Berger-Shaw's theorem for $p$-hyponormal operators

Inequalities in operator theory and its related topics

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For a $n$-multicyclic $p$-hyponormal operator $T$, we shall show that $|T|^{2p} - |T^*|^{2p}$ belongs to the Schatten $\frac{1}{p}$-class $C_{\frac{1}{p}}$ and that $\text{tr}\left(\left(|T|^{2p} - |T^*|^{2p}\right)^{\frac{1}{p}}\right) \leq \frac{n}{\pi} \text{Area}(\sigma(T))$.

1. **Introduction** For a bounded linear operator $T$ on Hilbert space $\mathcal{H}$, $\mathcal{R}(\sigma(T))$ denotes the set of all rational functions analytic on $\sigma(T)$, where $\sigma(T)$ is the spectrum of $T$. The operator $T$ is said to be $n$-multicyclic if there are $n$ vectors $x_1, \ldots, x_n \in \mathcal{H}$, called generating vectors, such that $\vee \{g(T)x_i; i = 1, \ldots, n, g \in \mathcal{R}(\sigma(T))\} = \mathcal{H}$. For a $p$ such as $0 < p \leq 1$, $T$ is said to be $p$-hyponormal, if $(T^*T)^p \geq (TT^*)^p$. In particular, 1-hyponormal is called hyponormal and $\frac{1}{2}$-hyponormal is called semihyponormal. Xia([7]) gave an example which is not hyponormal but semihyponormal. Thus, the class of $p$-hyponormal operators properly contains 1-hyponormal operators. Putnam([6]) obtained the norm estimation for the self-commutator of a hyponormal operator, so called Putnam's inequality. This inequality is extended for a $p$-hyponormal operator by Xia([8]) and Cho-Itoh([3]). Berger-Shaw([2]) showed the trace norm estimation for the self-commutator of $n$-multicyclic hyponormal operator, so called Berger-Shaw's inequality. In this paper we shall extend this inequality to the case of a $n$-multicyclic $p$-hyponormal operator.

2. **Preliminary lemmas**

For $p$-hyponormal operator $T$ with its polar decomposition $T = U |T|$, the operator $\bar{T} = |T|^{\frac{1}{2}}U |T|^{\frac{1}{2}}$ is said to be the Aluthge transform.
It is known, by Aluthge([1]), that \( \widetilde{T} \) is hyponormal if \( \frac{1}{2} \leq p \leq 1 \) and \((p + \frac{1}{2})\)-hyponormal if \( 0 < p \leq \frac{1}{2} \) and if \( U \) is unitary. In this paper, we deal with the operator \( \widetilde{T} = |T|^\frac{1}{2^m}U \) for \( \frac{1}{2^m+1} \leq p \leq \frac{1}{2^m} \), where \( m \) is non-negative integer.

**Lemma 1.** If \( T \) is \( p \)-hyponormal operator for a \( p \) such as \( \frac{1}{2^m+1} \leq p \leq \frac{1}{2^m} \), then

\[
(\widetilde{T} \widetilde{T}^*)^\frac{1}{2^m} \leq |T|^{\frac{1}{2^m} - p} |T^*|^2 |T|^{\frac{1}{2^m} - p} \\
\leq |T|^{\frac{1}{2^m-1}} \leq (\widetilde{T}^* \widetilde{T})^\frac{1}{2^m},
\]

and hence \( \widetilde{T} \) is \( \frac{1}{2^m} \)-hyponormal.

**Proof.** Since \( \frac{1}{2^m+1} \leq p \leq \frac{1}{2^m} \), \( \frac{1}{2^m} - p \leq p \) and \( T \) is \( (\frac{1}{2^m} - p) \)-hyponormal by Heinz's inequality and hence

\[
\widetilde{T}^* \widetilde{T} = |T|^{-1} \frac{1}{2^m} + p U^* |T| \frac{1}{2^m} - p \leq |T| \frac{1}{2^m} - p |T^*| \frac{1}{2^m} + p \\
= |T| \frac{1}{2^m} - p |T^*| \frac{1}{2^m} - p |T| \frac{1}{2^m} - p = |T|^2.
\]

Thus, by Heinz's inequality, we have the inequality,

\[
|\widetilde{T}|^s \geq |T|^s \quad \forall s \in (0, 2].
\]

Since, by the \( (\frac{1}{2^m} - p) \)-hyponormality of \( T \),

\[
\widetilde{T} \widetilde{T}^* = |T| \frac{1}{2^m} - p U |T| \frac{1}{2^m} - p \leq |T| \frac{1}{2^m} - p |T^*| \frac{1}{2^m} - p |T| \frac{1}{2^m} - p \\
= (|T| \frac{1}{2^m} - p |T^*| \frac{1}{2^m} - p |T| \frac{1}{2^m} - p)^2,
\]
We have, by Heinz's inequality and by \((\frac{1}{2^m} - p)\)-hyponormality of \(T\),
\[
(\overline{T}T^*)^{\frac{1}{2}} \leq |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2^m} - p}
\]
\[
= |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{(\frac{1}{2^m} - p)} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2^m} - p}
\]
\[
\leq |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T|^{\frac{1}{2^m} - p}
\]
\[
= (|T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T|^{\frac{1}{2^m} - p})^2,
\]
and, by repeating the same arguments as above, we obtain
\[
(\overline{T}T^*)^{\frac{1}{2^2}}
\]
\[
\leq |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2^m} - p}
\]
\[
= |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{(\frac{1}{2^m} - p)} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2^m} - p}
\]
\[
\leq |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2^m} - p}
\]
\[
=(|T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2^m} - p})^2,
\]
and
\[
(\overline{T}T^*)^{\frac{1}{2^3}}
\]
\[
\leq |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2^m} - p}
\]
\[
= |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{(\frac{1}{2^m} - p)} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2^m} - p}
\]
\[
\leq |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2^m} - p}
\]
\[
=(|T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2} - \frac{1}{2^m} + 2p} |T^*|^{\frac{1}{2^m} - p})^2.
\]
Eventually, we have
\[
(\overline{T}T^*)^{\frac{1}{2^m-1}}
\]
\[
\leq |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2^m-2} - \frac{1}{2^m-1} + 2p} |T^*|^{\frac{1}{2^m-1} - p}
\]
\[
= |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2^m-1} - \frac{1}{2^m-1} + 2p} |T^*|^{(\frac{1}{2^m} - p)} |T^*|^{\frac{1}{2^m-1} - \frac{1}{2^m-1} + 2p} |T^*|^{\frac{1}{2^m-1} - p}
\]
\[
= |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2^m} - 2p} |T^*|^{\frac{1}{2^m-2} - \frac{1}{2^m-1} + 2p} |T^*|^{\frac{1}{2^m-1} - 2p} |T^*|^{\frac{1}{2^m-1} - p}
\]
\[
\leq |T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2^m} - 2p} |T^*|^{\frac{1}{2} - \frac{1}{2^m-1} + 2p} |T^*|^{\frac{1}{2^m} - 2p} |T^*|^{\frac{1}{2^m} - p}
\]
\[
= (|T|^{\frac{1}{2^m}} |T^*|^{\frac{1}{2^m} - 2p} |T^*|^{\frac{1}{2^m} - 2p} |T^*|^{\frac{1}{2^m} - p})^2,
\]
and hence

\begin{align*}
(\overline{T}\overline{T}^{*})^{\frac{1}{2^{m}}} & \leq |T|^\frac{1}{2^{m}-p} |T^{*}|^{2p} |T|^{\frac{1}{2^{m}-p}} \\
& \leq |T|^\frac{1}{2^{m}-p} |T^{*}|^{2p} |T|^{\frac{1}{2^{m}-p}} \\
& = |T|^\frac{1}{2^{m-1}} \leq (\overline{T}^{*}\overline{T})^{\frac{1}{2^{m}}}.
\end{align*}

Therefore \( \overline{T} \) is \( \frac{1}{2^{m}} \)-hyponormal.

**Lemma 2.** If \( T \) is \( n \)-multicyclic \( p \)-hyponormal, then \( \overline{T} \) is also \( n \)-multicyclic and \( \sigma(\overline{T}) = \sigma(T) \).

**Proof.** For a \( p \)-hyponormal operator \( T \), Ker\( T \) is a reducing subspace of \( T \) and also \( \overline{T} \). Hence, we may assume that Ker\( T \) = \( \{0\} \). Put \( s = \frac{1}{2^{m}} - p \), where \( \frac{1}{2^{m+1}} \leq p \leq \frac{1}{2^{m}} \).

\[ \sigma(\overline{T}) = \sigma(|T|^{s}U|T|^{1-s}) \subset \sigma(U|T|^{1-s}|T|^{s}) \cup \{0\} = \sigma(T) \cup \{0\}. \]

Similarly

\[ \sigma(T) = \sigma(U|T|^{1-s}|T|^{s}) \subset \sigma(|T|^{s}U|T|^{1-s}) \cup \{0\} = \sigma(\overline{T}) \cup \{0\}. \]

Since \( T \) is invertible if and only if \( \overline{T} \) is invertible, we have \( \sigma(\overline{T}) = \sigma(T) \) and \( \mathcal{R}(\sigma(T)) = \mathcal{R}(\sigma(\overline{T})) \). Since \( T \) is \( n \)-multicyclic,

\[ \exists x_{1}, \ldots, x_{n} \in \mathcal{H} \text{ s.t. } \vee \{g(T)x_{i}; i = 1, \ldots, n, g \in \mathcal{R}(\sigma(T))\} = \mathcal{H}. \]

Put \( y_{i} = |T|^{s}x_{i}, i = 1, \ldots, n \). We shall show that \( \{y_{i}\}_{i=1}^{n} \) are \( n \)-multicyclic vectors for \( T \).

\[ \overline{T}^{k}|T|^{s} = \{|T|^{s}U|T|^{1-s}\}^{k}|T|^{s} = |T|^{s}\{U|T|\}^{k} = |T|^{s}T^{k}. \]

If \( \lambda \in \rho(T) \), then \( \lambda - U(|T| + \epsilon) \) is invertible for sufficiently small \( \epsilon > 0 \). Therefore,

\[ (|T| + \epsilon)^{s}(\lambda - U(|T| + \epsilon))^{-1} = ((\lambda - U(|T| + \epsilon))(|T| + \epsilon)^{-s})^{-1} = ((|T| + \epsilon)^{-s}(\lambda - (|T| + \epsilon)^{s}U(|T| + \epsilon)^{1-s}))^{-1} = (\lambda - (|T| + \epsilon)^{s}U(|T| + \epsilon)^{1-s})^{-1}(|T| + \epsilon)^{s}. \]

Letting \( \epsilon \downarrow 0 \), we have

\[ (\lambda - \overline{T})^{-1}|T|^{s} = |T|^{s}(\lambda - T)^{-1}. \]
Hence, we have
\[ g(\overline{T})|T|^s = |T|^sg(T), \quad \forall g \in \mathcal{R}(\sigma(T)), \]
and
\[ g(\overline{T})y_i = |T|^sg(T)x_i, \quad \forall g \in \mathcal{R}(\sigma(T)) \quad i = 1, \ldots, n. \]
Thus,
\[ \vee \{ g(\overline{T})y_i; i = 1, \ldots, n, g \in \mathcal{R}(\sigma(\overline{T})) \} = \vee \{ g(\overline{T})y_i; i = 1, \ldots, n, g \in \mathcal{R}(\sigma(T)) \} = \{|T|^sg(T)Xi;i=1, \ldots, n, g \in \mathcal{R}(\sigma(T))\} = [\{|T|^s\mathcal{H}\} = -\mathcal{H} \text{ because } \text{Ker}T = \{0\}. \]
This implies that \( \overline{T} \) is \( n \)-multicyclic.

3. **Main theorem**

**Berger-Shaw’s Theorem.** If \( T \) is a \( n \)-multicyclic hyponormal operator, then \([T^*, T] = T^*T - TT^*\) is in the trace class, and \( \text{tr}\left([T^*, T]\right) \leq \frac{n}{\pi}\text{Area}(\sigma(T)) \), where \text{Area} means the planar Lebesgue measure.

The following result is our main theorem.

**Theorem.** If \( T \) is a \( n \)-multicyclic \( p \)-hyponormal operator for \( p \) such as \( 0 < p \leq 1 \), then for \( p \) such as \( 0 < p \leq 1 \), then \( |T|^{2p} - |T^*|^{2p} \) belongs to the Schatten \( \frac{1}{p} \)-class \( C_{\frac{1}{p}} \) and
\[ \text{tr}\left(|T|^{2p} - |T^*|^{2p}\right)^{\frac{1}{p}} \leq \frac{n}{\pi}\text{Area}(\sigma(T)). \]
When \( p = 1 \), this theorem is exactly Berger-Shaw’s theorem.

The following is the key for our purpose.

**Lemma 3.** If \( T \) is \( n \)-multicyclic \( p \)-hyponormal, then
\[ \text{tr}\left(|T|^{1-p}(|T|^{2p} - |T^*|^{2p})|T|^{1-p}\right) \leq \frac{n}{\pi}\text{Area}(\sigma(T)). \]

**Proof.** We shall show this lemma for \( p \) such as \( \frac{1}{2m+1} \leq p \leq \frac{1}{2m} \), \( m = 0, 1, 2, \ldots \), by the induction in \( m \).
If $m = 0$, then $\widetilde{T}$ is a $n$-multicyclic hyponormal operator by Lemmas 1 and 2. Thus Berger-Shaw’s theorem implies that

$$\text{tr}(\overline{T^*T - \overline{T}\overline{T}^*}) \leq \frac{n}{\pi} \text{Area}(\sigma(\overline{T})) = \frac{n}{\pi} \text{Area}(\sigma(T)),$$

because $\sigma(\overline{T}) = \sigma(T)$ by Lemma 2. Since, by Lemma 1,

$$\overline{T^*T - \overline{T}\overline{T}^*} \geq |T|^2 - |T|^{1-p}|T^*|^{2p}T|^{1-p}$$

$$= |T|^{1-p}(|T|^{2p} - |T^*|^{2p}) T|^{1-p},$$

we have

$$\text{tr}(|T|^{1-p}(|T|^{2p} - |T^*|^{2p}) T|^{1-p}) \leq \text{tr}(\overline{T^*T - \overline{T}\overline{T}^*})$$

$$\leq \frac{n}{\pi} \text{Area}(\sigma(T)).$$

Hence, the assertion holds for $m = 0$.

Next, we assume that the assertion holds for $m = k$ ($k \geq 0$). If $m = k + 1$, then $\overline{T}$ is $\frac{1}{2^{k+1}}$-hyponormal by Lemma 1. Hence by the assumption and by Lemmas 1 and 2, we have

$$\frac{n}{\pi} \text{Area}(\sigma(T)) = \frac{n}{\pi} \text{Area}(\sigma(\overline{T}))$$

$$\geq \text{tr}(|\overline{T}|^{1-\frac{1}{2^{k+1}}}(|\overline{T}|^{2^{1-\frac{1}{2^{k+1}}}} - |\overline{T}^*|^{2^{1-\frac{1}{2^{k+1}}}})|\overline{T}|^{1-\frac{1}{2^{k+1}}})$$

$$= \text{tr}(|T|^{2^{1-\frac{1}{2^{k+1}}}} - |T|^{2^{1-\frac{1}{2^{k+1}}} - p} T^* |^{2p} T |^{\frac{1}{2^{k+1}}} | T|^{\frac{1}{2^{k+1}}})$$

$$\leq \text{tr}(|T|^{2^{1-\frac{1}{2^{k+1}}} - |T|^{2^{1-\frac{1}{2^{k+1}}} - p} T^* |^{2p} T |^{\frac{1}{2^{k+1}}} | T|^{\frac{1}{2^{k+1}}})$$

$$\leq \text{tr}(|T|^{1-\frac{1}{2^{k+1}}}(|T|^{2^{1-\frac{1}{2^{k+1}}} - |T|^{\frac{1}{2^{k+1}}} - p} T^* |^{2p} T |^{\frac{1}{2^{k+1}}} | T|^{\frac{1}{2^{k+1}}})$$

Hence, the assertion holds for $m = k + 1$. This completes the proof of Lemma 3.
Corollary 1. If $T$ is an invertible $n$-multicyclic $p$-hyponormal operator, then $(T^*T)^p - (TT^*)^p \in C_1$ and

$$\text{tr}((T^*T)^p - (TT^*)^p) \leq \|T^{-1}\|^{2(1-p)} \frac{n}{\pi} \text{Area}(\sigma(T)).$$

**Proof.** Since $T$ is invertible, $T^*T \geq \|T^{-1}\|^{-2}$, and $n$-multicyclic $p$-hyponormality of $T$ implies that

$$\frac{n}{\pi} \text{Area}(\sigma(T)) \geq \text{tr}\left(\frac{1}{|T|^{1-p}} \left\{ (T^*T)^p - (TT^*)^p \right\} |T|^{1-p} \right)$$

by Lemma 3.

$$= \text{tr}\left(\left\{ (T^*T)^p - (TT^*)^p \right\} \frac{1}{2} (T^*T)^{1-p} \left\{ (T^*T)^p - (TT^*)^p \right\}^{\frac{1}{2}} \right)$$

$$\geq \|T^{-1}\|^{-2(1-p)} \text{tr}\left( (T^*T)^p - (TT^*)^p \right).$$

We have $(T^*T)^p - (TT^*)^p \in C_1$ and

$$\text{tr}\left( (T^*T)^p - (TT^*)^p \right) \leq \|T^{-1}\|^{2(1-p)} \frac{n}{\pi} \text{Area}(\sigma(T)).$$

This completes the proof of Corollary 1.

**Proof of Theorem.** By Lemma 3,

$$\text{tr}\left(\frac{1}{|T|^{1-p}} \left\{ |T|^2 - |T^*|^2 \right\} |T|^{1-p} \right) \leq \frac{n}{\pi} \text{Area}(\sigma(T)).$$

And by the property of trace,

$$\text{tr}\left(\frac{1}{|T|^{1-p}} \left\{ |T|^2 - |T^*|^2 \right\} |T|^{1-p} \right)$$

$$= \text{tr}\left(\left\{ |T|^2 - |T^*|^2 \right\} \frac{1}{2} \left\{ |T|^2 - |T^*|^2 \right\}^{\frac{1}{2}} \right)$$

$$= \text{tr}\left(\left\{ |T|^2 - |T^*|^2 \right\} \frac{1-p}{p} \left\{ |T|^2 - |T^*|^2 \right\}^{\frac{1}{2}} \right).$$

If $\frac{1}{2} \leq p \leq 1$, then $0 \leq \frac{1-p}{p} \leq 1$. Thus, by Heinz's inequality, we obtain

$$\text{tr}\left(\left\{ |T|^2 - |T^*|^2 \right\} \frac{1}{p} \left\{ |T|^2 - |T^*|^2 \right\}^{\frac{1}{2}} \right)$$

$$\geq \text{tr}\left(\left\{ |T|^2 - |T^*|^2 \right\} \frac{1}{p} \left\{ |T|^2 - |T^*|^2 \right\}^{\frac{1}{2}} \right)$$

$$= \text{tr}\left(\left\{ |T|^2 - |T^*|^2 \right\}^{1 + \frac{1-p}{p}} \right)$$

$$= \text{tr}\left(\left\{ |T|^2 - |T^*|^2 \right\}^{\frac{1}{p}} \right).$$
Therefore, we have
\[
\text{tr} \left( \left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{p}} \right) \leq \frac{n}{\pi} \text{Area}(\sigma(T)).
\]
Thus, the assertion of Theorem holds for \( p \in \left[ \frac{1}{2}, 1 \right] \).

If \( 0 < p \leq \frac{1}{2} \), then by Furuta's inequality([4]),
\[
(\left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{2}} \{ |T|^{2p} \}^{\frac{1-p}{p}} \left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{2}})^{2p} \\
\geq (\left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{2}} \{ |T|^{2p} \}^{\frac{1-p}{p}} \left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{2}})^{2p} \\
= (|T|^{2p} - |T^*|^{2p})^2.
\]
Thus \( (\left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{2}} \{ |T|^{2p} \}^{\frac{1-p}{p}} \left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{2}})^{2p} \) and \( (|T|^{2p} - |T^*|^{2p})^2 \) are both compact positive operators. Let \( s_n[A] \) be the \( n \)-th singular number of a positive compact operator \( A \). Then,
\[
s_n\left[ (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}} \{ |T|^{2p} \}^{\frac{1-p}{p}} \left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{2}} \right]^{2p} \\
\geq s_n\left[ (|T|^{2p} - |T^*|^{2p})^2 \right]^{\frac{1}{2p}} \\
= s_n\left[ (|T|^{2p} - |T^*|^{2p})^{\frac{1}{p}} \right].
\]
and hence,
\[
s_n\left[ (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}} \{ |T|^{2p} \}^{\frac{1-p}{p}} \left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{2}} \right] \\
\geq s_n\left[ (|T|^{2p} - |T^*|^{2p})^2 \right]^{\frac{1}{2}} \\
= s_n\left[ (|T|^{2p} - |T^*|^{2p})^{\frac{1}{p}} \right].
\]
Hence,
\[
\text{tr} \left( \left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{2}} \{ |T|^{2p} \}^{\frac{1-p}{p}} \left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{2}} \right) \\
\geq \text{tr} \left( \left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{p}} \right).
\]
Therefore, we have
\[
\text{tr} \left( \left( |T|^{2p} - |T^*|^{2p} \right)^{\frac{1}{p}} \right) \leq \frac{n}{\pi} \text{Area}(\sigma(T)).
\]
The assertion of Theorem also holds for $p \in (0, \frac{1}{2}]$. This completes the proof of Theorem.

For the restriction of a $p$-hyponormal operator to invariant subspace, we have the following.

**Lemma 4.** Let $\mathcal{M}$ be an invariant subspace for a $p$-hyponormal operator $T$, and $T'$ be the restriction of $T$ to $\mathcal{M}$. Then

$$\{T'^*T'\}^p \leq P(TT^*)^p P$$

$$\leq P(T^*T)^p P \leq \{T'^*T'\}^p,$$

and $T'$ is also $p$-hyponormal, where $P$ denotes the projection onto $\mathcal{M}$.

**Proof.** Since $T' = TP$,

$$T'^*T' = PT^*TP,$$

and hence, for any $s \in (0, 1]$,

$$\{T'^*T'\}^s = \{PT^*TP\}^s \geq P(T^*T)^s P \text{ by Hansen's inequality(5)).}$$

While,

$$T'^*T' = TPT^* = PTPT^* P,$$

we have, for any $s \in (0, 1]$,

$$\{T'^*T'\}^s = (TPT^*)^s$$

$$= P(TPT^*)^s P$$

$$\leq P(TT^*)^s P \text{ by Heinz's inequality.}$$

Therefore, if $T$ is $p$-hyponormal for $p$ such as $0 < p \leq 1$, then

$$\{T'^*T'\}^p \leq P(TT^*)^p P$$

$$\leq P(T^*T)^p P$$

$$\leq \{T'^*T'\}^p.$$

Thus, $T'$ is also $p$-hyponormal.

**Corollary 2.** If $T$ is $p$-hyponormal operator, then

$$\|(T^*T)^p - (TT^*)^p\| \leq \left\{\frac{1}{\pi} \text{Area}(\sigma(T))\right\}^p.$$
Proof. Let $x$ be an arbitrary unit vector in $\mathcal{H}$. We define

$$\mathcal{H}_0 = \vee \{g(T)x; \ g \in \mathcal{R}(\sigma(T))\}.$$ 

Since $\mathcal{H}_0$ is an invariant subspace for $T$, Lemma 4 implies that $T' = T|_{\mathcal{H}_0}$ is a $(1$-multicyclic) $p$-hyponormal operator. If $\lambda \in \rho(T)$, then, for any $y \in \mathcal{H}_0$, $(T - \lambda)^{-1}y \in \mathcal{H}_0$. Therefore, $\lambda \in \rho(T')$. Hence, $\sigma(T') \subset \sigma(T)$. By Theorem,

$$\text{tr}\left(\{(T''^*T')^p - (T'T'^*)^p\}^\frac{1}{p}\right) \leq \frac{1}{\pi} \text{Area}(\sigma(T'))$$

and the maximal eigenvalue of positive trace class operator $\{(T''^*T')^p - (T'T'^*)^p\}^\frac{1}{p}$ is equal to or less than $\frac{1}{\pi} \text{Area}(\sigma(T))$. Thus, the maximal eigenvalue of $(T''^*T')^p - (T'T'^*)^p$ is equal to or less than $\{\frac{1}{\pi} \text{Area}(\sigma(T))\}^p$. Therefore,

$$\|(T''^*T')^p - (T'T'^*)^p\| \leq \{\frac{1}{\pi} \text{Area}(\sigma(T))\}^p.$$ 

Let $P$ be the projection onto $\mathcal{H}_0$. Then, by Lemma 4,

$$\begin{align*}
\{\frac{1}{\pi} \text{Area}(\sigma(T))\}^p \\
\geq \langle\{(T''^*T')^p - (T'T'^*)^p\}x, x\rangle \\
\geq \langle\{P(T''^*T')^pP - P(T'T'^*)^pP\}x, x\rangle \\
= \langle\{(T''^*T')^p - (T'T'^*)^p\}x, x\rangle.
\end{align*}$$

Since $x \in \mathcal{H}$ is arbitrary unit vector,

$$\|(T''^*T')^p - (T'T'^*)^p\| \leq \{\frac{1}{\pi} \text{Area}(\sigma(T))\}^p.$$ 

This inequality is an extension of the Putnam inequality to the case of $p$-hyponormal operator.

Corollary 3. If $T$ is an invertible $p$-hyponormal operator, then

$$\|(T''^*T')^p - (T'T'^*)^p\| \leq \|T^{-1}\|^{2(1-p)}\frac{1}{\pi} \text{Area}(\sigma(T)).$$
Proof. Put $\mathcal{H}_0, T' = T|_{\mathcal{H}_0}$ and $P$ as Corollary 2, then $T'$ is an invertible (1-multicyclic) $p$-hyponormal operator. Therefore by Lemma 3,

$$\frac{1}{\pi} \text{Area}(\sigma(T)) \geq \frac{1}{\pi} \text{Area}(\sigma(T'))$$

$$\geq \text{tr} \left( | T' |^{1-p} \{ (T'^* T')^p - (T'^* T'^*)^p \} | T' |^{1-p} \right)$$

$$= \text{tr} \left( \{ (T'^* T')^p - (T'^* T'^*)^p \} \frac{1}{2} (T'^* T'^*)^{1-p} \{ (T'^* T'^*)^p - (T'^* T'^*)^p \} \frac{1}{2} \right)$$

$$= \text{tr} \left( \{ (T'^* T')^p - (T'^* T'^*)^p \} \frac{1}{2} (PT* TP)^{1-p} \{ (T'^* T'^*)^p - (T'^* T'^*)^p \} \frac{1}{2} \right)$$

$$\geq \text{tr} \left( \{ (T'^* T'^*)^p - (T'^* T'^*)^p \} \frac{1}{2} P(T*T')^{1-p} P\{ (T'^* T'^*)^p - (T'^* T'^*)^p \} \frac{1}{2} \right)$$

(by Hansen’s inequality)

$$= \text{tr} \left( \{ (T'^* T')^p - (T'^* T'^*)^p \} \frac{1}{2} (T*T')^{1-p} \{ (T'^* T'^*)^p - (T'^* T'^*)^p \} \frac{1}{2} \right)$$

$$\geq ||T^{-1}||^{-2(1-p)} \text{tr} \left( (T'^* T')^p - (T'^* T'^*)^p \right) .$$

Therefore

$$||T^{-1}||^{2(1-p)} \frac{1}{\pi} \text{Area}(\sigma(T))$$

$$\geq \text{tr} \left( (T'^* T')^p - (T'^* T'^*)^p \right)$$

$$\geq ||(T'^* T'^*)^p - (T'^* T'^*)^p||$$

$$\geq \langle \{ (T'^* T'^*)^p - (T'^* T'^*)^p \} x, x \rangle$$

$$\geq \langle \{ P(T* T')^p P - P(TT*)^p P \} x, x \rangle$$

(by Lemma 4)

$$= \langle \{ (T* T')^p - (TT*)^p \} x, x \rangle .$$

Since $x \in \mathcal{H}$ is arbitrary unit vector,

$$||(T'^* T')^p - (TT*)^p|| \leq ||T^{-1}||^{2(1-p)} \frac{1}{\pi} \text{Area}(\sigma(T)).$$

Remark. Putnam inequality was extended to the $p$-hyponormal operator by Xia in the case of $\frac{1}{2} \leq p \leq 1$, and by Cho-Itoh in the case of $0 < p \leq \frac{1}{2}$. Their estimation is different from ours.
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