

and, by repeating the same arguments as above, we obtain

$$
\begin{aligned}
&(\widehat{T}\widehat{T}^*)^{\frac{1}{2^2}} \\
&\leq | T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p} \\
&= | T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} | T^* |^{2(\frac{1}{2^m}-p)} | T^* |^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p} \\
&\leq | T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} | T |^{2(\frac{1}{2^m}-p)} | T^* |^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p} \\
&= (| T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p})^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\widehat{T}\widehat{T}^*)^{\frac{1}{2^3}} \\
&\leq | T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p} \\
&= | T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2^3}-\frac{1}{2^{m-1}}+2p} | T^* |^{2(\frac{1}{2^m}-p)} | T^* |^{\frac{1}{2^3}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p} \\
&\leq | T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2^3}-\frac{1}{2^{m-1}}+2p} | T |^{2(\frac{1}{2^m}-p)} | T^* |^{\frac{1}{2^3}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p} \\
&= (| T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2^3}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p})^2.
\end{aligned}
$$

Eventually, we have

$$
\begin{aligned}
&(\widehat{T}\widehat{T}^*)^{\frac{1}{2^{m-1}}} \\
&\leq | T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2^{m-2}}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p} \\
&= | T |^{\frac{1}{2^m}-p} | T^* |^{\frac{1}{2^{m-1}}-\frac{1}{2^{m-1}}+2p} | T^* |^{2(\frac{1}{2^m}-p)} | T^* |^{\frac{1}{2^{m-1}}-\frac{1}{2^{m-1}}+2p} | T |^{\frac{1}{2^m}-p} \\
&= | T |^{\frac{1}{2^m}-p} | T^* |^{2p} | T^* |^{2(\frac{1}{2^m}-p)} | T^* |^{2p} | T |^{\frac{1}{2^m}-p} \\
&\leq | T |^{\frac{1}{2^m}-p} | T^* |^{2p} | T |^{2(\frac{1}{2^m}-p)} | T^* |^{2p} | T |^{\frac{1}{2^m}-p} \\
&= (| T |^{\frac{1}{2^m}-p} | T^* |^{2p} | T |^{\frac{1}{2^m}-p})^2,
\end{aligned}
$$

and hence

$$\begin{aligned}(\widehat{T}\widehat{T}^*)^{\frac{1}{2^m}} &\leq |\,T\,|^{\frac{1}{2^m}-p}|\,T^*\,|^{2p}|\,T\,|^{\frac{1}{2^m}-p} \\ &\leq |\,T\,|^{\frac{1}{2^m}-p}|\,T\,|^{2p}|\,T\,|^{\frac{1}{2^m}-p} \\ &= |\,T\,|^{\frac{1}{2^{m-1}}} \leq (\widehat{T}^*\widehat{T})^{\frac{1}{2^m}}.\end{aligned}$$

Therefore $\widehat{T}$ is $\frac{1}{2^m}$-hyponormal.

Lemma 2. *If T is n-multicyclic p-hyponormal, then $\widehat{T}$ is also n-multicyclic and $\sigma(\widehat{T}) = \sigma(T)$.*

Proof. For a p-hyponormal operator T, $\mathrm{Ker}T$ is a reducing subspace of T and also $\widehat{T}$. Hence, we may assume that $\mathrm{Ker}T = \{0\}$. Put $s = \frac{1}{2^m} - p$, where $\frac{1}{2^{m+1}} \leq p \leq \frac{1}{2^m}$.

$$\sigma(\widehat{T}) = \sigma(|\,T\,|^s U|\,T\,|^{1-s}) \subset \sigma(U|\,T\,|^{1-s}|\,T\,|^s)\bigcup\{0\} = \sigma(T)\bigcup\{0\}.$$

Similarly

$$\sigma(T) = \sigma(U|\,T\,|^{1-s}|\,T\,|^s) \subset \sigma(|\,T\,|^s U|\,T\,|^{1-s})\bigcup\{0\} = \sigma(\widehat{T})\bigcup\{0\}.$$

Since T is invertible if and only if $\widehat{T}$ is invertible, we have $\sigma(\widehat{T}) = \sigma(T)$ and $\mathcal{R}(\sigma(T)) = \mathcal{R}(\sigma(\widehat{T}))$. Since T is n-multicyclic,

$\exists\ x_1, \dots, x_n \in \mathcal{H}$ s.t.

$$\vee\{g(T)x_i;\ i = 1, \dots, n,\ g \in \mathcal{R}(\sigma(T))\} = \mathcal{H}.$$

Put $y_i = |\,T\,|^s x_i$, $i = 1, \dots, n$. We shall show that $\{y_i\}_{i=1}^n$ are n-multicyclic vectors for T.

$$\widehat{T}^k|\,T\,|^s = \{|\,T\,|^s U|\,T\,|^{1-s}\}^k|\,T\,|^s = |\,T\,|^s\{U\,|\,T\,|\}^k = |\,T\,|^s T^k.$$

If $\lambda \in \rho(T)$, then $\lambda - U(|\,T\,| + \epsilon)$ is invertible for sufficiently small $\epsilon > 0$. Therefore,

$$\begin{aligned}&(|\,T\,| + \epsilon)^s\big(\lambda - U(|\,T\,| + \epsilon)\big)^{-1} \\ =&\{\big(\lambda - U(|\,T\,| + \epsilon)\big)(|\,T\,| + \epsilon)^{-s}\}^{-1} \\ =&\{(|\,T\,| + \epsilon)^{-s}\big(\lambda - (|\,T\,| + \epsilon)^s U(|\,T\,| + \epsilon)^{1-s}\big)\}^{-1} \\ =&\big(\lambda - (|\,T\,| + \epsilon)^s U(|\,T\,| + \epsilon)^{1-s}\big)^{-1}(|\,T\,| + \epsilon)^s\end{aligned}$$

Letting $\epsilon \downarrow 0$, we have

$$(\lambda - \widehat{T})^{-1}|\,T\,|^s = |\,T\,|^s(\lambda - T)^{-1}.$$

Hence, we have

$$g(\widehat{T})|T|^s = |T|^s g(T), \quad \forall\, g \in \mathcal{R}(\sigma(T)),$$

and

$$g(\widehat{T})y_i = |T|^s g(T)x_i, \quad \forall\, g \in \mathcal{R}(\sigma(T)) \quad i = 1, \dots, n.$$

Thus,

$$\begin{aligned} & \vee \{g(\widehat{T})y_i;\ i = 1, \dots, n,\ g \in \mathcal{R}(\sigma(\widehat{T}))\} \\ = & \vee \{g(\widehat{T})y_i;\ i = 1, \dots, n,\ g \in \mathcal{R}(\sigma(T))\} \\ = & \vee \{|T|^s g(T)x_i;\ i = 1, \dots, n,\ g \in \mathcal{R}(\sigma(T))\} \\ = & [|T|^s \mathcal{H}]^{\sim} = \mathcal{H} \ \text{ because } \mathrm{Ker}T = \{0\}. \end{aligned}$$

This implies that $\widehat{T}$ is n-multicyclic.

3. Main theorem

Berger-Shaw's Theorem. If T is a n-multicyclic hyponormal operator, then $[T^*, T] = T^*T - TT^*$ is in the trace class, and $tr\Big([T^*, T]\Big) \leq \frac{n}{\pi}\mathrm{A}rea(\sigma(T))$, where $\mathrm{A}rea$ means the planar Lebesgue measure.

The following result is our main theorem.

Theorem. If T is a n-multicyclic p-hyponormal operator for p such as $0 < p \leq 1$, then for p such as $0 < p \leq 1$, then $|T|^{2p} - |T^*|^{2p}$ belongs to the Schatten $\frac{1}{p}$-class $\mathcal{C}_{\frac{1}{p}}$ and

$$tr\Big((|T|^{2p} - |T^*|^{2p})^{\frac{1}{p}}\Big) \leq \frac{n}{\pi}\mathrm{A}rea(\sigma(T)).$$

When $p = 1$, this theorem is exactly Berger-Shaw's theorem.

The following is the key for our purpose.

Lemma 3. *If T is n-multicyclic p-hyponormal, then*

$$tr\Big(|T|^{1-p}(|T|^{2p} - |T^*|^{2p})|T|^{1-p}\Big) \leq \frac{n}{\pi}\mathrm{A}rea(\sigma(T)).$$

Proof. We shall show this lemma for p such as $\frac{1}{2^{m+1}} \leq p \leq \frac{1}{2^m}$, $m = 0, 1, 2. \dots$, by the induction in m.

If $m=0$, then $\widehat{T}$ is a n-multicyclic hyponormal operator by Lemmas 1 and 2. Thus Berger-Shaw's theorem implies that

$$tr\left(\widehat{T}^*\widehat{T}-\widehat{T}\widehat{T}^*\right)\leq\frac{n}{\pi}Area(\sigma(\widehat{T}))=\frac{n}{\pi}Area(\sigma(T)),$$

because $\sigma(\widehat{T})=\sigma(T)$ by Lemma 2. Since, by Lemma 1,

$$\begin{aligned}\widehat{T}^*\widehat{T}-\widehat{T}\widehat{T}^*&\geq|T|^2-|T|^{1-p}|T^*|^{2p}|T|^{1-p}\\&=|T|^{1-p}(|T|^{2p}-|T^*|^{2p})|T|^{1-p},\end{aligned}$$

we have

$$\begin{aligned}tr\left(|T|^{1-p}(|T|^{2p}-|T^*|^{2p})|T|^{1-p}\right)&\leq tr(\widehat{T}^*\widehat{T}-\widehat{T}\widehat{T}^*)\\&\leq\frac{n}{\pi}Area(\sigma(T)).\end{aligned}$$

Hence, the assertion holds for $m=0$.

Next, we assume that the assertion holds for $m=k$ $(k\geq 0)$. If $m=k+1$, then $\widehat{T}$ is $\frac{1}{2^{k+1}}$-hyponormal by Lemma 1. Hence by the assumption and by Lemmas 1 and 2, we have

$$\begin{aligned}&\frac{n}{\pi}Area(\sigma(T))=\frac{n}{\pi}Area(\sigma(\widehat{T}))\\
&\geq tr\left(|\widehat{T}|^{1-\frac{1}{2^{k+1}}}(|\widehat{T}|^{2\frac{1}{2^{k+1}}}-|\widehat{T}^*|^{2\frac{1}{2^{k+1}}})|\widehat{T}|^{1-\frac{1}{2^{k+1}}}\right)\\
&\geq tr\left(|\widehat{T}|^{1-\frac{1}{2^{k+1}}}(|T|^{2\frac{1}{2^{k+1}}}-|T|^{\frac{1}{2^{k+1}}-p}|T^*|^{2p}|T|^{\frac{1}{2^{k+1}}-p})|\widehat{T}|^{1-\frac{1}{2^{k+1}}}\right)\\
&=tr\left((|T|^{2\frac{1}{2^{k+1}}}-|T|^{\frac{1}{2^{k+1}}-p}|T^*|^{2p}|T|^{\frac{1}{2^{k+1}}-p})^{\frac{1}{2}}|\widehat{T}|^{2(1-\frac{1}{2^{k+1}})}\right.\\
&\qquad\left.\times(|T|^{2\frac{1}{2^{k+1}}}-|T|^{\frac{1}{2^{k+1}}-p}|T^*|^{2p}|T|^{\frac{1}{2^{k+1}}-p})^{\frac{1}{2}}\right)\\
&\leq tr\left((|T|^{2\frac{1}{2^{k+1}}}-|T|^{\frac{1}{2^{k+1}}-p}|T^*|^{2p}|T|^{\frac{1}{2^{k+1}}-p})^{\frac{1}{2}}|T|^{2(1-\frac{1}{2^{k+1}})}\right.\\
&\qquad\left.\times(|T|^{2\frac{1}{2^{k+1}}}-|T|^{\frac{1}{2^{k+1}}-p}|T^*|^{2p}|T|^{\frac{1}{2^{k+1}}-p})^{\frac{1}{2}}\right)\\
&=tr\left(|T|^{1-\frac{1}{2^{k+1}}}(|T|^{2\frac{1}{2^{k+1}}}-|T|^{\frac{1}{2^{k+1}}-p}|T^*|^{2p}|T|^{\frac{1}{2^{k+1}}-p})|T|^{1-\frac{1}{2^{k+1}}}\right)\\
&=tr\left(|T|^{1-p}(|T|^{2p}-|T^*|^{2p})|T|^{1-p}\right).\end{aligned}$$

Hence, the assertion holds for $m=k+1$. This completes the proof of Lemma 3.

Corollary 1. *If T is an invertible n-multicyclic p-hyponormal operator, then $(T^*T)^p - (TT^*)^p \in \mathcal{C}_1$ and*

$$tr\Big((T^*T)^p - (TT^*)^p\Big) \leq \|T^{-1}\|^{2(1-p)}\frac{n}{\pi}\text{A}rea(\sigma(T)).$$

Proof. Since T is invertible, $T^*T \geq \|T^{-1}\|^{-2}$, and n-multicyclic p-hyponormality of T implies that

$$\begin{aligned}
&\frac{n}{\pi}\text{A}rea(\sigma(T)) \\
\geq & tr\Big(\mid T\mid^{1-p}\{(T^*T)^p - (TT^*)^p\}\mid T\mid^{1-p}\Big) \quad \text{by Lemma 3} \\
= & tr\Big(\{(T^*T)^p - (TT^*)^p\}^{\frac{1}{2}}(T^*T)^{1-p}\{(T^*T)^p - (TT^*)^p\}^{\frac{1}{2}}\Big) \\
\geq & \|T^{-1}\|^{-2(1-p)}tr\Big((T^*T)^p - (TT^*)^p\Big).
\end{aligned}$$

We have $(T^*T)^p - (TT^*)^p \in \mathcal{C}_1$ and $tr\Big((T^*T)^p - (TT^*)^p\Big) \leq \|T^{-1}\|^{2(1-p)}\frac{n}{\pi}\text{A}rea(\sigma(T))$. This completes the proof of Corollary 1.

Proof of Theorem. By Lemma 3,

$$tr\Big(\mid T\mid^{1-p}(\mid T\mid^{2p} - \mid T^*\mid^{2p})\mid T\mid^{1-p}\Big) \leq \frac{n}{\pi}\text{A}rea(\sigma(T)).$$

And by the property of trace,

$$\begin{aligned}
& tr\Big(\mid T\mid^{1-p}(\mid T\mid^{2p} - \mid T^*\mid^{2p})\mid T\mid^{1-p}\Big) \\
= & tr\Big((\mid T\mid^{2p} - \mid T^*\mid^{2p})^{\frac{1}{2}}\mid T\mid^{2-2p}(\mid T\mid^{2p} - \mid T^*\mid^{2p})^{\frac{1}{2}}\Big) \\
= & tr\Big((\mid T\mid^{2p} - \mid T^*\mid^{2p})^{\frac{1}{2}}\{\mid T\mid^{2p}\}^{\frac{1-p}{p}}(\mid T\mid^{2p} - \mid T^*\mid^{2p})^{\frac{1}{2}}\Big).
\end{aligned}$$

If $\frac{1}{2} \leq p \leq 1$, then $0 \leq \frac{1-p}{p} \leq 1$. Thus, by Heinz's inequality, we obtain

$$\begin{aligned}
& tr\Big((\mid T\mid^{2p} - \mid T^*\mid^{2p})^{\frac{1}{2}}\{\mid T\mid^{2p}\}^{\frac{1-p}{p}}(\mid T\mid^{2p} - \mid T^*\mid^{2p})^{\frac{1}{2}}\Big) \\
\geq & tr\Big((\mid T\mid^{2p} - \mid T^*\mid^{2p})^{\frac{1}{2}}(\mid T\mid^{2p} - \mid T^*\mid^{2p})^{\frac{1-p}{p}}(\mid T\mid^{2p} - \mid T^*\mid^{2p})^{\frac{1}{2}}\Big) \\
= & tr\Big((\mid T\mid^{2p} - \mid T^*\mid^{2p})^{1+\frac{1-p}{p}}\Big) \\
= & tr\Big((\mid T\mid^{2p} - \mid T^*\mid^{2p})^{\frac{1}{p}}\Big).
\end{aligned}$$

Therefore, we have

$$\mathrm{tr}\Big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{p}}\Big)\leq\frac{n}{\pi}\mathrm{A}rea(\sigma(T)).$$

Thus, the assertion of Theorem holds for $p\in[\frac{1}{2},1]$.

If $0<p\leq\frac{1}{2}$, then by Furuta's inequality([4]),

$$\begin{aligned}&\Big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\{|T|^{2p}\}^{\frac{1-p}{p}}(|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\Big)^{2p}\\ \geq&\Big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}(|T|^{2p}-|T^*|^{2p})^{\frac{1-p}{p}}(|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\Big)^{2p}\\ =&\big(|T|^{2p}-|T^*|^{2p}\big)^{2}.\end{aligned}$$

Thus $\Big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\{|T|^{2p}\}^{\frac{1-p}{p}}(|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\Big)^{2p}$ and $(|T|^{2p}-|T^*|^{2p})^2$ are both compact positive operators. Let $s_n\Big[A\Big]$ be the n-th singular number of a positive compact operator A. Then,

$$\begin{aligned}&s_n\Big[\Big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\{|T|^{2p}\}^{\frac{1-p}{p}}(|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\Big)^{2p}\Big]\\ =&s_n\Big[(|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\{|T|^{2p}\}^{\frac{1-p}{p}}(|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\Big]^{2p}\\ \geq&s_n\Big[\big(|T|^{2p}-|T^*|^{2p}\big)^{2}\Big],\end{aligned}$$

and hence,

$$\begin{aligned}&s_n\Big[(|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\{|T|^{2p}\}^{\frac{1-p}{p}}(|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\Big]\\ \geq&s_n\Big[\big(|T|^{2p}-|T^*|^{2p}\big)^{2}\Big]^{\frac{1}{2p}}\\ =&s_n\Big[\big(|T|^{2p}-|T^*|^{2p}\big)^{2\frac{1}{2p}}\Big]\\ =&s_n\Big[\big(|T|^{2p}-|T^*|^{2p}\big)^{\frac{1}{p}}\Big].\end{aligned}$$

Hence,

$$\begin{aligned}&\mathrm{tr}\Big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\{|T|^{2p}\}^{\frac{1-p}{p}}(|T|^{2p}-|T^*|^{2p})^{\frac{1}{2}}\Big)\\ \geq&\mathrm{tr}\Big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{p}}\Big).\end{aligned}$$

Therefore, we have

$$\mathrm{tr}\Big((|T|^{2p}-|T^*|^{2p})^{\frac{1}{p}}\Big)\leq\frac{n}{\pi}\mathrm{A}rea(\sigma(T)).$$

The assertion of Theorem also holds for $p \in (0, \frac{1}{2}]$. This completes the proof of Theorem.

For the restriction of a p-hyponormal operator to invariant subspace, we have the following.

Lemma 4. *Let $\mathcal{M}$ be an invariant subspace for a p-hyponormal operator T, and T' be the restriction of T to $\mathcal{M}$. Then*

$$\begin{aligned} \{T'T'^*\}^p &\le P(TT^*)^p P \\ &\le P(T^*T)^p P \le \{T'^*T'\}^p, \end{aligned}$$

and T' is also p-hyponormal, where P denotes the projection onto $\mathcal{M}$.

Proof. Since $T' = TP$,

$$T'^*T' = PT^*TP,$$

and hence, for any $s \in (0, 1]$,

$$\{T'^*T'\}^s = \{PT^*TP\}^s \ge P(T^*T)^s P \text{ by Hansen's inequality([5]).}$$

While,

$$T'T'^* = TPT^* = PTPT^*P,$$

we have, for any $s \in (0, 1]$,

$$\begin{aligned} \{T'T'^*\}^s &= (TPT^*)^s \\ &= P(TPT^*)^s P \\ &\le P(TT^*)^s P \text{ by Heinz's inequality.} \end{aligned}$$

Therefore, if T is p-hyponormal for p such as $0 < p \le 1$, then

$$\begin{aligned} \{T'T'^*\}^p &\le P(TT^*)^p P \\ &\le P(T^*T)^p P \\ &\le \{T'^*T'\}^p. \end{aligned}$$

Thus, T' is also p-hyponormal.

Corollary 2. *If T is p-hyponormal operator, then*

$$\|(T^*T)^p - (TT^*)^p\| \le \{\frac{1}{\pi}\text{Area}(\sigma(T))\}^p.$$

Proof. Let x be an arbitrary unit vector in $\mathcal{H}$. We define

$$\mathcal{H}_0 = \vee\{g(T)x;\ g \in \mathcal{R}(\sigma(T))\}.$$

Since $\mathcal{H}_0$ is an invariant subspace for T, Lemma 4 implies that $T' = T|_{\mathcal{H}_0}$ is a (1-multicyclic) p-hyponormal operator. If $\lambda \in \rho(T)$, then, for any $y \in \mathcal{H}_0$, $(T-\lambda)^{-1}y \in \mathcal{H}_0$. Therefore, $\lambda \in \rho(T')$. Hence, $\sigma(T') \subset \sigma(T)$. By Theorem,

$$\begin{aligned} tr\Big(\{(T'^*T')^p - (T'T'^*)^p\}^{\frac{1}{p}}\Big) &\leq \frac{1}{\pi}\text{Area}(\sigma(T')) \\ &\leq \frac{1}{\pi}\text{Area}(\sigma(T)) \end{aligned}$$

and the maximal eigenvalue of positive trace class operator $\{(T'^*T')^p - (T'T'^*)^p\}^{\frac{1}{p}}$ is equal to or less than $\frac{1}{\pi}\text{Area}(\sigma(T))$. Thus, the maximal eigenvalue of $(T'^*T')^p-(T'T'^*)^p$ is equal to or less than $\{\frac{1}{\pi}\text{Area}(\sigma(T))\}^p$. Therefore,

$$\|(T'^*T')^p - (T'T'^*)^p\| \leq \{\frac{1}{\pi}\text{Area}(\sigma(T))\}^p.$$

Let P be the projection onto $\mathcal{H}_0$. Then, by Lemma 4,

$$\begin{aligned} &\{\frac{1}{\pi}\text{Area}(\sigma(T))\}^p \\ &\geq \big\langle\{(T'^*T')^p - (T'T'^*)^p\}x, x\big\rangle \\ &\geq \langle\{P(T^*T)^pP - P(TT^*)^pP\}x, x\rangle \\ &= \langle\{(T^*T)^p - (TT^*)^p\}x, x\rangle\,. \end{aligned}$$

Since $x \in \mathcal{H}$ is arbitrary unit vector,

$$\|(T^*T)^p - (TT^*)^p\| \leq \{\frac{1}{\pi}\text{Area}(\sigma(T))\}^p.$$

This inequality is an extension of the Putnam inequality to the case of p-hyponormal operator.

Corollary 3. *If T is an invertible p-hyponormal operator, then*

$$\|(T^*T)^p - (TT^*)^p\| \leq \|T^{-1}\|^{2(1-p)}\frac{1}{\pi}\text{Area}(\sigma(T)).$$

Proof. Put $\mathcal{H}_0$, $T' = T|_{\mathcal{H}_0}$ and P as Corollary 2, then T' is an invertible (1-multicyclic) p-hyponormal operator. Therfore by Lemma 3,

$$
\begin{aligned}
&\frac{1}{\pi}\text{Area}(\sigma(T)) \geq \frac{1}{\pi}\text{Area}(\sigma(T'))\\
\geq&\text{tr}\Big(\mid T' \mid^{1-p} \{(T'^*T')^p - (T'T'^*)^p\} \mid T' \mid^{1-p}\Big)\\
=&\text{tr}\Big(\{(T'^*T')^p - (T'T'^*)^p\}^{\frac{1}{2}}(T'^*T')^{1-p}\{(T'^*T')^p - (T'T'^*)^p\}^{\frac{1}{2}}\Big)\\
=&\text{tr}\Big(\{(T'^*T')^p - (T'T'^*)^p\}^{\frac{1}{2}}(PT^*TP)^{1-p}\{(T'^*T')^p - (T'T'^*)^p\}^{\frac{1}{2}}\Big)\\
\geq&\text{tr}\Big(\{(T'^*T')^p - (T'T'^*)^p\}^{\frac{1}{2}}P(T^*T)^{1-p}P\{(T'^*T')^p - (T'T'^*)^p\}^{\frac{1}{2}}\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad(\text{ by Hansen's inequality })\\
=&\text{tr}\Big(\{(T'^*T')^p - (T'T'^*)^p\}^{\frac{1}{2}}(T^*T)^{1-p}\{(T'^*T')^p - (T'T'^*)^p\}^{\frac{1}{2}}\Big)\\
\geq&\|T^{-1}\|^{-2(1-p)}\text{tr}\Big((T'^*T')^p - (T'T'^*)^p\Big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\|T^{-1}\|^{2(1-p)}\frac{1}{\pi}\text{Area}(\sigma(T))\\
\geq&\text{tr}\Big((T'^*T')^p - (T'T'^*)^p\Big)\\
\geq&\|(T'^*T')^p - (T'T'^*)^p\|\\
\geq&\big\langle\{(T'^*T')^p - (T'T'^*)^p\}x, x\big\rangle\\
\geq&\langle\{P(T^*T)^pP - P(TT^*)^pP\}x, x\rangle \qquad \text{by Lemma 4}\\
=&\langle\{(T^*T)^p - (TT^*)^p\}x, x\rangle\,.
\end{aligned}
$$

Since $x \in \mathcal{H}$ is arbitrary unit vector,

$$\|(T^*T)^p - (TT^*)^p\| \leq \|T^{-1}\|^{2(1-p)}\frac{1}{\pi}\text{Area}(\sigma(T)).$$

Remark. Putnam inequality was extended to the p-hyponormal operator by Xia in the case of $\frac{1}{2} \leq p \leq 1$, and by Cho-Itoh in the case of $0 < p \leq \frac{1}{2}$. Their estimation is different from ours.

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