Berger-Shaw's theorem for $p$-hyponormal operators (Inequalities in operator theory and its related topics)

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For a $n$-multicyclic $p$-hyponormal operator $T$, we shall show that $|T|^{2p} - |T^*|^{2p}$ belongs to the Schatten $\frac{1}{p}$-class $\mathcal{C}_{\frac{1}{p}}$ and that $\text{tr} \left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{p}} \leq \frac{n}{\pi} \text{Area}(\sigma(T))$.

1. **Introduction** For a bounded linear operator $T$ on Hilbert space $\mathcal{H}$, $\mathcal{R}(\sigma(T))$ denotes the set of all rational functions analytic on $\sigma(T)$, where $\sigma(T)$ is the spectrum of $T$. The operator $T$ is said to be $n$-multicyclic if there are $n$ vectors $x_1, \ldots, x_n \in \mathcal{H}$, called generating vectors, such that $\bigvee \{g(T)x_i; i = 1, \ldots, n, g \in \mathcal{R}(\sigma(T))\} = \mathcal{H}$. For a $p$ such as $0 < p \leq 1$, $T$ is said to be $p$-hyponormal, if $(T^*T)^p \geq (TT^*)^p$. In particular, 1-hyponormal is called hyponormal and $\frac{1}{2}$-hyponormal is called semihyponormal. Xia([7]) gave an example which is not hyponormal but semihyponormal. Thus, the class of $p$-hyponormal operators properly contains 1-hyponormal operators. Putnam([6]) obtained the norm estimation for the self-commutator of a hyponormal operator, so called Putnam’s inequality. This inequality is extended for a $p$-hyponormal operator by Xia([8]) and Cho-Itoh([3]). Berger-Shaw([2]) showed the trace norm estimation for the self-commutator of $n$-multicyclic hyponormal operator, so called Berger-Shaw’s inequality. In this paper we shall extend this inequality to the case of a $n$-multicyclic $p$-hyponormal operator.

2. **Preliminary lemmas**

For $p$-hyponormal operator $T$ with its polar decomposition $T = U \ |T \ |$, the operator $\overline{T} = |T|^{\frac{1}{2}}U \ |T \ |^{\frac{1}{2}}$ is said to be the Aluthge transform.
It is known, by Aluthge([1]), that $\overline{T}$ is hyponormal if $\frac{1}{2} \leq p \leq 1$ and $(p + \frac{1}{2})$-hyponormal if $0 < p \leq \frac{1}{2}$ and if $U$ is unitary. In this paper, we deal with the operator $\overline{T} = |T|^\frac{1}{2m}U|T|^{1-\frac{1}{2m}+p}$ for $\frac{1}{2m+1} \leq p \leq \frac{1}{2m}$, where $m$ is non-negative integer.

**Lemma 1.** If $T$ is $p$-hyponormal operator for a $p$ such as $\frac{1}{2m+1} \leq p \leq \frac{1}{2}$, then

$$
(\overline{T}\overline{T}^*)^{\frac{1}{2m}} \leq |T|^\frac{1}{2m}-p |T^*|^2 |T|^{\frac{1}{2m}-p}
\leq |T|^\frac{1}{2m-1} \leq (\overline{T}^*\overline{T})^{\frac{1}{2m}},
$$

and hence $\overline{T}$ is $\frac{1}{2m}$-hyponormal.

**Proof.** Since $\frac{1}{2m+1} \leq p \leq \frac{1}{2}$, $\frac{1}{2m}-p \leq p$ and $T$ is $(\frac{1}{2m}-p)$-hyponormal by Heinz’s inequality and hence

$$
\overline{T}^*\overline{T} = |T|^\frac{1}{2m}+p U* |T|^{2(\frac{1}{2m}-p)} U |T|^{1-\frac{1}{2m}+p}
\geq |T|^\frac{1}{2m}+p U* |T^*|^{2(\frac{1}{2m}-p)} U |T|^{1-\frac{1}{2m}+p}
= |T|^\frac{1}{2m}+p |T|^{2(\frac{1}{2m}-p)} |T^*|^1|T|^{1-\frac{1}{2m}+p} = |T|^2.
$$

Thus, by Heinz’s inequality, we have the inequality,

$$
|\overline{T}|^s \geq |T|^s \quad \forall s \in (0, 2].
$$

Since, by the $(\frac{1}{2m} - p)$-hyponormality of $T$,

$$
\overline{T}^*\overline{T} = |T|^\frac{1}{2m}U |T|^{2-\frac{1}{2m-1}+2p} U* |T|^{\frac{1}{2m}-p}
= |T|^\frac{1}{2m}-p |T^*|^2 |T^*|^{\frac{1}{2m}-1+2p} |T|^{\frac{1}{2m}-p}
= |T|^\frac{1}{2m}-p |T^*|^{\frac{1}{2m}-1+2p} |T^*|^{2(\frac{1}{2m}-p)} |T^*|^1|T|^{1-\frac{1}{2m}+2p} |T|^{\frac{1}{2m}-p}
\leq |T|^\frac{1}{2m}-p |T^*|^{\frac{1}{2m}-1+2p} |T|^{2(\frac{1}{2m}-p)} |T^*|^{1-\frac{1}{2m}+2p} |T|^{\frac{1}{2m}-p}
= |T|^\frac{1}{2m}-p |T^*|^{\frac{1}{2m}-1+2p} |T|^{\frac{1}{2m}-p}|^2,
$$
We have, by Heinz’s inequality and by \((\frac{1}{2^{m}} - p)\)-hyponormality of \(T\),

\[
(\overline{T}T^*)^{\frac{1}{2}} \leq |T|^{\frac{1}{2^{m}}-p} T^* |^{\frac{1}{2^{m}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
= |T|^{\frac{1}{2^{m}}-p} |T^* |^{\frac{1}{2^{m}}-\frac{1}{2^{m-1}}+2p} |T^* |^{2(\frac{1}{2^{m}}-p)} |T^* |^{\frac{1}{2^{m}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
\leq |T|^{\frac{1}{2^{m}}-p} |T^* |^{\frac{1}{2^{m}}-\frac{1}{2^{m-1}}+2p} |T|^{2(\frac{1}{2^{m}}-p)} |T^* |^{\frac{1}{2^{m}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
= (|T|^{\frac{1}{2^{m}}-p} |T^* |^{\frac{1}{2^{m}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p})^2,
\]

and, by repeating the same arguments as above, we obtain

\[
(\overline{T}T^*)^{\frac{1}{2^{2}}}
\]

\[
\leq |T|^{\frac{1}{2^{m}}-p} T^* |^{\frac{1}{2^{2}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
= |T|^{\frac{1}{2^{m}}-p} |T^* |^{\frac{1}{2^{2}}-\frac{1}{2^{m-1}}+2p} |T^* |^{2(\frac{1}{2^{m}}-p)} |T^* |^{\frac{1}{2^{2}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
\leq |T|^{\frac{1}{2^{m}}-p} |T^* |^{\frac{1}{2^{2}}-\frac{1}{2^{m-1}}+2p} |T|^{2(\frac{1}{2^{m}}-p)} |T^* |^{\frac{1}{2^{2}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
= (|T|^{\frac{1}{2^{m}}-p} |T^* |^{\frac{1}{2^{2}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p})^2,
\]

and

\[
(\overline{T}T^*)^{\frac{1}{2^{3}}}
\]

\[
\leq |T|^{\frac{1}{2^{m}}-p} T^* |^{\frac{1}{2^{3}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
= |T|^{\frac{1}{2^{m}}-p} |T^* |^{\frac{1}{2^{3}}-\frac{1}{2^{m-1}}+2p} |T^* |^{2(\frac{1}{2^{m}}-p)} |T^* |^{\frac{1}{2^{3}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
\leq |T|^{\frac{1}{2^{m}}-p} |T^* |^{\frac{1}{2^{3}}-\frac{1}{2^{m-1}}+2p} |T|^{2(\frac{1}{2^{m}}-p)} |T^* |^{\frac{1}{2^{3}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
= (|T|^{\frac{1}{2^{m}}-p} |T^* |^{\frac{1}{2^{3}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p})^2.
\]

Eventually, we have

\[
(\overline{T}T^*)^{\frac{1}{2^{m-1}}}
\]

\[
\leq |T|^{\frac{1}{2^{m}}-p} T^* |^{\frac{1}{2^{m-2}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
= |T|^{\frac{1}{2^{m}}-p} |T^* |^{\frac{1}{2^{m-1}}-\frac{1}{2^{m-2}}+2p} |T^* |^{2(\frac{1}{2^{m}}-p)} |T^* |^{\frac{1}{2^{m-1}}-\frac{1}{2^{m-2}}+2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
= |T|^{\frac{1}{2^{m}}-p} |T^* |^{2p} |T^* |^{2(\frac{1}{2^{m}}-p)} |T^* |^{2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
\leq |T|^{\frac{1}{2^{m}}-p} T^* |^{2p} |T|^{\frac{1}{2^{m}}-p}
\]

\[
= (|T|^{\frac{1}{2^{m}}-p} T^* |^{2p} |T|^{\frac{1}{2^{m}}-p})^2,
\]
and hence
\[ \left( \overline{T} \overline{T}^* \right)^{\frac{1}{2^m}} \leq |T|^{\frac{1}{2^m} - p} \left| T^* \right|^{2p} |T|^{\frac{1}{2^m} - p} \]
\[ \leq |T|^{\frac{1}{2^m} - p} |T|^{2p} |T|^{\frac{1}{2^m} - p} \]
\[ = |T|^{\frac{1}{2^m - 1}} \leq (\overline{T}^* \overline{T})^{\frac{1}{2^m}}. \]

Therefore $\overline{T}$ is $\frac{1}{2^m}$-hyponormal.

**Lemma 2.** If $T$ is $n$-multicyclic $p$-hyponormal, then $\overline{T}$ is also $n$-multicyclic and $\sigma(\overline{T}) = \sigma(T)$.

**Proof.** For a $p$-hyponormal operator $T$, $\text{Ker}T$ is a reducing subspace of $T$ and also $\overline{T}$. Hence, we may assume that $\text{Ker}T = \{0\}$. Put $s = \frac{1}{2^m} - p$, where $\frac{1}{2^{m+1}} \leq p \leq \frac{1}{2^m}$.

$$\sigma(\overline{T}) = \sigma(|T|^s U |T|^{1-s}) \subset \sigma(U |T|^{1-s} |T|^s) \cup \{0\} = \sigma(T) \cup \{0\}.$$

Similarly

$$\sigma(T) = \sigma(U |T|^{-s} |T|^s) \subset \sigma(|T|^s U |T|^{1-s}) \cup \{0\} = \sigma(\overline{T}) \cup \{0\}.$$

Since $T$ is invertible if and only if $\overline{T}$ is invertible, we have $\sigma(\overline{T}) = \sigma(T)$ and $\mathcal{R}(\sigma(T)) = \mathcal{R}(\sigma(\overline{T}))$. Since $T$ is $n$-multicyclic,

$$\exists x_1, \ldots, x_n \in \mathcal{H} \text{ s.t.} \vee \{g(T)x_i; i = 1, \ldots, n, g \in \mathcal{R}(\sigma(T))\} = \mathcal{H}.$$ 

Put $y_i = |T|^s x_i, i = 1, \ldots, n$. We shall show that $\{y_i\}_{i=1}^n$ are $n$-multicyclic vectors for $T$.

$$\overline{T}^k |T|^s = \{|T|^s U |T|^{1-s} \}^k |T|^s = |T|^s \{U |T| \}^k = |T|^s T^k.$$

If $\lambda \in \rho(T)$, then $\lambda - U(|T| + \epsilon)$ is invertible for sufficiently small $\epsilon > 0$. Therefore,

$$|T| + \epsilon)^s (\lambda - U(|T| + \epsilon))^{-1}$$

$$= \{(\lambda - U(|T| + \epsilon))(|T| + \epsilon)^{-s}\}^{-1}$$

$$= \{(|T| + \epsilon)^{-s} (\lambda - (|T| + \epsilon)^s U(|T| + \epsilon)^{1-s})\}^{-1}$$

$$= (\lambda - (|T| + \epsilon)^s U(|T| + \epsilon)^{1-s})^{-1} (|T| + \epsilon)^s$$

Letting $\epsilon \downarrow 0$, we have

$$(\lambda - \overline{T})^{-1} |T|^s = |T|^s (\lambda - T)^{-1}.$$
Hence, we have
\[ g(\overline{T}) | T |^s = | T |^s g(T), \ \forall \ g \in \mathcal{R} (\sigma(T)), \]
and
\[ g(\overline{T}) y_i = | T |^s g(T) x_i, \ \forall \ g \in \mathcal{R} (\sigma(T)) \ i = 1, \ldots, n. \]
Thus,
\[ \vee \{ g(\overline{T}) y_i; \ i = 1, \ldots, n, \ g \in \mathcal{R} (\sigma(\overline{T})) \} = \vee \{ g(\overline{T}) y_i; \ i = 1, \ldots, n, \ g \in \mathcal{R} (\sigma(T)) \} = \{|T|^s g(T) x_i; i = 1, \ldots, n, \ g \in \mathcal{R} (\sigma(T)) \} = \{|T|^s \mathcal{H}\} = -\mathcal{H} \]
because \( \ker T = \{0\} \).

This implies that \( \overline{T} \) is \( n \)-multicyclic.

3. Main theorem

Berger-Shaw’s Theorem. If \( T \) is a \( n \)-multicyclic hyponormal operator, then \( [T^*, T] = T^*T - TT^* \) is in the trace class, and \( \text{tr} \left( [T^*, T] \right) \leq \frac{n}{\pi} \text{Area}(\sigma(T)) \), where \( \text{Area} \) means the planar Lebesgue measure.

The following result is our main theorem.

Theorem. If \( T \) is a \( n \)-multicyclic \( p \)-hyponormal operator for \( p \) such as \( 0 < p \leq 1 \), then for \( p \) such as \( 0 < p \leq 1 \), then \( |T|^{2p} - |T^*|^{2p} \) belongs to the Schatten \( \frac{1}{p} \)-class \( C_{\frac{1}{p}} \) and
\[ \text{tr} \left( (|T|^{2p} - |T^*|^{2p})^{\frac{1}{p}} \right) \leq \frac{n}{\pi} \text{Area}(\sigma(T)). \]
When \( p = 1 \), this theorem is exactly Berger-Shaw’s theorem.

The following is the key for our purpose.

Lemma 3. If \( T \) is \( n \)-multicyclic \( p \)-hyponormal, then
\[ \text{tr} \left( |T|^{1-p} (|T|^{2p} - |T^*|^{2p}) |T|^{1-p} \right) \leq \frac{n}{\pi} \text{Area}(\sigma(T)). \]

Proof. We shall show this lemma for \( p \) such as \( \frac{1}{2m+1} \leq p \leq \frac{1}{2m}, \ m = 0, 1, 2, \ldots \), by the induction in \( m \).
If $m = 0$, then $\overline{T}$ is a $n$-multicyclic hyponormal operator by Lemmas 1 and 2. Thus Berger-Shaw's theorem implies that

$$\text{tr}\left((\overline{T}^{*}\overline{T} - \overline{T}\overline{T}^{*})\right) \leq \frac{n}{\pi}\text{Area}(\sigma(\overline{T})) = \frac{n}{\pi}\text{Area}(\sigma(T)), $$

because $\sigma(\overline{T}) = \sigma(T)$ by Lemma 2. Since, by Lemma 1,

$$\overline{T}^{*}\overline{T} - \overline{T}\overline{T}^{*} \geq |T|^{2} - |T|^{1-p}T^{*}2p|T|^{1-p},$$

we have

$$\text{tr}\left(|T|^{1-p}(|T|^{2p} - |T^{*}|^{2p})|T|^{1-p}\right) \leq \text{tr}(\overline{T}^{*}\overline{T} - \overline{T}\overline{T}^{*}) \leq \frac{n}{\pi}\text{Area}(\sigma(T)).$$

Hence, the assertion holds for $m = 0$.

Next, we assume that the assertion holds for $m = k$ ($k \geq 0$). If $m = k + 1$, then $\overline{T}$ is $\frac{1}{2^{k+1}}$-hyponormal by Lemma 1. Hence by the assumption and by Lemmas 1 and 2, we have

$$\frac{n}{\pi}\text{Area}(\sigma(T)) = \frac{n}{\pi}\text{Area}(\sigma(\overline{T})), $$

$$\geq \text{tr}\left(|\overline{T}|^{1-\frac{1}{2^{k+1}}}(|T|^{2^{1-\frac{1}{2^{k+1}}}} - |\overline{T}^{*}|^{2^{1-\frac{1}{2^{k+1}}}})|\overline{T}|^{1-\frac{1}{2^{k+1}}})\right),$$

$$\geq \text{tr}\left(|\overline{T}|^{1-\frac{1}{2^{k+1}}}(|T|^{2^{1-\frac{1}{2^{k+1}}}} - |T|^{2^{1-\frac{1}{2^{k+1}}}-p}|T^{*}|^{2p}|T|^{2^{1-\frac{1}{2^{k+1}}}-p})|\overline{T}|^{1-\frac{1}{2^{k+1}}})\right),$$

$$= \text{tr}\left(|T|^{2^{1-\frac{1}{2^{k+1}}}} - |T|^{2^{1-\frac{1}{2^{k+1}}}-p}|T^{*}|^{2p}|T|^{2^{1-\frac{1}{2^{k+1}}}-p})|\overline{T}|^{2(1-\frac{1}{2^{k+1}})}\right) \times \text{tr}\left(|T|^{2^{1-\frac{1}{2^{k+1}}}} - |T|^{2^{1-\frac{1}{2^{k+1}}}-p}|T^{*}|^{2p}|T|^{2^{1-\frac{1}{2^{k+1}}}-p})\right),$$

$$\leq \text{tr}\left(|T|^{2^{1-\frac{1}{2^{k+1}}}} - |T|^{2^{1-\frac{1}{2^{k+1}}}-p}|T^{*}|^{2p}|T|^{2^{1-\frac{1}{2^{k+1}}}-p})|\overline{T}|^{2(1-\frac{1}{2^{k+1}})}\right) \times \text{tr}\left(|T|^{2^{1-\frac{1}{2^{k+1}}}} - |T|^{2^{1-\frac{1}{2^{k+1}}}-p}|T^{*}|^{2p}|T|^{2^{1-\frac{1}{2^{k+1}}}-p})\right),$$

$$= \text{tr}\left(|T|^{1-\frac{1}{2^{k+1}}}(|T|^{2^{1-\frac{1}{2^{k+1}}}} - |T|^{2^{1-\frac{1}{2^{k+1}}}-p}|T^{*}|^{2p}|T|^{2^{1-\frac{1}{2^{k+1}}}-p})|T|^{1-\frac{1}{2^{k+1}}})\right)$$

$$= \text{tr}\left(|T|^{1-p}(|T|^{2p} - |T^{*}|^{2p})|T|^{1-p}\right).$$

Hence, the assertion holds for $m = k + 1$. This completes the proof of Lemma 3.
Corollary 1. If $T$ is an invertible $n$-multicyclic $p$-hyponormal operator, then $(T^*T)^p - (TT^*)^p \in C_1$ and

$$\text{tr} \left( (T^*T)^p - (TT^*)^p \right) \leq \|T^{-1}\|^{2(1-p)} \frac{n}{\pi} \text{Area}(\sigma(T)).$$

**Proof.** Since $T$ is invertible, $T^*T \geq \|T^{-1}\|^{-2}$, and $n$-multicyclic $p$-hyponormality of $T$ implies that

$$\frac{n}{\pi} \text{Area}(\sigma(T)) \geq \text{tr} \left( \sum_{k=0}^{1-p} \left\{ (T^*T)^p - (TT^*)^p \right\} T |^{1-p} \right) \quad \text{by Lemma 3.}$$

$$= \text{tr} \left\{ \left( (T^*T)^p - (TT^*)^p \right)^{1/2} (T^*T)^{1-p} \left\{ (T^*T)^p - (TT^*)^p \right\}^{1/2} \right\}$$

$$\geq \|T^{-1}\|^{-2(1-p)} \text{tr} \left( (T^*T)^p - (TT^*)^p \right).$$

We have $(T^*T)^p - (TT^*)^p \in C_1$ and

$$\text{tr} \left( (T^*T)^p - (TT^*)^p \right) \leq \|T^{-1}\|^{2(1-p)} \frac{n}{\pi} \text{Area}(\sigma(T)).$$ This completes the proof of Corollary 1.

**Proof of Theorem.** By Lemma 3,

$$\text{tr} \left( \sum_{k=0}^{1-p} \left\{ (|T|^2 - |T^*|^2) T |^{1-p} \right\} \right) \leq \frac{n}{\pi} \text{Area}(\sigma(T)).$$

And by the property of trace,

$$\text{tr} \left( \sum_{k=0}^{1-p} \left\{ (|T|^2 - |T^*|^2) T |^{1-p} \right\} \right)$$

$$= \text{tr} \left( \sum_{k=0}^{1-p} \left\{ (|T|^2 - |T^*|^2) T |^{2-2p} \right\} \right)$$

$$= \text{tr} \left( \sum_{k=0}^{1-p} \left\{ (|T|^2 - |T^*|^2) T |^{2p-1} \right\} \right).$$

If $\frac{1}{2} \leq p \leq 1$, then $0 \leq \frac{1-p}{p} \leq 1$. Thus, by Heinz's inequality, we obtain

$$\text{tr} \left( \sum_{k=0}^{1-p} \left\{ (|T|^2 - |T^*|^2) T |^{2p-1} \right\} \right) \geq \text{tr} \left( \sum_{k=0}^{1-p} \left\{ (|T|^2 - |T^*|^2) T |^{2p-1} \right\} \right)$$

$$= \text{tr} \left( \sum_{k=0}^{1-p} \left\{ (|T|^2 - |T^*|^2) T |^{2p-1} \right\} \right)$$

$$= \text{tr} \left( \sum_{k=0}^{1-p} \left\{ (|T|^2 - |T^*|^2) T |^{2p-1} \right\} \right).$$


Therefore, we have
\[
\text{tr} \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{\frac{1}{p}} \leq \frac{n}{\pi} \text{Area}(\sigma(T)).
\]

Thus, the assertion of Theorem holds for \( p \in [\frac{1}{2}, 1] \).

If \( 0 < p \leq \frac{1}{2} \), then by Furuta's inequality([4]),
\[
\begin{align*}
(\left| T \right|^{2p} - \left| T^* \right|^{2p})^{\frac{1}{2}} \left\{ \left| T \right|^{2p} \right\}^{\frac{1-p}{p}} (\left| T \right|^{2p} - \left| T^* \right|^{2p})^{\frac{1}{2}} &= \left( \left| \tau \right|^{2p} - \left| \tau^* \right|^{2p} \right)^{2} \\
(\left| T \right|^{2p} - \left| T^* \right|^{2p})^{\frac{1}{2}} \left\{ \left| T \right|^{2p} \right\}^{\frac{1-p}{p}} (\left| T \right|^{2p} - \left| T^* \right|^{2p})^{\frac{1}{2}} &\geq \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{2}.
\end{align*}
\]

Thus \( (\left| T \right|^{2p} - \left| T^* \right|^{2p})^{\frac{1}{2}} \left\{ \left| T \right|^{2p} \right\}^{\frac{1-p}{p}} (\left| T \right|^{2p} - \left| T^* \right|^{2p})^{\frac{1}{2}} \) are both compact positive operators. Let \( s_n[A] \) be the \( n \)-th singular number of a positive compact operator \( A \). Then,
\[
\begin{align*}
\begin{aligned}
s_n \left[ \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{\frac{1}{2}} \left\{ \left| T \right|^{2p} \right\}^{\frac{1-p}{p}} (\left| T \right|^{2p} - \left| T^* \right|^{2p})^{\frac{1}{2}} \right]^2 &= s_n \left[ \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{\frac{1}{2}} \left\{ \left| T \right|^{2p} \right\}^{\frac{1-p}{p}} (\left| T \right|^{2p} - \left| T^* \right|^{2p})^{\frac{1}{2}} \right] \\
&\geq s_n \left[ \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{2} \right].
\end{aligned}
\end{align*}
\]

and hence,
\[
\begin{align*}
\begin{aligned}
s_n \left[ \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{\frac{1}{2}} \left\{ \left| T \right|^{2p} \right\}^{\frac{1-p}{p}} (\left| T \right|^{2p} - \left| T^* \right|^{2p})^{\frac{1}{2}} \right] &\geq s_n \left[ \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{\frac{1}{2}} \left\{ \left| T \right|^{2p} \right\}^{\frac{1-p}{p}} (\left| T \right|^{2p} - \left| T^* \right|^{2p})^{\frac{1}{2}} \right] \\
&= s_n \left[ \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{\frac{1}{2}} \right] \\
&= s_n \left[ \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{\frac{1}{2}} \right].
\end{aligned}
\end{align*}
\]

Hence,
\[
\text{tr} \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{\frac{1}{2}} \left\{ \left| T \right|^{2p} \right\}^{\frac{1-p}{p}} (\left| T \right|^{2p} - \left| T^* \right|^{2p})^{\frac{1}{2}} \right) \geq \text{tr} \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{\frac{1}{p}}.
\]

Therefore, we have
\[
\text{tr} \left( \left| T \right|^{2p} - \left| T^* \right|^{2p} \right)^{\frac{1}{p}} \leq \frac{n}{\pi} \text{Area}(\sigma(T)).
\]
The assertion of Theorem also holds for $p \in (0, \frac{1}{2}]$. This completes the proof of Theorem.

For the restriction of a $p$-hyponormal operator to invariant subspace, we have the following.

**Lemma 4.** Let $\mathcal{M}$ be an invariant subspace for a $p$-hyponormal operator $T$, and $T'$ be the restriction of $T$ to $\mathcal{M}$. Then

$$\{T'T^*\}^p \leq P(TT^*)^p P \leq P(T^*T)^p P \leq \{T^*T\}^p,$$

and $T'$ is also $p$-hyponormal, where $P$ denotes the projection onto $\mathcal{M}$.

**Proof.** Since $T' = TP$,

$$T'T^* = PT^*TP,$$

and hence, for any $s \in (0, 1]$,

$$\{T'^*T'\}^s = \{PT^*TP\}^s \geq P(T^*T)^s P$$

by Hansen's inequality ([5]).

While,

$$T'T'^* = TPT^* = PTPT^* P,$$

we have, for any $s \in (0, 1]$,

$$\{T'^*T'\}^s = (TPT^*)^s$$

$$= P(TPT^*)^s P$$

$$\leq P(TT^*)^s P$$

by Heinz's inequality.

Therefore, if $T$ is $p$-hyponormal for $p$ such as $0 < p \leq 1$, then

$$\{T'^*T'\}^p \leq P(TT^*)^p P$$

$$\leq P(T^*T)^p P$$

$$\leq \{T^*T\}^p.$$

Thus, $T'$ is also $p$-hyponormal.

**Corollary 2.** If $T$ is $p$-hyponormal operator, then

$$\| (T^*T)^p - (TT^*)^p \| \leq \left\{ \frac{1}{\pi} \text{Area}(\sigma(T)) \right\}^p.$$
**Proof.** Let $x$ be an arbitrary unit vector in $\mathcal{H}$. We define

$$ \mathcal{H}_0 = \vee \{g(T)x; g \in \mathcal{R}(\sigma(T))\}. $$

Since $\mathcal{H}_0$ is an invariant subspace for $T$, Lemma 4 implies that $T' = T|_{\mathcal{H}_0}$ is a (1-multicyclic) $p$-hyponormal operator. If $\lambda \in \rho(T)$, then, for any $y \in \mathcal{H}_0$, $(T - \lambda)^{-1}y \in \mathcal{H}_0$. Therefore, $\lambda \in \rho(T')$. Hence, $\sigma(T') \subseteq \sigma(T)$. By Theorem,

$$ \text{tr}\left(\{(T''^*T')^p - (T'T'^*)^p\}^{\frac{1}{p}}\right) \leq \frac{1}{\pi} \text{Area}(\sigma(T')) \leq \frac{1}{\pi} \text{Area}(\sigma(T)) $$

and the maximal eigenvalue of positive trace class operator $\{(T''^*T')^p - (T'T'^*)^p\}^{\frac{1}{p}}$ is equal to or less than $\frac{1}{\pi} \text{Area}(\sigma(T))$. Thus, the maximal eigenvalue of $(T''^*T')^p - (T'T'^*)^p$ is equal to or less than $\{\frac{1}{\pi} \text{Area}(\sigma(T))\}^p$. Therefore,

$$ \|(T''^*T')^p - (T'T'^*)^p\| \leq \{\frac{1}{\pi} \text{Area}(\sigma(T))\}^p. $$

Let $P$ be the projection onto $\mathcal{H}_0$. Then, by Lemma 4,

$$ \begin{align*}
\{\frac{1}{\pi} \text{Area}(\sigma(T))\}^p \\
\geq \langle\{(T''^*T')^p - (T'T'^*)^p\}x, x\rangle \\
\geq \langle\{P(T''^*T')^pP - P(T'T'^*)^pP\}x, x\rangle \\
= \langle\{(T''T')^p - (TT'^*)^p\}x, x\rangle.
\end{align*} $$

Since $x \in \mathcal{H}$ is arbitrary unit vector,

$$ \|(T''T')^p - (TT'^*)^p\| \leq \{\frac{1}{\pi} \text{Area}(\sigma(T))\}^p. $$

This inequality is an extension of the Putnam inequality to the case of $p$-hyponormal operator.

**Corollary 3.** If $T$ is an invertible $p$-hyponormal operator, then

$$ \|(T''T')^p - (TT'^*)^p\| \leq \|T^{-1}\|^{2(1-p)}\frac{1}{\pi} \text{Area}(\sigma(T)). $$
Proof. Put $\mathcal{H}_0, T' = T|_{\mathcal{H}_0}$ and $P$ as Corollary 2, then $T'$ is an invertible $(1\text{-multicyclic}) p$-hyponormal operator. Therefore by Lemma 3,

$$\frac{1}{\pi} \text{Area}(\sigma(T)) \geq \frac{1}{\pi} \text{Area}(\sigma(T'))$$

$$\geq \text{tr} \left( |T'|^{1-p} \left\{ (T'^*T')^p - (T'T'^*)^p \right\} \right)$$

$$= \text{tr} \left( \left\{ (T'^*T')^p - (T'T'^*)^p \right\}^{\frac{1}{2}} (T'^*T')^{1-p} \left\{ (T'^*T')^p - (T'T'^*)^p \right\}^{\frac{1}{2}} \right)$$

$$= \text{tr} \left( \left\{ (T'^*T')^p - (T'T'^*)^p \right\}^{\frac{1}{2}} (PT^*TP)^{1-p} \left\{ (T'^*T')^p - (T'T'^*)^p \right\}^{\frac{1}{2}} \right)$$

$$\geq \text{tr} \left( \left\{ (T'^*T')^p - (T'T'^*)^p \right\}^{\frac{1}{2}} P(PT^*TP)^{1-p} \left\{ (T'^*T')^p - (T'T'^*)^p \right\}^{\frac{1}{2}} \right)$$

(by Hansen’s inequality)

$$= \text{tr} \left( \left\{ (T'^*T')^p - (T'T'^*)^p \right\}^{\frac{1}{2}} (T^*T)^{1-p} \left\{ (T'^*T')^p - (T'T'^*)^p \right\}^{\frac{1}{2}} \right)$$

$$\geq \|T^{-1}\|^{2(1-p)} \text{tr} \left( (T'^*T')^p - (T'T'^*)^p \right).$$

Therefore

$$\|T^{-1}\|^{2(1-p)} \frac{1}{\pi} \text{Area}(\sigma(T)) \geq \text{tr} \left( (T'^*T')^p - (T'T'^*)^p \right)$$

$$\geq \| (T'^*T')^p - (T'T'^*)^p \|$$

$$\geq \langle \{ (T'^*T')^p - (T'T'^*)^p \}, x, x \rangle$$

$$\geq \langle \{ P(T^*T)^p P - P(TT^*)^p P \} x, x \rangle \quad \text{by Lemma 4}$$

$$= \langle \{ (T^*T)^p - (TT^*)^p \}, x, x \rangle.$$

Since $x \in \mathcal{H}$ is arbitrary unit vector,

$$\|(T'^*T')^p - (T'T'^*)^p \| \leq \|T^{-1}\|^{2(1-p)} \frac{1}{\pi} \text{Area}(\sigma(T)) .$$

Remark. Putnam inequality was extended to the $p$-hyponormal operator by Xia in the case of $\frac{1}{2} \leq p \leq 1$, and by Cho-Itoh in the case of $0 < p \leq \frac{1}{2}$. Their estimation is different from ours.
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